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# EVERY ABELIAN GROUP IS DETERMINED BY A SUBGROUP LATTICE

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**Abstract**

We prove that every Abelian group $G$ is determined up to an isomorphism by the subgroup lattice of the group $\mathbb{Z} \times G$ and some other similar results.

It is well known that the lattice $L(G)$ of all subgroups of a group $G$ does not determine the group, that is, there exist non-isomorphic groups with isomorphic subgroup lattices. However, sometimes, specific subgroup lattices may determine some groups. For example, it was proved in [2] that every Abelian group which has the square root property is determined by the subgroup lattice of its square. This result generalizes an earlier result of Lukács and Pálfy [4]. The main result of this note is the following theorem which actually can be proved in a more general setting (see Lemma 2).

THEOREM 1. *Let $A$ be an Abelian group and $G$ a group. The following statements are true.*

(i) *If $\mathbb{Z} \times A$ and $\mathbb{Z} \times G$ have isomorphic subgroup lattices then $A \cong G$.*

(ii) *If $\mathbb{Q} \times A$ and $\mathbb{Q} \times G$ have isomorphic subgroup lattices then $A \cong G$.*

(iii) *If $A$ is an Abelian $p$-group and $G$ is a $p$-group such that $\mathbb{Z}(p^\infty) \times A$ and $\mathbb{Z}(p^\infty) \times G$ have isomorphic subgroup lattices then $A \cong G$.*

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Lemma 1. *Let $K$ and $G$ be groups such that for every $g \in G$ there exists $k \in K$ such that* $\operatorname{ord}(g)$ *divides* $\operatorname{ord}(k)$ *(here every non-zero positive integer is considered to be a divisor of $\infty$).*

a) *If $L(K \times G)$ is modular then every subgroup of $G$ is normal.*

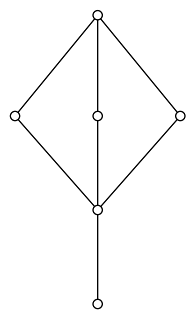
b) *If for an Abelian group $A$, $L(K \times G)$ is isomorphic to $L(A)$, then $G$ is an Abelian group.*

Proof. a) Let $H$ be a subgroup of $G$, $g \in G$ and $H^g = g^{-1}Hg$. There exists $k \in K$ such that $\operatorname{ord}(g)$ divides $\operatorname{ord}(k)$. Consider the following subgroups of $K \times G$: $X = 0 \times \langle H, H^g \rangle$, $Y = \langle (k,g) \rangle$ and $Z = 0 \times H$. Then $Z \leqq X$, and so $X \wedge (Y \vee Z) = (X \wedge Y) \vee Z = \{(0,e)\} \vee Z = Z$ (here $e$ denotes the identity of $G$). Moreover, for every $h \in H$ we have

$$(0, g^{-1}hg) = (k^{-1}, g^{-1})(0,h)(k,g) \in Y \vee Z,$$

and $X \leqq Y \vee Z$ follows. Therefore, $X = Z$ and this implies that every subgroup of $G$ is normal.

b) Since $L(A)$ is modular then $L(K \times G)$ is also modular, and, by a), every subgroup of $G$ is normal. Using [5, Exercise **1**, p. 68], two possibilities do exist: $G$ is Abelian or $G$ is isomorphic to a direct product of the quaternion group $H_8$ and an Abelian group, say $B$. In the second case we first observe that $K \times G$ contains a "top" interval isomorphic with $H_8$ (namely $K \times G / K \times B$). Further, notice that the subgroup lattice of $H_8$ is of the following form



that is, has a smallest nontrivial subgroup. Thus, the subgroup lattice of the Abelian group $A$ contains a "top" interval which is not a chain, but has a smallest element, i.e., $A$ has a quotient group with such a subgroup lattice. According to [3, Theorem 3.1], such a group is cocyclic and so, has a chain as subgroup lattice, a contradiction. Therefore $G$ must be Abelian. $\square$

Lemma 2. *Let $A$ be an Abelian group. If $G$ is a group and $B$ is a non-torsion Abelian group such that $L(B \times A) \cong L(B \times G)$ then $B \times A \cong B \times G$. Moreover, if $B$ has the cancellation property then $A \cong G$.*

PROOF. Since $B \times A$ is Abelian, by the previous Lemma, $G$ is also Abelian. Using results given in [5, Theorem 2.6.10] (actually due to Baer), if $A$ is not a torsion group, or [5, Theorem 2.6.15], if $A$ is a torsion group, we obtain $B \times A \cong B \times G$. $\square$

For the sake of completeness, in order to justify some ingredients below, we recall that divisible Abelian groups of finite rank have semilocal endomorphism rings; as such, these rings have 1 in the stable range and so these groups share the cancellation property. In particular, $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$ cancel from direct sums. $\mathbb{Z}$ too has the cancellation property (see [1, Corollary 8.8]).

PROOF OF THEOREM 1. Since $\mathbb{Z}$ and $\mathbb{Q}$ have the cancellation property, for (i) and (ii) it suffices to apply Lemma 2.

For (iii) we observe that $\mathbb{Z}(p^\infty)$ and $A$ satisfy the hypothesis of Lemma 1, hence $G$ must be Abelian. Finally we apply [5, Theorem 2.6.8] since $\mathbb{Z}(p^\infty)$ has the cancellation property. $\square$

COROLLARY 1. *Let $\mathcal{C}$ be the class of all non-reduced Abelian groups whose divisible part is not torsion. In the class $\mathcal{C}$, groups are determined by their subgroup lattice.*

PROOF. Let $A$, $A' \in \mathcal{C}$. Since these groups are not reduced, their divisible parts are not zero. These divisible parts being not torsion, by the structure theorem of Abelian divisible groups, each $A$ and $A'$ has a direct summand isomorphic with $\mathbb{Q}$: $A \cong \mathbb{Q} \oplus B$, $A' \cong \mathbb{Q} \oplus B'$. Hence we can apply Lemma 2 and $A \cong A'$. $\square$

As a by-product, the proof of the theorem also gives:

COROLLARY 2. *For an arbitrary group $G$, the subgroup lattice $L(\mathbb{Z} \times G)$ is modular if and only if $G$ is Abelian.*

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