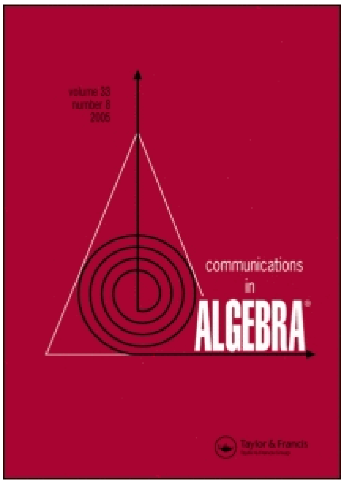

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

THE FULLY INVARIANT EXTENDING PROPERTY FOR ABELIAN GROUPS

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Online Publication Date: 31 January 2001

To cite this Article Birkenmeier, Gary F., Clugreanu, Girgore, Fuchs, Laszlo and Goeters, H. Pat(2001)'THE FULLY INVARIANT EXTENDING PROPERTY FOR ABELIAN GROUPS',Communications in Algebra,29:2,673 — 685

To link to this Article: DOI: 10.1081/AGB-100001532

URL: http://dx.doi.org/10.1081/AGB-100001532

COMMUNICATIONS IN ALGEBRA, 29(2), 673–685 (2001)

THE FULLY INVARIANT EXTENDING PROPERTY FOR ABELIAN GROUPS

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ABSTRACT

An abelian group has the *FI-extending* property if every fully invariant subgroup is essential in a direct summand. A mixed abelian group has the FI-extending property if and only if it is a direct sum of a torsion and a torsion-free abelian group, both with the FI-extending property. A full characterization is obtained for the abelian groups with the FI-extending property which are either torsion-free of finite rank or torsion.

Key Words: Abelian group; Fully invariant; Torsion; Torsion-free; Irreducible; Rank; Mixed group.

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A (left) module M over a ring R is said to have the *extending property* if every submodule of M is contained as an essential submodule in a direct summand of M. Modules with the extension property have been investigated by several authors (see, e.g., (1) and the literature listed there).

In a recent paper, Birkenmeier-Müller-Rizvi (2) defined a module M to have the *FI-extending property* (fully invariant extending) if every fully invariant submodule of M is contained as an essential submodule in a direct summand of M. They have obtained several results on the behavior of such modules. In this note, we are dealing with the case $R = \mathbb{Z}$, i.e., with abelian groups which enjoy the FI-extending property, and trying to find the structure of such groups.

Our main results can be summarized as follows.

1. A mixed group has the FI-extending property if and only if it is a direct sum of a torsion and a torsion-free group, both with the FI-extending property.
2. A torsion group has the FI-extending property exactly if it is a direct sum of a divisible group and a separable torsion group (i.e., its p-components have no elements $\neq 0$ of infinite height).
3. A torsion-free group of finite rank has the FI-extending property if and only if it is a finite direct sum of torsion-free groups that are irreducible in the sense of J. Reid. They can also be characterized via their quasiendomorphism rings which are in this case full matrix rings over division rings.

Actually, the characterization of the torsion-free case does not rely on properties of the integers, so it can easily be extended to torsion-free modules over integral domains.

We have no satisfactory results in the infinite rank case. We shall make a few comments on the homogeneous summands and characterize the vector groups with the FI-extending property.

An important corollary to our results is that the FI-extending property is inherited by direct summands—this does not seem to hold for general domains (though no counterexample is known to us).

1. PRELIMINARY LEMMAS

In this note, "group" will mean "abelian group," or equivalently, a $\mathbb{Z}$-module. For unexplained terminology and facts, we refer to Fuchs (3).

We start our discussion with a few lemmas on groups with the FI-extending property. They are valid for modules over arbitrary commutative domains (the first three lemmas even over any ring), so we phrase them to cover the module case.

The following result is in (2), but we include its proof for the sake of completeness.

Lemma 1.1. *Direct sums of modules with the FI-extending property have again the FI-extending property.*

Proof: Suppose the modules $A_i (i \in I)$ have the FI-extending property. If F is a fully invariant submodule of the direct sum $A = \oplus_{i \in I} A_i$, then $F = \oplus_{i \in I}(F \cap A_i)$ by full invariance. Clearly, $F \cap A_i$ is fully invariant in A_i for each $i \in I$, so it is contained as an essential submodule in a summand H_i of A_i. Then $H = \oplus_{i \in I} H_i$ is a summand of A that is an essential extension of the submodule F. □

Lemma 1.2. *If the module $A = B \oplus C$ has the FI-extending property and B is a fully invariant summand, then both B and C have the FI-extending property.*

Proof: A fully invariant submodule F of$B$ is fully invariant in A. If H is a summand of A containing F as an essential submodule, then $B \cap H$ will be a summand of B that contains F as an essential submodule.

To conclude that C has the FI-extending property, pick a fully invariant submodule F of C, and apply the FI-extending property of A to its fully invariant submodule $B \oplus F$. We infer that a summand H of A contains $B \oplus F$ as an essential submodule. Then $H = B \oplus (H \cap C)$, where $H \cap C$ is summand of C with F as an essential submodule. □

The following simple observation (which also holds for quasiinjectives) is well known, and we state it for easy reference.

Lemma 1.3. *Injective modules share the FI-extending property.*

Recall that by an RD-submodule (relatively divisible) of an R-module M (R a domain) is meant a submodule N such that $rN = N \cap rM$ for all $r \in R$. (For abelian groups, this is the same notion as purity.) In a torsion-free module over a domain, every submodule is contained as an essential submodule in a unique relatively divisible submodule, called the RD-hull—this is the intersection of all RD-submodules containing it.

Lemma 1.4. *A torsion-free module over an integral domain has the FI-extending property exactly if every fully invariant RD-submodule is a summand.*

Proof: This is an immediate consequence of the fact that the RD-hull of a fully invariant submodule of a torsion-free module is again fully invariant. □

2. TORSION GROUPS WITH THE FI-EXTENDING PROPERTY

From now on we focus our attention on abelian groups. We start with torsion groups.

As is very often the case in the theory of abelian groups, the discussion of torsion groups can be reduced at once to the case of p-groups. Indeed, as the

p-components of a torsion abelian group are fully invariant subgroups, from Lemmas 1.1 and 1.2 we can conclude at once:

Proposition 2.1. *A torsion group has the FI-extending property exactly if all of its p-components enjoy the FI-extending property.*

Consequently, we may restrict our discussion to p-groups A. A crucial fact is concerned with the first Ulm subgroup $A^1 = \bigcap_{n\geq 0} p^n A$ of A (in other words, this is the set of elements of infinite height in A). A p-group with trivial first Ulm subgroup is called *separable.*

Lemma 2.2. *Let A be a p-group. If A has the FI-extending property, then its first Ulm subgroup is divisible. Hence*

$$A = D \oplus C,$$

where D is a divisible group and C is a separable p-group.

Proof: By hypothesis, A^1 is essential in a summand D of A. Clearly, the socle $A^1[p] = \{a \in A^1 \mid pa = 0\}$ of A^1 satisfies $A^1[p] = D[p]$, thus every element of the socle $D[p]$ has infinite height in A, and hence in D. Consequently, D is divisible (3, 20(C)), so it must be equal to A^1.

As a divisible subgroup, D is a summand of A, so we can write $A = D \oplus C$ with C reduced. Clearly, $A^1 = D$ implies that $C^1 = 0$. □

In order to proceed, we rely on Kaplansky's structure theorem for fully invariant subgroups of separable p-groups. This theorem states that a fully invariant subgroup F of a separable p-group A is of the form

$$F = A(n_0, n_1, \ldots, n_k, \ldots) = \{a \in A \mid h(p^k a) \geq n_k\},$$

where $h(x)$ denotes the height of $x \in A$, and the n_k form a strictly increasing sequence of integers ≥ 0 and possible symbols ∞.

Theorem 2.3. *A torsion group has the FI-extending property if and only if it is a direct sum of a divisible group and separable p-groups.*

Proof: By Lemmas 2.1 and 2.2, only the sufficiency requires a proof. In view of Lemmas 1.1 and 1.3, it is enough to show that every separable p-group A has the FI-extending property.

Let F be a fully invariant subgroup of A; as indicated, it is of the form $F = A(n_0, n_1, \ldots, n_k, \ldots)$. From the definition it is evident that

$$p^{n_0} A[p] \leq F \leq p^{n_0} A.$$

Thus, F is essential in $p^{n_0} A$, and the proof is reduced to the case $F = p^{n_0} A$.

By a theorem of Khabbaz (3, Thm 27.7), a subgroup C of A that is maximal with respect to the property of being disjoint from $p^{n_0} A$ is a summand of A. C is a pure subgroup bounded by p^{n_0}, thus there is a decomposition $A = C \oplus B$ with $B \geq p^{n_0} A$ (actually B can be any subgroup maximal with respect to the properties $B \geq p^{n_0}A$ and $B \cap C = 0$). From the construction it is evident that $p^{n_0}A$ is essential in summand B. □

3. THE MIXED CASE

It is easy to dispose of the mixed case by reducing the problem to the torsion and torsion-free cases.

Proposition 3.1. *A mixed group has the FI-extending property if and only if it is the direct sum of a torsion group and a torsion-free group, both with the FI-extending property.*

Proof: Let A be a mixed group with the FI-extending property. Its torsion subgroup T is fully invariant and has no essential extension in A, so it must be a summand of A. The rest follows straightforwardly from Lemmas 1.1 and 1.2. □

By making use of our results on the structure of mixed and torsion groups with the FI-extending property, we can now easily conclude that the FI-extending property is inherited by summands.

Theorem 3.2. *Every summand of a group with the FI-extending property enjoys the FI-extending property.*

Proof: It is well known (and easy to prove) that summands of splitting mixed groups are splitting, so the proof is reduced at once to the torsion and torsion-free cases. Furthermore, in the torsion case it is obviously enough to consider p-groups.

For p-groups observe that any summand of a direct sum of a divisible and a separable p-group is again of the same kind. For torsion-free groups A the claim is easy to verify: if $A = B \oplus C$ and X is an RD-submodule of B which is fully invariant in B, then consider the fully invariant submodule F of A generated by X. Its RD-hull F_* is a summand of A and satisfies $F_* = (B \cap F_*) \oplus (C \cap F_*)$. As the first summand is equal to X, X is a summand of B. □

We do not know how to prove the preceding theorem without relying on our result on the structure of torsion groups with the FI-extending property.

4. FINITE RANK TORSION-FREE GROUPS WITH THE FI-EXTENDING PROPERTY

It is considerably more difficult to characterize torsion-free groups which possess this property. We do not have any satisfactory characterization in the general case, but a fairly informative result is available whenever the groups are of finite rank.

Following J.D. Reid, a torsion-free group A is called *irreducible* when A has no pure fully invariant subgroups other than 0 and A. Irreducible torsion-free groups have been studied a great deal; in particular, we refer the reader to Reid's papers (4–6).

Let $E(A)$ denote the endomorphism ring of the torsion-free group A. $\mathbb{Q} \otimes E(A)$ is called the *quasiendomorphism ring* of A (we shall simply write $QE(A)$). This is a Q-vector space whose dimension is finite whenever the group A has finite rank.

In the following result—which is concerned with the finite rank case—the condition of having finite rank can be replaced by the more general condition that the quasiendomorphism ring of the group is left or right artinian. Moreover, instead of groups we could have stated it for modules over domains.

Theorem 4.1. *A torsion-free group whose quasiendomorphism ring is left or right artinian has the FI-extending property if and only if it is a finite direct sum of irreducible groups.*

Proof: Owing to the above definition, irreducible groups have the FI-extending property. Hence, in view of Lemma 1.1 it is evident that their (finite) direct sum also has this property.

If the quasiendomorphism ring of a torsion-free group is artinian on either side, then it contains no infinite set of orthogonal idempotents. Hence, it follows at once that the group must be a finite direct sum of indecomposable groups. By Theorem 3.2, each of these summands inherits the FI-extending property. Furthermore, it is pretty obvious that an indecomposable group possesses the FI-extending property if and only if each of its nonzero fully invariant subgroups is an essential subgroup. Consequently, an indecomposable torsion-free group has the FI-extending property exactly if it is irreducible. □

We require more information about irreducible groups of finite rank. Fortunately, they have been satisfactorily characterized by Reid (5).

We now recall a couple of definitions needed in the next theorem. Two torsion-free groups of finite rank are called *quasiisomorphic* if each is isomorphic to a subgroup of finite index of the other group. The group A is *strongly indecomposable* if A is not quasiisomorphic to the direct sum of two nonzero groups.

Theorem 4.2 (Reid (5)). *For a torsion-free group A of finite rank*, *the following are equivalent*:

(i) *A is irreducible*;
(ii) *A is quasiisomorphic to a finite direct sum of isomorphic*, *strongly indecomposable irreducible groups*;

(iii) *the quasiendomorphism ring* $QE(A)$ *of* A *is the full* $m \times m$ *matrix ring* (*for some integer* m) *over a division ring* D; *and the rank of* A *is* md *where* d *denotes the* $\mathbb{Q}$-*dimension of* D.

To complement this theorem, it should be observed that an irreducible group of finite rank is strongly indecomposable exactly if its quasiendomorphism ring is a division ring; see Reid [5].

Irreducible groups exist for every finite rank, as is shown by free abelian groups. To convince ourselves that there are indecomposable irreducible groups as well, we exhibit such examples.

Example 4.3. Let R denote the group of integers in a finite algebraic extension of $\mathbb{Q}$, say, of degree n. By Zassenhaus (7), there exists a torsion-free group G of rank n, whose endomorphism ring is isomorphic to R. As $R \otimes \mathbb{Q}$ is a field, all the endomorphisms of G are monic, thus the subgroup Rg must have rank n for every nonzero $g \in G$. We conclude that Rg is an essential fully invariant subgroup of G. Manifestly, G is an indecomposable irreducible group.

5. THE INFINITE RANK CASE

Though the FI-extending property imposes quite a restriction on infinite rank torsion-free groups, their classification seems to be beyond reach at this time. As Rüdiger Göbel pointed out, all homogeneous separable torsion-free groups of type Z are irreducible, and hence share the FI-extending property, but they cannot be classified satisfactorily.

A torsion-free module G over an integral domain R is called *strongly irreducible*, if for any nonzero fully invariant submodule F, there is a nonzero $r \in R$ such that $rG \leq F$. Irreducible modules are not usually strongly irreducible, but a collection of strongly irreducible modules arises in the study of vector groups with the FI-extending property.

Example 5.1. For any domain R, any direct product of copies of R is strongly irreducible, thus it has the FI-extending property. To prove this, let $P = \prod_{i \in I} X_i$ with $X_i \cong R$ for all $i \in I$, and $F \neq 0$ a fully invariant submodule of P. Suppose $f \in F$ has ith component $r \neq 0$. Since F is fully invariant, F contains rX_i. Therefore, F contains $\mathrm{Hom}(X_i, G)rX_i = r\mathrm{Hom}(X_i, G)X_i = rG$ due to the fact that X_i is isomorphic to R.

We wish to establish a few relevant properties which might help to understand the effect of the FI-extending property on the structure of infinite rank torsion-free groups.

Suppose G is a torsion-free group with the FI-extending property. For any type $\mathbf{t}$, the subgroups

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$$G(\mathbf{t}) = \{a \in G \mid \mathbf{t}(a) \geq \mathbf{t}\} \quad \text{and} \quad G^*(\mathbf{t})_* = \{a \in G \mid \mathbf{t}(a) > \mathbf{t}\}_*$$

are fully invariant pure subgroups, and hence summands of G (lower stars denote purification). We conclude that there is a direct decomposition

$$G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$$

for some $\mathbf{t}$-homogeneous summand $G_{\mathbf{t}}$ of G. It should be emphasized that $G_{\mathbf{t}}$ is not uniquely determined, though it is unique within isomorphism.

Observe that $G_{\mathbf{t}}$ is a fully invariant subgroup in G only if either $G_{\mathbf{t}} = 0$ or $G^*(\mathbf{t})_* = 0$.

Theorem 5.2. *Let G be a torsion-free group with the FI-extending property, and let $\mathbf{T}$ denote its typeset.*

(i) *The subgroups $G_{\mathbf{t}}$ $(\mathbf{t} \in \mathbf{T})$ have the FI-extending property.*

(ii) *For every finite subset $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}$ of $\mathbf{T}$, the direct sum*

$$G_{\mathbf{t}_1} \oplus \cdots \oplus G_{\mathbf{t}_k}$$

is a summand of G.

(iii) *$\bar{G} = \sum_{\mathbf{t} \in \mathbf{T}} G_{\mathbf{t}}$ is the direct sum of the $G_{\mathbf{t}}$. It is a pure subgroup in G with the FI-extending property, unique up to isomorphism.*

(iv) *If the typeset $\mathbf{T}$ satisfies the ascending chain condition, then*

$$G = \bigoplus_{\mathbf{t} \in \mathbf{T}} G_{\mathbf{t}}.$$

Proof:

(i) Since both $G(\mathbf{t})$ and $G^*(\mathbf{t})_*$ are fully invariant and pure in G, the claim follows from Lemma 1.2 at once.

(ii) The proof of (i) shows that each $G_{\mathbf{t}_i}$ is a summand of G. Pick a minimal type, say $\mathbf{t}_1$, in the given set of types, and write $G = G_{\mathbf{t}_1} \oplus H_1$ for a subgroup H_1 of G. If $\mathbf{t}_2$ is a minimal in the set of remaining types, then the full invariance of $G(\mathbf{t}_2)$ implies that it is contained in, and hence a summand of, the summand H_1. Thus, we can write $G = G_{\mathbf{t}_1} \oplus G_{\mathbf{t}_2} \oplus H_2$ for a subgroup H_2 of G. Continuing in this way, we obtain a decomposition of G, as desired.

(iii) It is an immediate consequence of (ii) that the $G_{\mathbf{t}}$ generate their direct sum $\bar{G}$ in G and $\bar{G}$ is pure in G. The uniqueness of $\bar{G}$ within isomorphism is obvious from the uniqueness of the $G_{\mathbf{t}}$ up to isomorphism.

(iv) Suppose that $\mathbf{T}$ satisfies the ascending chain condition, and there are elements not included in this direct sum. Choose one, $g \in G$, of maximal type from among the missing elements. If $\mathbf{t}$ denotes the type of g, then $g \in G(\mathbf{t}) = G_{\mathbf{t}} \oplus G^*(\mathbf{t})_*$. By the choice of g, both summands are contained in the direct sum of the $G_{\mathbf{s}}$ $(\mathbf{s} \in \mathbf{T})$. This contradiction proves the claim. $\square$

It is easy to show that the subgroup $\bar{G} = \sum_{\mathbf{t}\in\mathbf{T}} G_{\mathbf{t}}$ need not be all of G. For instance,

Example 5.3. Let $\mathbf{t}_0 < \mathbf{t}_1 < \cdots < \mathbf{t}_n < \cdots$ be a strictly increasing sequence of idempotent types, and A_n a torsion-free rank 1 group of type $\mathbf{t}_n$. Define

$$G = \prod_{n<w} A_m.$$

Then the typeset of G is just $\mathbf{T} = \{\mathbf{t}_n \mid n < \omega\}$. Furthermore, we have $G(\mathbf{t}_n) = \prod_{j\geq n} A_j$, so that we can choose $G_{\mathbf{t}_n} = A_n$ for each n, in which case $\bar{G}$ is just the direct sum of the A_n. (Theorem 6.5 later will show that G has the FI-extending property.)

By the way, the correspondence $G \to \bar{G}$ may be viewed as a *grading functor* (see Bourbaki (8)) from the category of torsion-free groups with the FI-extending property to the subcategory of graded torsion-free groups with the same property. The grading is provided by the lattice of all types. We are not going to refer to this functor in the sequel.

We would like to point out that (unlike the direct sum) the direct product of torsion-free irreducible groups does not in general have the FI-extending property.

Example 5.4. Let R be a subring of an algebraic number field such that $R/pR \cong \mathbb{Z}/p\mathbb{Z}$ for all p such that $pR \neq R$, and assume there are infinitely many such p's. Partition the set $S = \{p \mid pR \neq R\}$ into infinitely many mutually disjoint sets $S_1, S_2, \ldots$. Take $G_j = X_j \otimes R$ where $X_j = \langle 1/p \mid p \in S_j\rangle$ and $G = \prod_j G_j$. Note that G_j is irreducible (actually quasipure injective, hence strongly homogeneous). Let $F_j \cong \mathrm{End}(G_j)$ and so $F = \prod_j F_j$ is fully invariant in G. Then, F_* contains $\oplus_j G_j$, and so G/F_* is divisible. But $a = (1/p_1, 1/p_2, \ldots) \in G \backslash F_*$ for any $p_i \in S_i$. Therefore, F_* cannot be a summand of G since G is reduced while G/F_* is nonzero divisible.

6. VECTOR GROUPS

By a *vector group* is meant the cartesian product of groups each of which is isomorphic to a subgroup of $\mathbb{Q}$. We wonder which vector groups enjoy the FI-extending property.

We associate with a torsion-free group X a set of primes: $\pi(X) = \{p \mid pX \neq X\}$.

Lemma 6.1. *If X is a subgroup of $\mathbb{Q}$, not of idempotent type, then the cartesian power of X, $G = \prod_I X$, does not have the FI-extending property whenever I is an infinite set.*

Proof: Without loss of generality we may assume that $\mathbb{Z} \leq X$, and—by passing to a summand—that $I = \{1, 2, \ldots\}$. The type of X is not idempotent, so $\pi(X)$ is an infinite set: $\{p_1, p_2, \ldots\}$. Consider the fully invariant subgroup

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$F = E(G)e_1$ where $e_1 = (1, 0, 0, \ldots)$ and $E(G)$ is the endomorphism ring of G. Then $F = \mathrm{Hom}(X_1, G)e_1$ where $X_1 = \{(x, 0, 0, \ldots) \mid x \in X\}$, and $\mathrm{Hom}(X, G) = \prod_{j=1}^{\infty} \mathrm{Hom}(X, X)$ implies $F = \prod_j E(X)$.

But then, $G/F = \prod_j T$ where $T = X/Z$. Since T is reduced and unbounded, $G \backslash F_*$ is not empty. Also, it is easy to see that $T/t(T)$ is divisible, where $t(T)$ denotes the torsion subgroup of T. So, G/F_* is nonzero divisible, and consequently, F_* cannot be a summand of G. □

Lemma 6.2. *If X_j $(j = 1, 2, \ldots)$ are reduced rank 1 torsion-free groups such that* type $X_1 >$ type $X_2 > \cdots$, *then $G = \prod_j X_j$ fails to have the FI-extending property.*

Proof: Let p be a prime such that $pX_1 \neq X_1$. Then $pX_j \neq X_j$ for all j. The subgroup $F = \prod_j p^j X_j$ is fully invariant in G, and so it suffices to show that F_* cannot be a summand of G. Note that $T = G/F$ is isomorphic to the direct product $T \cong \prod_j (\mathbb{Z}/p^j\mathbb{Z})$. To simplify the argument, we will identify T with this direct product.

Now, G/F_* is isomorphic to the group $T/t(T)$ where $t(T)$ represents the torsion subgroup of T. But T is cotorsion, hence so is $T/t(T)$, while G—as a product of slender groups—cannot contain such a summand. Therefore, F_* is not a summand of G. □

Lemma 6.3. *If $X_1, X_2, \ldots,$ are rank 1 torsion-free groups with pair-wise incomparable types, then $G = \prod_j X_j$ does not have the FI-extending property.*

Proof: It is enough to show that some direct summand of G does not have the FI-extending property. Set $\pi_j = \pi(X_j)$ and let $\mathbf{t}_j$ denote the type of X_j. Choose $p_1 \in \pi_1$. Because $\mathbf{t}_2$ is incomparable to $\mathbf{t}_1$, there is a $p_2 \in \pi_2$ different from p_1. It is possible that $\pi_3 = \{p_1, p_2\}$; if this is the case, then discard X_3 and reindex. Then, there exists $p_3 \in \pi_3$ different from p_1, p_2. Continuing, having found distinct primes $p_1, \ldots, p_n$ such that $p_i \in \pi_i$ for $i = 1, \ldots, n$, there are only finitely many $\mathbf{t}_i$'s with $\pi_i \subseteq \{p_1, \ldots, p_n\}$; discard the corresponding X_i's and reindex, in order to find a $p_{n+1} \in \pi_{n+1}$ different from $p_1, \ldots, p_n$.

Let $F = \prod_j (p_j X_j)$. Then F is fully invariant in G, since there are no homomorphisms between different terms X_i and X_j. So, $C = G/F$ is isomorphic to the direct product $\prod_j (\mathbb{Z}/p_j\mathbb{Z})$. Hence, modulo its torsion subgroup, C is divisible, and so, as in the previous lemma, we conclude that F_* cannot be a summand of the reduced group G. □

Lemma 6.4. *If $X_1, X_2, \ldots$ are rank one torsion-free groups whose types are non-idempotent and satisfy $\mathbf{t}_1 < \mathbf{t}_2 < \cdots$, then $G = \prod_j X_j$ does not have the FI-extending property.*

Proof: As before, let $\pi_j = \pi(X_j)$ and for $T = X_1/\mathbb{Z}$, let $T_j = T_{\pi_j}$ be the localization of T at the set of primes π_j. We will assume that $\mathbb{Z} \leq X_1 \leq X_2 \leq \cdots$. Let h_j denote the height sequence for $1 \in X_j$. It is easy to see that $Y_j = \mathrm{Hom}(X_1, X_j)$

is isomorphic to the subgroup of X_j containing $\mathbb{Z}$ in which the height sequence of 1 is h'_j where $h'_j(p) = h_j(p) - h_1(p)$ when $h_j(p)$ is finite, and $h'_j(p) = \infty$, when $h_j(p) = \infty$. We may identify Y_j with that subgroup by viewing an element of $\mathrm{Hom}(X_1, X_j)$ as multiplication by the appropriate rational.

Consider the fully invariant subgroup $F = E(G)e_1$ where e_1 denotes the vector $(1, 0, 0, \dots)$ and $E(G)$ is the endomorphism ring of G. Then $F = \mathrm{Hom}(X_1, G)e_1 = \prod_j Y_j$. By looking at height sequences, we see that $X_j/Y_j \cong T_j$ for each $j = 1, 2, \dots$. Therefore, $C = G/F$ is isomorphic to the product $C = \prod_j T_j$. But in this case, C modulo its torsion subgroup is divisible, and consequently, F_* cannot be a summand of G. □

A word of caution: if $\mathbf{t}_1 < \mathbf{t}_2 < \cdots$ are idempotent types, then—as we will show—the group $G = \prod_j X_j$ does have the FI-extending property.

Theorem 6.5. *Let G be a vector group, and write $G = \prod_{t-\boldsymbol{T}} \in \boldsymbol{T} G_{\mathbf{t}}$ where $G_{\mathbf{t}}$ is a direct product of isomorphic rank one groups of type $\mathbf{t}$. Then G has the FI-extending property if and only if the following hold*:

(a) *the set $\boldsymbol{T}$ of types contains but finitely many non-idempotent types, and $G_{\mathbf{t}}$ has finite rank whenever $\mathbf{t} \in \boldsymbol{T}$ is non-idempotent*;
(b) *$\boldsymbol{T}$ satisfies the descending chain condition, and*
(c) *$\boldsymbol{T}$ contains no infinite set of incomparable types.*

Proof: Assume G has the FI-extending property. Since summands of G also have this property, Lemmas 6.2 and 6.3 imply (b) and (c), respectively. These along with Lemma 6.4 show that $\boldsymbol{T}$ cannot contain infinitely many nonidempotent types $\mathbf{t}$, and by Lemma 6.1, $G_{\mathbf{t}}$ has finite rank for each such $\mathbf{t} \in \boldsymbol{T}$.

Conversely, suppose G satisfies (a)–(c). Write $G = C \oplus G_0$ where C is finite rank completely decomposable and $G_0 = \prod_{\boldsymbol{T}_0} G_{\mathbf{t}}$ ($\boldsymbol{T}_0$ denotes the set of idempotent types in $\boldsymbol{T}$). Given a nonzero fully invariant subgroup F of G_0, we consider the subset S of $\boldsymbol{T}_0$ consisting of those types $\mathbf{t} \in \boldsymbol{T}_0$ for which F contains an element with nonzero $G_{\mathbf{t}}$-coordinate.

Clearly, $F \le \prod_{\mathbf{t}\in S} G_{\mathbf{t}}$. By (b), S has minimal elements, but only finitely many—as guaranteed by (c): let $\mathbf{s}_1, \dots, \mathbf{s}_n \in S$ be such that any $\mathbf{t} \in S$ satisfies $\mathbf{t} \ge \mathbf{s}_j$ for some j. For each $j = 1, \dots, n$, the intersection $F \cap G_{\mathbf{s}_j}$ is a nonzero fully invariant subgroup in $G_{\mathbf{s}_j}$, and so by Example 5.1, there is an integer $0 \ne m_j$ such that $m_j G_{\mathbf{s}_j} \le F$. Let $S_j = \{\mathbf{t} \in S \mid \mathbf{t} \ge \mathbf{s}_j\}$. Now, if X is a rank-1 summand of $G_{\mathbf{s}_j}$ (of type $\mathbf{s}_j$), then $\mathrm{Hom}(X, \prod_{\mathbf{t}\in S_j} G_{\mathbf{t}}) = \prod_{\mathbf{t}\in S_j} G_{\mathbf{t}}$, since X is a ring and $\prod_{\mathbf{t}\in S_j} G_{\mathbf{t}}$ is an X-module.

Hence, it follows that F contains $m_j \prod_{\mathbf{t}\in S_j} G_{\mathbf{t}}$, and therefore it also contains $m \prod_{\mathbf{t}\in S} G_{\mathbf{t}}$ where m is the least common multiple of the m_j's. We now have $m \prod_{\mathbf{t}\in S} G_{\mathbf{t}} \le F \le \prod_{\mathbf{t}\in S} G_{\mathbf{t}}$ for $m \ne 0$, and so $F_* = \prod_{\mathbf{t}\in S} G_{\mathbf{t}}$ is in fact a direct summand of G. □

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7. STRONGLY FI-EXTENDING GROUPS

Call the module M *strongly FI-extending* if every fully invariant submodule embeds as an essential submodule in a fully invariant summand. By making use of our results on the FI-extending property, it is easy to obtain a characterization of groups subject to this stronger condition.

Theorem 7.1. *An abelian group A is strongly FI-extending if and only if*

$$A = B \oplus C \oplus D,$$

where
(a) *B is a direct sum of p-groups each of which is the direct sum of cyclic groups of the same order*;
(b) *C is a torsion-free FI-extending group*;
(c) *D is a divisible group*;

such that if B has a nontrivial p-component, *then C is p-divisible.*

Proof: A strongly FI-extending group has the FI-extending property, thus our task consists in eliminating those which do not enjoy the stronger property.

First assume A is a strongly FI-extending group. Thus, we can write $A = B \oplus C \oplus D$ with B separable torsion and C, D as in (b)–(c). Clearly, the summands inherit the strongly FI-extending property, so B is a separable p-group that is strongly FI-extending. If, for some integer $n > 0$, $p^n A \neq 0$, then the socle of $p^n A$ must be equal to the socle of A, since otherwise a summand of A containing $p^n A$ as an essential subgroup would not be fully invariant. Hence it follows readily that A must be bounded and, moreover, the direct sum of cyclic groups of the same order.

If, for some prime p, $pC \neq C$ and B has a nontrivial p-component B_p, say, of exponent p^k, then $p^k A \cap B_p = 0$, but every fully invariant summand of A containing $p^k A$ will intersect B_p.

To prove the converse, note that the groups listed in (a)–(c) are strongly FI-extending. Indeed, we know this for (b) and (c) from Lemmas 1.3 and 1.4, while for (a) the claim is immediate. It remains to show that a direct sum of three groups, each of different type listed in (a)–(c) is strongly FI-extending whenever the additional condition is fulfilled.

Let F be fully invariant in $A = B \oplus C \oplus D$, subject to the stated conditions. Thus, $F = (B \cap F) \oplus (C \cap F) \oplus (D \cap F)$, where the components are fully invariant in the respective summands. Let B_0, C_0, D_0 be fully invariant summands of B, C, D, respectively, containing $B \cap F$, $C \cap F$, $D \cap F$ as essential subgroups. Then $F_0 = B_0 \oplus C_0 \oplus D_0$ is a summand of A containing F as an essential subgroup. D_0 and $D_0 \oplus B_0$ are fully invariant in A. Therefore, it suffices to prove that the fully invariant subgroup of A generated by B_0 (by C_0) does not intersect D_0 (resp. $B_0 \oplus D_0$), unless $B \cap F$ (resp. $C \cap F$) does so. This is trivial

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for B_0 and follows from the additional condition for C_0. Hence A is strongly FI-extending. □

ACKNOWLEDGMENT

The paper was completed while the second named author was visiting Tulane University as a Fulbright fellow, Nr.22836. He expresses his thanks to the J. William Fulbright Foreign Scholarship Board and to Tulane University.

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Received July 1999
Revised November 1999