MATHEMATICA, Tome 45 (68), N° 1, 2003, pp. 19–24

FULLY INVARIANT ELEMENTS IN LATTICES

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Abstract. For a fully invariant subgroup A of an abelian group $G = H \oplus K$ the equality $A = (A \cap H) \oplus (A \cap K)$ holds. This leads to a weaker definition of fully invariant elements in lattices. Among other things, it is proved that, even in decent conditions, the socle of a lattice and the terms of the so-called Loewy series are (in this sense) fully invariant.

1. PRELIMINARY DEFINITIONS AND RESULTS

In this paper, L will be a complete lattice, 0 and 1 denoting as usually the smallest and the largest elements. L is called *upper continuous* if $a \wedge (\bigvee D) = \bigvee\limits_{d \in D} (a \wedge d)$ holds for every $a \in L$ and every upper directed subset (or, equivalently, for every chain) $D \subseteq L$. An element a is called *atom* if it covers the smallest element. A lattice is called *atomic* if each quotient sublattice $a/0$ (our notations follow [2]) contains an atom for each $a > 0$ respectively *atom generated* if each element is a join of atoms. If $a \in L$, an element $a' \in L$ is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. In this situation we shall also use the notation $a \oplus a' = 1$ and will call this a *direct sum*. A join (also called *sum*) $\bigvee\limits_{i \in I} b_i$ is called *direct* if $b_j \wedge (\bigvee\limits_{i \in I, j \neq i} b_i) = 0$, holds for each $j \in I$. The lattice L is called *complemented* if each element in L has at least one complement.

Following [1], a lattice L satisfies *the condition (B)* if for any chain $\{b_i\}_{i \in I}$ and for any $a \in L$ such that $a \wedge b_i = 0$, $\forall i \in I$, the following holds $a \wedge (\bigvee\limits_{i \in I} b_i) = 0$. It is called *inductive* if all its quotient sublattices (intervals) satisfy the condition (B). Obviously, intervals of inductive lattices are inductive and each upper continuous lattice is inductive. Moreover, lattices of finite length are inductive.

An element c is called a *pseudocomplement* of b in L if $b \wedge c = 0$ and c is maximal with this property. The lattice L is called *pseudocomplemented* if every element in L has at least one pseudocomplement. Every inductive lattice is pseudocomplemented.

For the sake of completeness recall the following known results.

THEOREM 1.1. *A lattice L is modular if and only if $a \leq c$ and $a \wedge b = c \wedge b$, $a \vee b = c \vee b \Rightarrow a = c$ for every $a, b, c \in L$.*

LEMMA 1.1. ([5]) *In a modular lattice with zero, $p = (p \vee a) \wedge (p \vee b)$, $p \wedge a = p \wedge b = 0$ imply $p \wedge (a \vee b) = 0$.*

LEMMA 1.2. ([1]) *In a modular lattice L if $a \leq b$ and a is a direct summand in L then a is also a direct summand in $b/0$.*

2. WEAKLY FULLY INVARIANT ELEMENTS

DEFINITION. We call $a \in L$ a *weakly fully invariant element* if for each direct decomposition $1 = \bigoplus_{i \in I} b_i$ one has $a = \bigoplus_{i \in I}(a \wedge b_i)$.

Clearly, $a \geq \bigvee_{i \in I}(a \wedge b_i)$ holds for arbitrary elements a, $\{b_i\}_{i \in I}$ in every lattice.

PROPERTIES. 1) *The weakly fully invariant property is transitive.*

Indeed, let a be weakly fully invariant in $b/0$ ($a \leq b$) and b be weakly fully invariant in L. Then a is weakly fully invariant in L because

$$1 = \bigoplus_{i \in I} b_i \Rightarrow b = \bigoplus_{i \in I}(b \wedge b_i) \in b/0 \Rightarrow a = \bigoplus_{i \in I}(a \wedge (b \wedge b_i)) = \bigoplus_{i \in I}(a \wedge b_i).$$

2) *A weakly fully invariant element contained in a direct summand is weakly fully invariant in it.*

Indeed, let a be a weakly fully invariant element in L and for $a \leq b$, let $b \oplus c = 1$. Then if $b = \bigoplus_{i \in I} c_i$ we derive

$$1 = (\bigoplus_{i \in I} c_i) \oplus c \Rightarrow a = (a \wedge c) \oplus (\bigoplus_{i \in I}(a \wedge c_i)) = \bigoplus_{i \in I}(a \wedge c_i).$$

3) *In a distributive lattice, the weakly fully invariant elements form an upper semilattice.*

Indeed, if a, c are weakly fully invariant in L then for each direct decomposition $1 = \bigoplus_{i \in I} b_i$ we have $a = \bigoplus_{i \in I}(a \wedge b_i)$ and $c = \bigoplus_{i \in I}(c \wedge b_i)$. Hence $a \vee c = (\bigoplus_{i \in I}(a \wedge b_i)) \vee (\bigoplus_{i \in I}(c \wedge b_i)) = \bigvee_{i \in I}((a \wedge b_i) \vee (c \wedge b_i)) = \bigvee_{i \in I}((a \vee c) \wedge b_i) = \bigoplus_{i \in I}((a \vee c) \wedge b_i)$, using the distributivity.

DEFINITION. A complete lattice L is called *meet infinitely distributive* if $a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I}(a \wedge b_i)$ holds for arbitrary elements a, $\{b_i\}_{i \in I}$ in L.

4) *In a meet infinitely distributive lattice the weakly fully invariant elements form a sublattice.*

Indeed, with the above notations $a \wedge c = (\bigoplus_{i \in I}(a \wedge b_i)) \wedge (\bigoplus_{j \in I}(c \wedge b_j)) =$

$$\bigvee_{i\in I}(\bigoplus_{j\in I}(a\wedge b_i))\wedge(c\wedge b_j)=\bigvee_{i\in I}(a\wedge c\wedge b_i)=\bigoplus_{i\in I}((a\wedge c)\wedge b_i).$$

PROPOSITION 2.1. *In a bounded, modular lattice the join of two weakly fully invariant direct summands is also a direct summand.*

Proof. Let a, b be weakly fully invariant direct summands in L. Then $1 = a\oplus a' = b\oplus b'$ for suitable elements $b, b' \in L$ and $b = b\wedge 1 = b\wedge(a\oplus a') = (a\wedge b)\oplus(a'\wedge b)$ and symmetrically $a = (a\wedge b)\oplus(a\wedge b')$.

Now $a\vee b = (a\wedge b)\vee(a\wedge b')\vee(a'\wedge b)$ and we verify that this sum is direct.

(a) $(a\wedge b')\wedge((a\wedge b)\oplus(a'\wedge b))\leq b'\wedge b=0$;

(b) $(a'\wedge b)\wedge((a\wedge b)\oplus(a\wedge b'))\leq a'\wedge a=0$;

(c) $(a\wedge b)\wedge((a'\wedge b)\oplus(a\wedge b'))=0$ follows using the Lemma 1.1; indeed, $((a\wedge b)\oplus(a\wedge b'))\wedge((a\wedge b)\oplus(a'\wedge b))=(a\wedge(b\oplus b'))\wedge((a\oplus a')\wedge b)=a\wedge b$ and $(a\wedge b)\wedge(a'\wedge b)\leq a\wedge a'=0, (a\wedge b)\wedge(a\wedge b')\leq b\wedge b'=0$.

Hence $a\vee b=(a\wedge b)\oplus(a\wedge b')\oplus(a'\wedge b)=a\oplus(a'\wedge b)$. But $a'\wedge b$ is a direct summand in $b/0$ and so also in L (b being a direct summand in L). Hence by Lemma 1.2, $a'\wedge b$ is also a direct summand in a' so that $a\vee b=a\oplus(a'\wedge b)$ is a direct summand in L. □

PROPOSITION 2.2. *If the weakly fully invariant elements form a sublattice of L then the set FS of all the weakly fully invariant direct summands forms a distributive sublattice of L.*

Proof. Let a, b, c be weakly fully invariant direct summands in L. Only the distributivity requires a proof. If we take $c'=(a\vee b)\wedge c$, $a\vee b$ being a direct summand in L (see the previous Proposition) and c' being weakly fully invariant in L, c' is also weakly fully invariant in $a\vee b$. Hence $c'=(a\wedge c')\oplus(a'\wedge b\wedge c')\leq(a\wedge c)\vee(b\wedge c)$ (using $c'\leq c$, the modularity and the equality $a\vee b=a\oplus(a'\vee b)$ obtained in the previous Proposition). □

We can partly recover a result from [4].

THEOREM 2.1. *In a complete lattice L let a be a direct summand and $\{c_i\}_{i\in I}$ all the elements of L such that $a\oplus c_i=1$. Then $\bigwedge_{i\in I} c_i$ contains all the weakly fully invariant elements e such that $a\wedge e=0$.*

Proof. Indeed, e being weakly fully invariant from $a\oplus c_i=1$ we derive $e=(e\wedge a)\oplus(e\wedge c_i)=e\wedge c_i$ and so $e\leq c_i$ for every $i\in I$. Hence $e\leq\bigwedge_{i\in I} c_i$.

Clearly $a\wedge c_i=0, \forall i\in I$ imply $a\wedge(\bigwedge_{i\in I} c_i)=0$. Observe that $a\vee(\bigwedge_{i\in I} c_i)=1$ does not generally hold. □

3. WEAKLY FULLY INVARIANT SOCLE

Let L be a lattice with zero.

DEFINITION. The join of all the atoms of L, denoted $s(L)$, is called the *socle* of the lattice L.

PROPERTIES. (a) $s(a/0) \leq a$;
(b) $a \leq b \Rightarrow s(a/0) \leq s(b/0)$;
(c) $s((\bigwedge_{i\in I} a_i)/0) \leq (\bigwedge_{i\in I} s(a_i/0))$;
(d) $(\bigvee_{i\in I} s(a_i/0)) \leq s((\bigvee_{i\in I} a_i)/0)$.

DEFINITION. A lattice L is called *with enough (or ample) pseudocomplements* if $a \wedge b = 0$ implies that a has a pseudocomplement $c \in L$ such that $b \leq c$. Then

LEMMA 3.1. *Every inductive lattice has enough pseudocomplements.*

Proof. Indeed, set $C = \{x \in L | a \wedge x = 0, b \leq x\}$; this is a nonempty subset of L containing $b \in C$. L being inductive, it is readily checked that each chain in C has its upper bound in C. Hence, by Zorn's Lemma, C has at least a maximal element c. Manifestly, this is also maximal *only* relative to the property $a \wedge c = 0$, and hence it is a pseudocomplement of a in L. $\square$

DEFINITION. ([1]) A complete lattice L is called *reducible* (or *semiatomic*) if its socle $s(L) = 1$.

THEOREM 3.1. ([1]) *A modular, reducible and inductive lattice is complemented and atom generated.*

LEMMA 3.2. *Let L be inductive and $a \leq b$. Then $s(a/0) = s(b/0) \wedge a$.*

Proof. Clearly $s(a/0) \leq s(b/0)$ and $s(a/0) \leq a$ and so $s(a/0) \leq s(b/0) \wedge a$. But $s(b/0)/0$ is reducible and inductive and hence (by Theorem 3.1) $s(b/0) \wedge a$ is a join of atoms. If s is such an atom, then $s \leq s(b/0) \wedge a$ implies $s \leq a$ or $s \leq s(a/0)$. Hence $s(b/0) \wedge a \leq s(a/0)$ and the required equality. $\square$

LEMMA 3.3. *In a modular, inductive lattice L, for arbitrary elements $a, b \in L$ the equality $a \vee (s(1/a) \wedge b) = s((a \vee b)/b)$ holds.*

Proof. One uses the modularity and the previous Lemma: $a \vee (s(1/a) \wedge b) = (a \vee b) \wedge s(1/a) = s((a \vee b)/b)$. $\square$

Recall an important result:

PROPOSITION 3.1. ([1]) *Let $\{a_i\}_{i\in I}$ be an independent subset of an inductive, modular lattice L. Then* $s((\bigoplus_{i\in I} a_i)/0) = \bigoplus_{i\in I} s(a_i/0)$.

LEMMA 3.4. *For a weakly fully invariant element $b \in L$ in a modular lattice, if $\bigoplus_{i\in I} a_i = 1$ then $\{a_i \vee b\}_{i\in I}$ is independent in $1/b$.*

Proof. We present the case $|I| = 2$ that is, $a \oplus c = 1, b = (b \wedge a) \oplus (b \wedge c)$ imply $b = (b \vee a) \wedge (b \vee c)$.

Indeed, by modularity $(b \vee a) \wedge (b \vee c) = b \vee (a \wedge (b \vee c)) = b$ is equivalent with $a \wedge (b \vee c) \leq b$. This is verified using the hypothesis and again the modularity, as follows:

$$\begin{aligned} a \wedge (b \vee c) &= a \wedge ((b \wedge a) \oplus (b \wedge c) \vee c) = a \wedge ((b \wedge a) \vee c) \\ &= (b \wedge a) \vee (c \wedge a) = (b \wedge a) \vee 0 = b \wedge a \leq b. \end{aligned}$$

□

The construction of the socle in an arbitrary lattice with zero yields by transfinite induction an ascending chain of elements, named the *Loewy series associated to L*,

$$s_0(L) \leq s_1(L) \leq \dots \leq s_\sigma(L) \leq s_{\sigma+1}(L) \leq \dots$$

defined as follows:

$s_0(L) = 0$, $s_1(L) = s(L)$ and for an arbitrary ordinal σ, $s_{\sigma+1}(L) = s(1/s_\sigma(L))$ or, if σ is a limit ordinal, $s_\sigma(L) = \bigvee\limits_{\alpha<\sigma} s_\alpha(L)$.

THEOREM 3.2. *The socle $s(L)$ of an inductive, modular lattice L is weakly fully invariant. Moreover, so are the terms of the Loewy sequence.*

Proof. First notice that $s(L) \wedge a = s(a/0)$ holds in an inductive lattice (see Lemma 3.2). Then, by Proposition 3.1, for $1 = \bigoplus\limits_{i\in I} a_i$ we have

$$s(L) = s(L) \wedge (\bigoplus_{i\in I} a_i) = s((\bigoplus_{i\in I} a_i)/0) = (\bigoplus_{i\in I} s(a_i/0)) = \bigoplus_{i\in I}(s(L) \wedge a_i),$$

i.e., $s(L)$ is weakly fully invariant.

This is true also for each term of the Loewy series. By the way of contradiction, suppose that this is not true. Then there exists a least ordinal σ such that $s_\sigma(L)$ is not weakly fully invariant. Notice that $\sigma > 0$ and σ is not a limit ordinal. Then for each $1 = \bigoplus\limits_{i\in I} a_i$ the equality

$$s_{\sigma-1}(L) = s_{\sigma-1}(L) \wedge (\bigoplus_{i\in I} a_i) = \bigoplus_{i\in I}(s_{\sigma-1}(L) \wedge a_i)$$

holds. The contradiction we obtain is $s_\sigma(L) = \bigoplus\limits_{i\in I}(s_\sigma(L) \wedge a_i)$ i.e.

$$b = s(1/s_{\sigma-1}(L)) = \bigoplus_{i\in I}(s(1/s_{\sigma-1}(L)) \wedge a_i) = a.$$

Trivially $b \geq a$ holds. Using the modularity of the lattice L and Theorem 1.1, it suffices to verify $a \vee c = b \vee c, a \wedge c = b \wedge c$ where $c = s_{\sigma-1}(L)$. The equalities

$b \vee c = s(1/s_{\sigma-1}(L))$ and $b \wedge c = s_{\sigma-1}(L)$ follow immediately by $c \leq b$. Next,

$$\begin{aligned} a \vee c &= (\bigoplus_{i\in I}(s(1/s_{\sigma-1}(L)) \wedge a_i)) \vee s_{\sigma-1}(L) \\ &= \bigvee_{i\in I}((s(1/s_{\sigma-1}(L)) \wedge a_i) \vee s_{\sigma-1}(L)) \\ &= \bigvee_{i\in I}(s(1/s_{\sigma-1}(L)) \wedge (a_i \vee s_{\sigma-1}(L))) \\ &= \bigvee_{i\in I} s((a_i \vee s_{\sigma-1}(L))/s_{\sigma-1}(L)) \\ &= s(1/s_{\sigma-1}(L)) \end{aligned}$$

using the case $\sigma = 1$ in the quotient sublattice $1/s_{\sigma-1}(L)$ (indeed, note that $\{a_i \vee s_{\sigma-1}(L)\}_{i\in I}$ is independent in $1/s_{\sigma-1}(L)$ – see the previous Lemma 3.4) and the Lemma 3.3.

Finally, $a \wedge c = (\bigoplus_{i\in I}(s(1/s_{\sigma-1}(L)) \wedge a_i)) \wedge s_{\sigma-1}(L) = s_{\sigma-1}(L)$ is equivalent with $s_{\sigma-1}(L) \leq \bigoplus_{i\in I}(s(1/s_{\sigma-1}(L)) \wedge a_i)$ which follows at once from $s_{\sigma-1}(L) = \bigoplus_{i\in I}(s_{\sigma-1}(L) \wedge a_i)$. □

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Received January 5, 2001

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