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# Subgroups which admit extensions of homomorphisms

Simion Breaz, Grigore Călugăreanu and Phill Schultz

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**Abstract.** We classify by numerical invariants the finite subgroups $H$ of a primary abelian group $G$ for which every homomorphism or monomorphism of $H$ into $G$, or every endomorphism of $H$, extends to an endomorphism of $G$. We apply these results to show that for finitely generated subgroups of general abelian groups, the extendibility of monomorphisms implies the extendibility of all homomorphisms.

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## 1 Introduction

We begin by characterizing in module theoretic terms the extension properties described above. The problem of characterizing the subgroups satisfying these properties forms part of the more general question of characterizing special classes of submodules of a module. For example, Birkhoff observed in [2] that although for some rings $R$ (and in particular for $R = \mathbb{Z}$) finitely generated modules can be completely characterized using numerical invariants, in general it is difficult to describe even cyclic modules as submodules of a given module. A complex study in this direction was initiated by Ringel and Schmidmeier [16] for artinian algebras, and it was continued by several authors who show that in many cases the category of submodules is 'wild' (cf. the introduction of [15]). For the case of abelian groups, we mention the studies realized in [1] and [14].

Let $R$ be a unital ring and $M$ an $R$-module, and let $\mathrm{Sub}(M)$ be the set of all submodules of $M$.

If $N \in \mathrm{Sub}(M)$, and $\iota : N \to M$ is the inclusion map, by applying the contravariant functors $\mathrm{Hom}(-, M)$ and $\mathrm{Hom}(-, N)$, we obtain a commutative diagram:

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$$\begin{array}{ccc} \mathrm{Hom}(M,N) & \xrightarrow{\iota_M^*} & \mathrm{End}(M) \\ \Big\downarrow{\scriptstyle \mathrm{res}_N} & & \Big\downarrow{\scriptstyle \mathrm{res}_M} \\ \mathrm{End}(N) & \xrightarrow{\iota_N^*} & \mathrm{Hom}(N,M), \end{array}$$

where $\iota_X^* : \mathrm{Hom}(X,N) \to \mathrm{Hom}(X,M)$ are the induced inclusion maps and $\mathrm{res}_X : \mathrm{Hom}(M,X) \to \mathrm{Hom}(N,X)$ the induced restriction maps.

Several module theoretic properties that appear in the literature in other guises can be described in terms of this diagram: $\iota_M^*$ is an isomorphism if and only if $N = M$; if $\mathrm{res}_M$ is an isomorphism then $M$ is a localization of $N$ (see [3]); $\mathrm{res}_N$ is an isomorphism if and only if $N$ is a direct summand with a unique complement; $\mathrm{res}_M$ factors through $\iota_N^*$ if and only if $N$ is fully invariant in $M$; for a given $N$, $\mathrm{res}_M$ and $\mathrm{im}_N^*$ have the same image for all $M$ if and only if $N$ is rigid [4].

Now consider, for a given module $M$, the following sets of submodules:

$$\begin{aligned} \mathcal{S}(M) &= \{N \in \mathrm{Sub}(M) : \mathrm{res}_N \text{ is epic}\}, \\ \mathcal{Q}(M) &= \{N \in \mathrm{Sub}(M) : \mathrm{res}_M \text{ is epic}\}, \\ \mathcal{W}(M) &= \{N \in \mathrm{Sub}(M) : \mathrm{im}(\mathrm{res}_M) = \mathrm{im}(\iota_N^*)\}, \\ \mathcal{P}(M) &= \{N \in \mathrm{Sub}(M) : \mathrm{im}(\mathrm{res}_M) \supseteq \mathrm{Mon}(N,M)\}, \end{aligned}$$

where $\mathrm{Mon}(N,M)$ is the set of monomorphisms of $N$ into $M$.

It is easy to see that $\mathcal{S}(M)$ is the set of all direct summands of $M$, $\mathcal{Q}(M)$ is the set of all submodules $N$ of $M$ such that all homomorphisms of $N$ into $M$ can be extended to endomorphisms of $M$, $\mathcal{W}(M)$ is the class of all submodules $N$ of $M$ such that all endomorphisms of $N$ can be extended to endomorphisms of $M$, and $\mathcal{P}(M)$ is the class of all submodules $N$ of $M$ for which all monomorphisms of $N$ into $M$ can be extended to endomorphisms of $M$.

Therefore, if one of these classes coincides with the set of all submodules of $M$, i.e., $\mathcal{X}(M) = \mathrm{Sub}(M)$ for $\mathcal{X} = \mathcal{S}, \mathcal{Q}, \mathcal{W}$, or $\mathcal{P}$ then $M$ is semi-simple, respectively quasi-injective [6], weakly-injective [12] or pseudo-injective [10]. Using known results about the structure of these modules, it is easy to see that in general $\mathcal{S}(M) \subseteq \mathcal{Q}(M) \subseteq \mathcal{P}(M) \subseteq \mathrm{Sub}(M)$ and all inclusions can be strict. With one exception (the inclusion $\mathcal{Q}(M) \subseteq \mathcal{P}(M)$) this strictness can be demonstrated in the class of abelian groups, using structure theorems in [11] and [12]. For $\mathcal{Q}(M) \subsetneqq \mathcal{P}(M)$, there are pseudo-injective modules which are not quasi-injective (see [10] or [18]). However, it is proved in [17] that over principal ideal domains quasi-injective and pseudo-injective modules coincide, and we do not know if there exists an abelian group $G$ such that $\mathcal{Q}(G) \neq \mathcal{P}(G)$.

Recently Er, Singh and Srivastava [5] showed that pseudo-injective modules are precisely those modules which are invariant under automorphisms of their injective hulls. This description should be compared with the well-known characterization of quasi-injective modules as those modules which are fully invariant, i.e., invariant under endomorphisms of their injective hulls.

The object of the paper is to characterize finitely generated subgroups which lie in these classes for the case of primary abelian groups. Let $G$ be a $p$-group. Cyclic subgroups in $\mathcal{Q}(G)$ are described in Theorem 2.5, while finitely generated subgroups in $\mathcal{Q}(G)$ are characterized in Theorem 3.4. Using these results we prove in Theorem 4.4 that for all abelian groups $G$ (not only for quasi-injective or pseudo-injective abelian groups) we have $\mathcal{Q}_f(G) = \mathcal{P}_f(G)$, where $\mathcal{X}_f(G)$ denotes the set of all finitely generated subgroups in $\mathcal{X}(G)$. Theorem 3.6 gives us information about finitely generated subgroups in $\mathcal{W}(G)$.

All groups in this paper are abelian. Unless specifically noted we use the standard notation of [7, 8].

*With the exception of the final section, in the rest of this paper we assume that $p$ is a fixed prime and $G$ a $p$-group.*

If $G$ is bounded, we denote by $\exp(G)$ the least positive integer $k$ such that $p^k G = 0$. If $G$ is not bounded then $\exp(G) = \infty$. The exponent of an element $x \in G$ denotes the positive integer $\exp(x)$ such that the order of $x$ is $p^{\exp(x)}$. If $x \in G$, then $h(x)$ denotes the height of $x$. A group $G$ is *homogeneous* if it is a direct sum of isomorphic quasi-cyclic groups, i.e., $G \cong \mathbb{Z}(p^k)^{(\lambda)}$ where $k$ is a positive integer or $\infty$ and $\lambda$ is a cardinal.

It is well known that the semi-simple groups are the direct sums of elementary groups. Moreover, quasi-injective (primary) groups are exactly the homogeneous groups.

## 2 Cyclic subgroups in $\mathcal{Q}(G)$

In this section, we consider the question: which cyclic subgroups of $G$ are in $\mathcal{Q}(G)$ or in $\mathcal{P}(G)$?

If $G$ is divisible, then $\langle x\rangle \in \mathcal{Q}(G)$ for all $x \in G$, so we need consider only non-divisible $G$. We recall that $G^1 = \bigcap_{n>0} p^n G$ denotes *the first Ulm subgroup* of $G$.

**Lemma 2.1.** *Let $G$ be non-divisible and $\langle x\rangle \in \mathcal{P}(G)$.*

(1) *If $G$ has an unbounded basic subgroup then $\langle x\rangle \cap G^1 = \{0\}$.*

(2) *If $G = B \oplus D$ with $B$ bounded and $D$ divisible then either*

(a) $\exp(x) \leq \exp(B)$ *and* $\langle x\rangle \cap D = 0$ *or*

(b) $\exp(x) > \exp(B)$ *and* $x = b + d$ *with* $b \in B$, $d \in D$ *such that* $\exp(B) = \exp(b)$ *and* $\exp(d) > \exp(B)$.

*Proof.* (1) If $G$ has an unbounded basic subgroup then $G$ has a cyclic direct summand $\langle a \rangle$ of order $\geq \operatorname{ord}(x)$. Hence there is a monomorphism $\langle x \rangle \to \langle a \rangle$ which can be extended to an endomorphism of $G$. It follows that the heights of all non-zero elements of $\langle x \rangle$ are finite.

(2) Suppose $G = B \oplus D$ with $\exp(B) = n$ and $D$ divisible.

(a) If $\exp(x) \leq n$ then as in (1) $G$ has a cyclic direct summand of order $\geq \operatorname{ord}(x)$ and non-zero elements from $\langle x \rangle$ have finite height.

(b) Suppose that $\exp(x) > n$, and $x = b + d$ with $b \in B$ and $d \in D$. Then $\exp(d) > n \geq \exp(b)$. Let $y \in B$ have exponent $n$. Then $\exp(y + d) = \exp(d) = \exp(x)$, and there exists $f \in \operatorname{End}(G)$ such that $f(x) = y + d$. Since $h(y) = 0$, it follows that $h(x) = 0$, hence $\exp(b) = n$. $\square$

**Lemma 2.2.** *If $G$ and $x$ are as in Lemma* 2.1 (2) (b) *then* $\langle x \rangle \in \mathcal{Q}(G)$.

*Proof.* Since $\exp(b) = \exp(B)$, $\langle b \rangle$ is a direct summand of $B$.

Let $f \in \operatorname{Hom}(\langle x \rangle, G)$ with $f(x) = a + e$, where $a \in B$ and $e \in D$. Since $\langle b \rangle$ is a summand of $B$ and $\exp(b) \geq \exp(a)$, the map $f_1$ defined by $f_1(b) = a$ extends to $g_1 \in \operatorname{Hom}(B, G)$. Since $D$ is injective and $\exp(d) = \exp(x) \geq \exp(e)$, the map $f_2$ defined by $f_2(d) = e$ extends to $g_2 \in \operatorname{Hom}(D, G)$. Hence $g = g_1 + g_2$ is an extension of $f$ to $\operatorname{End}(G)$. $\square$

It remains to find intrinsic criteria for cyclic groups satisfying the conditions of Lemma 2.1 (1) and (2) (a) to be in $\mathcal{Q}(G)$. We consider therefore cyclic groups $\langle x \rangle$ containing no elements of infinite height in $G$.

Recall (see [8, Section 65]) that for $n \in \mathbb{N}$, an *Ulm sequence of length $n$* is a strictly increasing infinite sequence $U = (h_0, h_1, \ldots, h_{n-1}, \infty, \ldots)$ with each $h_i$ an ordinal, under the conventions that each ordinal $h_i < \infty$, $\infty < \infty$ and the constant sequence $(\infty)$ is the unique Ulm sequence of length 0. The set of Ulm sequences is well-ordered pointwise with maximum $(\infty)$, no minimum but infimum $\mathbb{N} = (0, 1, \ldots, n, n+1, \ldots)$. This means in particular that if $U \leq V$ where $U$ has length $n$ and $V$ has length $m$, then $n \geq m$. An Ulm sequence $U$ is called *finite* if all its non-infinity entries are finite. In particular, $(\infty)$ is a finite Ulm sequence.

By Lemma 2.1 we have:

**Corollary 2.3.** *If $\langle x \rangle \in \mathcal{P}(G)$ then $U(x)$ is finite.* $\square$

We say that the Ulm sequence $U$ *has a gap before* $k$ if $h_k > h_{k-1} + 1$, where $h_{-1}$ denotes by definition the integer $-1$. The gap before $n$, where $n$ is the length of $U$, is called *the trivial gap*.

Let $x \in G$ with $\exp(x) = n$. Then $x$ determines an Ulm sequence of length $n$ by $U(x) = (h(x), h(px), \dots, h(p^{n-1}x), \infty, \dots)$. It is clear from this definition that $U(x)$ is finite if and only if $\langle x\rangle \cap G^1 = \{0\}$, $h(p^k x) = \infty$ if and only if $p^k x \in D$, the divisible part of $G$, $U(x) = (0, 1, \dots, n-1, \infty, \dots)$ if and only if $\langle x\rangle$ is a summand of exponent $n$ and for $x, y \in G$, $U(x + y) \geq \min\{U(x), U(y)\}$. Finally, note that by [8, Lemma 65.3], if $h(x) = 0$ and $U(x)$ has the first non-trivial gap before $k$, then $G$ has a direct summand of exponent $k$.

By [8, Lemma 65.5, Exercise 6] we have:

**Lemma 2.4.** *Let $G$ be a group and $x \in G$ such that $\langle x\rangle \cap G^1 = 0$.*

(1) *The following are equivalent:*

  (a) $\langle x\rangle \in \mathcal{Q}(G)$.

  (b) *If $y \in G$ such that* $\exp(x) \geq \exp(y)$ *then* $U(x) \leq U(y)$.

(2) *The following are equivalent:*

  (a) $\langle x\rangle \in \mathcal{P}(G)$.

  (b) *If $y \in G$ such that* $\exp(x) = \exp(y)$ *then* $U(x) \leq U(y)$. □

Using this result we can characterize cyclic groups $\langle x\rangle$ with no elements of infinite height in $\mathcal{Q}(G)$ as follows:

**Theorem 2.5.** *Let $G$ be a group and $x \in G$ an element of exponent $n$ such that $\langle x\rangle \cap G^1 = \{0\}$. The following are equivalent:*

(1) $\langle x\rangle \in \mathcal{Q}(G)$.

(2) $\langle x\rangle \in \mathcal{P}(G)$.

(3) *$U(x)$ has at most one non-trivial gap and if a gap occurs before the index $k \geq 0$ and $h(p^k x) = k + \ell$, then $G$ has no cyclic summands of exponents between $k + 1$ and $n + \ell - 1$.*

*Proof.* (1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Let $x \in G$ such that $\langle x\rangle \in \mathcal{P}(G)$. If $U(x)$ has no non-trivial gaps then $\langle x\rangle$ is a direct summand of $G$. Therefore we can assume that $U(x)$ has at least one non-trivial gap.

Suppose that $U(x) = (h_0, \dots, h_{n-1}, \infty, \dots)$ has at least two non-trivial gaps. Since all heights $h_i$ are integers, we can apply [8, Lemma 65.4], and it follows that there is a direct summand $C = \langle c_1\rangle \oplus \cdots \oplus \langle c_t\rangle$ of $G$ and a strictly increasing chain of positive integers $0 < k_1 < k_2 < \cdots < k_t$ such that

(i) $t \geq 3$,

(ii) $\exp(c_1) < \exp(c_2) < \cdots < \exp(c_t) = k_t + n$, and

(iii) $x = p^{k_1}c_1 + p^{k_2}c_2 + \cdots + p^{k_t}c_t$.

We observe that the exponent of

$$y = p^{k_1}c_1 + p^{k_2-1}c_2 + p^{k_3}c_3 \cdots + p^{k_t}c_t$$

is $n$. But

$$\begin{aligned} h(p^{\exp(c_1)-k_1}y) &= \exp(c_1) - k_1 + k_2 - 1 \\ &< \exp(c_1) - k_1 + k_2 = h(p^{\exp(c_1)-k_1}x), \end{aligned}$$

hence $U(y) \not\geq U(x)$, a contradiction.

Therefore $U(x)$ has exactly one non-trivial gap. Let $k$ be the index such that $U(x)$ has a gap before $k$. Hence $h(p^k x) = k + \ell$ with $\ell > 0$.

Suppose that $\langle z\rangle$ is a direct summand of $G$ of exponent $n \leq e \leq n + \ell - 1$. If $v = p^{e-n}z$ then $p^k v \neq 0$ since $n > k$. Moreover,

$$h(p^k v) = e - n + k \leq n + \ell - 1 - n + k = k + \ell - 1 < h(p^k x).$$

Therefore $U(v) \ngeq U(x)$, but $\exp(v) = \exp(x)$, a contradiction.

Suppose that $\langle z\rangle$ is a direct summand of $G$ of exponent $k + 1 \leq e \leq n - 1$. We observe that $v = x + z$ is of exponent $n$. But $h(p^k x) > k = h(p^k z)$, hence

$$h(p^k v) = h(p^k x + p^k z) = k < h(p^k x),$$

and it follows that $U(v) \ngeq U(x)$. This leads to a contradiction and the proof is complete.

(3) $\Rightarrow$ (1) Let $x$ be as in (3). If $U(x)$ has no nontrivial gaps then $\langle x\rangle$ is a direct summand of $G$.

Suppose that $U(x)$ has a gap before the index $k$, and we fix an element $y$ of exponent $e \leq \exp(x)$. We will prove that $U(x) \leq U(y)$.

We consider the Ulm sequence $U(y) = (r_0, \ldots, r_{e-1}, \infty, \ldots)$.

*Case I: $r_{e-1}$ is finite.* In order to prove that $U(x) \leq U(y)$, since $U(x)$ has only one gap and this occurs before $k$, it is enough to prove that $h(p^k x) \leq h(p^k y)$.

Suppose by contradiction that $h(p^k x) > h(p^k y)$. As in [8, Lemma 65.4], if $n_1, \ldots, n_t$ are the positive indexes before the gaps occur and we set $r_{n_i} = n_i + k_{i+1}$ and $k_1 = r_0$ then we have cyclic direct summands of exponent $n_i + k_i$, with $i = 1, \ldots, t$.

If $k = 0$ then we have no cyclic direct summands of exponent $1, \dots, n + \ell - 1$. Then every element $y$ of exponent $\leq n$ must have height $\geq \ell$.

If $k > 0$, let $n_j \leq k$ be the largest index $n_i \leq k$. Then $h(p^k y) = k + k_{j+1} < k + \ell$ and $G$ has a direct summand of exponent $n_{j+1} + k_{j+1}$. Since $k < n_{j+1} \leq n$, we obtain that $G$ has a cyclic direct summand of exponent $e$ with $k < e < n + \ell$, a contradiction.

*Case II: $r_{e-1}$ is infinite.* Let $u = h(p^{n-1}x)$. If $B = \bigoplus_{i>0} B_i$ is a basic subgroup of $G$, where $B_i$ are homogeneous subgroups of exponent $i$, we consider the direct decomposition $G = B_1 \oplus \cdots \oplus B_u \oplus p^u G$ and write $y = y_1 + \cdots + y_u + y^*$ with $y_i \in B_i$ for all $i = 1, \dots, u$ and $y^* \in p^u G$. It is obvious that $U(x) \leq U(y^*)$. Moreover, $y - y^*$ satisfies Case I and $\exp(y - y^*) < \exp(x)$. Therefore $U(x) \leq \min\{U(y - y^*), U(y^*)\}$, and it follows that $U(x) \leq U(y)$. □

**Corollary 2.6.** *Let $\langle x \rangle \in \mathcal{Q}(G)$. If $U(x)$ has no non-trivial gap then $\langle x \rangle \in \mathcal{S}(G)$. If $U(x)$ has a non-trivial gap at $k$, then $x \in H$, a summand of $G$, where $H$ is cyclic if $k = 0$ and finite of rank 2 otherwise.* □

It is easy to see that $\mathcal{Q}(G)$ is closed with respect to direct summands.

**Corollary 2.7.** *The set $\mathcal{Q}(G)$ is not closed under direct sums, even in the case that $G$ is a finite $p$-group.*

*Proof.* Let $x, y \in G$ such that $U(x)$ has a single non-trivial gap before index $k$ and $h(p^k x) = k + \ell$ with $\ell > 1$. Let $\langle y \rangle$ be cyclic of exponent $k + 1$. Then $\langle x \rangle$ and $\langle y \rangle \in \mathcal{Q}(G)$ but $\langle x \rangle \oplus \langle y \rangle \notin \mathcal{Q}(G)$. □

## 3 Finite subgroups in $\mathcal{Q}(G)$

We now extend the results of Section 2 from cyclic to finite subgroups. The main result of this section is that a finite subgroup $H$ of a group $G$ is in $\mathcal{Q}(G)$ if and only if it is a valuated direct sum of cyclic subgroups from $\mathcal{Q}(G)$.

Recall from [9] that if $H \subseteq G$, the *valuation of $H$ induced by heights in $G$* is defined by $v(x) = h(x)$, the height of $x$ in $G$, for all $x \in H$ and $H = K \oplus L$ is a *valuated direct sum* if $v(k + \ell) = \min\{v(k), v(\ell)\}$ for all $k \in K$ and $\ell \in L$. In the following results, the valuation of $H$ is always that induced by heights in $G$. Consequently, $f \in \operatorname{Hom}(H, G)$ does not decrease valuations if and only if $f$ does not decrease Ulm sequences in $G$.

**Lemma 3.1.** *Let $G$ be a group and let $K, L \leq \mathcal{Q}(G)$ with $K \cap L = 0$. If $K \oplus L \in \mathcal{W}(G)$ then $K \oplus L$ is a valuated direct sum.*

*Proof.* Suppose the direct sum $K \oplus L$ is not valuated. Then there exists a pair $(k, \ell) \in K \oplus L$ such that $h^G((k,\ell)) > \min\{h^G(k), h^G(\ell)\}$. For example, say $h^G((k,\ell)) > h^G(k)$. Let $f \in \mathrm{End}(K \oplus L)$ be the natural projection onto $K$. Then $h^G(f(k,\ell)) = h^G(k) < h^G((k,\ell))$ so $f$ cannot be extended to $\mathrm{End}(G)$, a contradiction. □

**Lemma 3.2.** *Let $G$ be a group, $K$ a pure subgroup of $G$ such that $K$ is a direct sum of cyclic groups and $G/K$ is divisible, and $H \le K$ a finite subgroup. If $f : H \to G$ is a homomorphism, the following are equivalent:*

(1) *$f$ can be extended to an endomorphism of $G$.*

(2) *$f$ can be extended to a homomorphism $\overline{f} : K \to G$.*

(3) *$f$ can be extended to a homomorphism $\overline{f} : K \to G$ such that $\overline{f}(K)$ is bounded.*

*Proof.* (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) We extend $f$ to a homomorphism $g : K \to G$. Since $H$ is finite and $K$ is a direct sum of cyclic groups, there is a finite direct summand $L$ of $K$ such that $H \le L$. If $L \oplus M = K$, we define $\overline{f} : K \to G$ by $\overline{f}(x + y) = g(x)$ for all $x \in L$ and $y \in M$.

(3) $\Rightarrow$ (1) Let $\overline{f}$ be as in (3). We have to extend $\overline{f}$ to an endomorphism of $G$. Let $k > 0$ be an integer such that $\overline{f}(K)$ is bounded by $p^k$. Since $G/K$ is divisible, it is easy to see that $G = K + p^k G$, hence for every $x \in G$ there are $y \in K$ and $z \in G$ such that $x = y + p^k z$. Since $K$ is pure in $G$, it is not hard to see that the map $g : G \to G$, $g(x) = \overline{f}(y)$ is well defined, and it represents an endomorphism of $G$ which extends $f$. □

**Corollary 3.3.** *Let $G$ be a group and let $K$ be a pure subgroup of $G$ such that $K$ is a direct sum of cyclic groups and $G/K$ is divisible. Let $H$ be a finite subgroup of $K$.*

(1) *The following are equivalent:*

   (a) *$H \in \mathcal{Q}(G)$.*

   (b) *Every homomorphism $f : H \to G$ can be extended to a homomorphism $\overline{f} : K \to G$.*

   (c) *Every homomorphism $f : H \to G$ can be extended to a homomorphism $\overline{f} : K \to G$ such that $\overline{f}(K)$ is bounded.*

(2) *The following are equivalent:*

   (a) *$H \in \mathcal{W}(G)$.*

(b) *Every endomorphism* $f : H \to H$ *can be extended to a homomorphism* $\overline{f} : K \to G$.

(c) *Every endomorphism* $f : H \to H$ *can be extended to a homomorphism* $\overline{f} : K \to G$ *such that* $\overline{f}(K)$ *is bounded.* □

We are now able to characterize finite subgroups of $G$ in $\mathcal{Q}(G)$, respectively in $\mathcal{W}(G)$.

**Theorem 3.4.** *Let $G$ be a group and let $H = \bigoplus_{i=1}^n H_i$ be a finite subgroup such that all $H_i$ are cyclic groups. The following are equivalent:*

(1) $H \in \mathcal{Q}(G)$.

(2) (a) $H_i \in \mathcal{Q}(G)$ *for all* $i = 1, \dots, n$;

(b) $H = \bigoplus_{i=1}^n H_i$ *is a valuated direct sum of cyclic groups.*

*Proof.* (1) $\Rightarrow$ (2) This follows from Lemma 3.1.

(2) $\Rightarrow$ (1) *Case I: $G$ has an unbounded basic subgroup.* Then $H_i \cap G^1 = 0$ for every $i \in \{1, \dots, n\}$. Since the direct sum $\bigoplus_{i=1}^n H_i$ is valuated, it follows that $H \cap G^1 = 0$, hence there is a basic subgroup $B \leq G$ such that $H \leq B$. By Lemma 3.2, it is enough to prove that every homomorphism $f : H \to G$ can be extended to a homomorphism $f' : B \to G$.

We consider a homomorphism $f : H \to G$. If $x \in B$ we denote by $h^B(x)$ the height of $x$ calculated in $B$ and by $h(x)$ the height of $x$ as an element of $G$.

Then the restrictions $f|_{H_i}$ can be extended to endomorphisms of $G$, and it follows that $h(x_i) \leq h(f(x_i))$ for all $i$ and all $x_i \in H_i$.

Let $x = x_1 + \cdots + x_n \in H$ with $x_i \in H_i$ for all $i$. We observe that

$$\begin{aligned} h^B(x) \leq h(x) &= \min\{h(x_i) : i = 1, \dots, n\} \leq \min\{h(f(x_i)) : i = 1, \dots, n\} \\ &\leq h(f(x_1) + \cdots + f(x_n)) = h(f(x)). \end{aligned}$$

Since $B/H$ is a direct sum of cyclic groups, and $H$ is a nice subgroup of $B$ as a consequence of [8, Property 79 (b)], we can apply [8, Corollary 81.4] to conclude that there is a homomorphism $f' : B \to G$ which extends $f$, and the proof is complete.

*Case II: $G = B \oplus D$ with $B$ bounded and $D$ divisible.* Let

$$X = \left\{ i \in \{1, \dots, n\} : H_i \cap G^1 = 0 \right\}$$

and $Y = \{1, \dots, n\} \setminus X$. Since the direct sum $\bigoplus_{i=1}^n H_i$ is valuated, it follows that $\bigoplus_{i \in X} H_i \cap G^1 = 0$, hence we can suppose $\bigoplus_{i \in X} H_i \leq B$.

For every $i \in Y$ we fix a generator $h_i$ for $H_i$, and write $h_i = b_i + d_i$ with $\langle b_i \rangle$ a direct summand of $B$, $d_i \in D$. Since $H_i \cap G^1 \neq 0$, it follows that $\exp(B) = \exp(b_i) < \exp(d_i) = \exp(h_i)$ by Lemma 2.1.

We claim that $\sum_{i \in Y} \langle b_i \rangle = \bigoplus_{i \in Y} \langle b_i \rangle$. In order to prove this, suppose by contradiction that there exist an index $j \in Y$ and a non-zero element $0 \neq k_j b_j = \sum_{i \in Y \setminus \{j\}} k_i b_i$. Then $k_j h_j - \sum_{i \in Y \setminus \{j\}} k_i h_i$ is of infinite height, hence the sum $\bigoplus_{i \in Y} H_i$ is not valuated since $k_j h_j$ is of finite height. This contradicts our hypothesis, so the claim is true. Moreover, $\bigoplus_{i \in Y} \langle b_i \rangle$ is a direct summand of $B$ as a bounded pure subgroup, and using [7, Exercise 9.8] we conclude that it is an absolute direct summand of $B$.

Using a similar argument, if we suppose that $(\bigoplus_{i \in Y} \langle b_i \rangle) \cap (\bigoplus_{i \in X} H_i) \neq 0$, we obtain that $\bigoplus_{i=1}^{n} H_i$ is not a valuated direct sum, a contradiction. Hence

$$\Big( \bigoplus_{i \in Y} \langle b_i \rangle \Big) \cap \Big( \bigoplus_{i \in X} H_i \Big) = 0,$$

and it follows that there is a direct complement $C$ of $(\bigoplus_{i \in Y} \langle b_i \rangle)$ in $B$ such that $\bigoplus_{i \in X} H_i \leq C$.

Moreover, since the sum $\bigoplus_{i \in Y} H_i$ is direct and $H_i[p] = \langle d_i \rangle [p]$ for all $i \in Y$, it follows that the sum $\sum_{i \in Y} \langle d_i \rangle$ is a direct sum. Hence we can find infinite quasi-cyclic subgroups $D_i$, $i \in Y$, such that $D = (\bigoplus_{i \in Y} D_i) \oplus D'$ and $d_i \in D_i$ for all $i \in Y$.

Let $f : H \to G$ be a homomorphism. Using the same argument as in the first case we observe that every homomorphism $\bigoplus_{i \in X} H_i \to G$ can be extended to a homomorphism $f_0 : C \to G$ (note that the valuation induced on $\bigoplus_{i \in X} H_i$ by $C$ is the same as the valuation induced by $G$).

For all $i \in I$, we have $\exp(f(h_i)) \leq \exp(h_i)$, hence $f(h_i) = a_i + z_i$ with $a_i \in B$ and $z_i \in D$ such that $\exp(z_i) \leq \exp(d_i)$. Therefore there exist homomorphisms $f_i' : \langle b_i \rangle \to G$ such that $f_i'(b_i) = a_i$ and $f_i'' : D_i \to D$ such that $f_i''(d_i) = z_i$.

Since

$$G = C \oplus \Big( \bigoplus_{i \in Y} \langle b_i \rangle \Big) \oplus \Big( \bigoplus_{i \in Y} D_i \Big) \oplus D',$$

the homomorphisms $f_0$, $f_i'$ and $f_i''$, $i \in Y$, induce an endomorphism $\overline{f} : G \to G$, and it is easy to see that $\overline{f}$ extends $f$. □

**Remark 3.5.** Cyclic valuated groups are characterized using invariants in [9, Theorem 3]. Therefore this result together with Theorem 3.4 and Theorem 2.5 give us a characterization by invariants for subgroups in $\mathcal{Q}(G)$.

We close this section with a characterization of some finite subgroups in $\mathcal{W}(G)$.

**Theorem 3.6.** *Let $G$ be a group and $H = \bigoplus_{i=1}^{n} H_i$ a finite subgroup such that $H \cap p^{\omega} G = 0$ and each $H_i = \langle z_i \rangle$ is a cyclic group of exponent $e_i$. The following are equivalent:*

(1) $H \in \mathcal{W}(G)$.

(2) (a) *If $e_j \le e_i$ then $U(z_i) \le U(z_j) \le U(p^{e_i - e_j} z_i)$;*

  (b) *$H = \bigoplus_{i=1}^{n} H_i$ is a valuated direct sum of cyclic groups.*

*Proof.* (1) $\Rightarrow$ (2) In order to prove (a), let $i, j$ be two indices such that $e_j \le e_i$. Then there are homomorphisms $f : H_i \to H_j$ with $f(z_i) = z_j$ and $g : H_j \to H_i$ with $f(z_j) = p^{e_i - e_j} z_i$. Since these homomorphisms can be extended to endomorphisms of $H$, they can be extended to endomorphisms of $G$. The inequalities $U(z_i) \le U(z_j) \le U(p^{e_i - e_j} z_i)$ follow from the fact that endomorphisms do not decrease heights.

The statement (b) is a consequence of Lemma 3.1.

(2) $\Rightarrow$ (1) As in the proof of Theorem 3.4, it is enough to prove that every homomorphism of $f : H_i \to H$ does not decrease the valuation.

Let $f : H_j \to H$ be a homomorphism defined for some $j \in \{1, \dots, n\}$. Since $U(mz) = U(z)$ for all integers $m$ with $(m, p) = 1$, it is enough to prove that $U(p^k z_j) \le U(p^k (f(z_j)))$ for all $0 \le k < e_j$. Since for every element $x$ and for every positive integer $k$ the indicator $U(p^k x)$ can be obtained by deleting the first $k$ components of $U(x)$, it is enough to prove $U(z_j) \le U(f(z_j))$.

Let $f(z_j) = \sum_{i=1}^{n} m_i z_i$. Note that if $e_j < e_i$ then $p^{e_i - e_j}$ divides $m_i$. Then

$$f(z_j) = \Big( \sum_{e_j < e_i} n_i p^{e_i - e_j} z_i \Big) + \Big( \sum_{e_i \le e_j} m_i z_i \Big),$$

hence

$$U(f(z_j)) = \min \big\{ U(n_i p^{e_i - e_j} z_i) : e_j < e_i \big\} \cup \big\{ U(m_i z_i) : e_i \le e_j \big\} \ge U(z_j),$$

and the proof is complete. □

In the following example we show that both pairs of conditions (a) and (b) in Theorem 3.4 and Theorem 3.6 respectively are necessary.

**Example 3.7.** Let $G = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$ be a group with $\exp(a) = 1$ and $\exp(b) = \exp(c) = 2$. Then $x = (a, pb, 0)$ and $y = (a, 0, pc)$ generate direct summands. We have $\langle x, y \rangle = \langle x \rangle \oplus \langle y \rangle = \langle x \rangle \oplus \langle (0, -pb, pc) \rangle$. The direct sum $\langle x \rangle \oplus \langle y \rangle$ is not a valuated direct sum, while $\langle x \rangle \oplus \langle (0, -pb, pc) \rangle$ is a valuated direct sum, but $\langle (0, -pb, pc) \rangle \notin \mathcal{Q}(G)$.

**Remark 3.8.** The result of Theorem 3.6 cannot be extended to infinite direct sums of cyclic groups. Pierce [13, Theorem 15.4] has constructed an example of a separable $p$-group $G$ with standard basic subgroup (i.e., $B = \bigoplus_{i=1}^{\infty} \mathbb{Z}(p^i)$) such that $\mathrm{End}(G) = J + E$ where $J$ is the rank 1 torsion-free complete $p$-adic module generated by the identity and $E$ is the ideal of small endomorphisms. On the other hand, $\mathrm{End}(B)$ has infinite torsion-free $p$-adic rank, so $B \notin \mathcal{W}(G)$.

## 4 Finitely generated subgroups in $\mathcal{P}(G)$

Jain and Singh proved in [10] that if $R$ is a principal ideal domain, then all pseudo-injective modules are quasi-injective. In this section we prove a stronger version of this for $R = \mathbb{Z}$: if $G$ is a group then all finitely generated subgroups in $\mathcal{P}(G)$ are in $\mathcal{Q}(G)$.

*In this section, G is an arbitrary abelian group.*

In order to prove $\mathcal{Q}_f(G) = \mathcal{P}_f(G)$ for all groups $G$ we start with the case of $p$-groups.

**Lemma 4.1.** *Let $G$ be a $p$-group and $H \in \mathcal{P}(G)$. If $K$ is a cyclic direct summand of $H$ then $K \in \mathcal{P}(G)$.*

*Proof.* Let $L$ be a direct complement of $K$ in $H$, so $H = K \oplus L$, and let $\varphi : K \to G$ be a monomorphism.

If $\varphi(K) \cap L = 0$, then the homomorphism $\psi : K \oplus L \to G$, $\psi(x, y) = \varphi(x) + y$ is a monomorphism, hence it can be extended to an endomorphism $\overline{\varphi} \in \mathrm{End}(G)$. It is easy to see that $\overline{\varphi}$ also extends $\varphi$.

If $\varphi(K) \cap L \neq 0$, we first observe that the socle $\varphi(K)[p]$ is contained in $L$ since $\varphi(K)$ is a cyclic $p$-group (hence its subgroup lattice is a finite chain). Let $\varphi' : K \to G$ be the homomorphism defined by $\varphi'(x) = \varphi(x) - x$. Suppose that $\varphi'$ is not a monomorphism. Then there exists a non-zero element $x \in K$ such that $\varphi'(x) = 0$. Then $\varphi(x) = x \in K$, hence $\varphi(K) \cap K \neq 0$. It follows that $\varphi(K)[p] \subseteq K$, and this contradicts $K \cap L = 0$. Hence $\varphi'$ is a monomorphism. Suppose that $\varphi'(K) \cap L \neq 0$. Then there exists $x \in K$ such that $0 \neq \varphi(x) - x \in L$. If $e$ is the exponent of $x$ then $p^{e-1}(\varphi(x) - x) \in L[p]$. But $p^{e-1}\varphi(x) \in \varphi(K)[p] \subseteq L[p]$, hence $p^{e-1}x \in L[p]$, a contradiction. Then $\varphi'(K) \cap L = 0$ and we can apply what we proved so far to observe that there exists an endomorphism $\psi$ of $G$ which extends $\varphi$. Then for every $x \in K$ we have $\varphi(x) = \psi(x) + x$, hence $\psi + 1_G$ extends $\varphi$. □

We need the following technical result:

**Lemma 4.2.** *Let $G$ be a $p$-group and $H = \bigoplus_{i=1}^{n} H_i$ a finite subgroup such that all $H_i = \langle h_i \rangle$ are cyclic groups such that*

(i) $\exp(h_1) \le \exp(h_2) \le \cdots \le \exp(h_n)$,

(ii) *for all $m \in \{1, \dots, n\}$ and for all $x \in \bigoplus_{i=1}^{m} H_i$ we have $U(h_m) \le U(x)$, and*

(iii) *if $\exp(h_i) = \exp(h_j)$ then $U(h_i) = U(h_j)$.*

*Then the direct sum $\bigoplus_{i=1}^{n} H_i$ is a valuated direct sum of cyclic $p$-groups.*

*Proof.* The proof is by induction. For $n = 1$ the property is obvious. Suppose that (ii) is valid for all $m < n$. Then $\bigoplus_{i=1}^{n-1} H_i$ is a valuated direct sum of cyclic groups. Let $k$ be the minimal index such that $\exp(H_n) = \exp(H_k)$.

We observe that the sequence $U(h_i)$, $i = 1, \dots, n$ is a decreasing sequence such that $U(h_i) = U(h_j)$ if and only if $\exp(H_i) = \exp(H_j)$. Moreover, it follows by (b) that $U(h_n) \le U(y)$ for all $y \in H$.

If $U$ is an Ulm sequence, we denote, as in [9, p. 100],

$$H(U) = \{x \in H : U(x) \ge U\}, \quad H(U)^* = \{x \in H : U(x) > U\}$$

and we consider the $\mathbb{Z}(p)$-vector space

$$H_U = \frac{H(U) + pH}{H(U)^* + pH}.$$

Recall that a *$v$-basis* for $H$ is constructed in the following way: for every $U$ we fix a basis in $H_U$, and we choose one representative whose Ulm sequence is $U$ for each element in this basis; the union of all these representatives is a $v$-basis for $H$. It is proved in [9, Theorem 3] that $H$ is a valuated direct sum of cyclics if and only if the cardinal of a $v$-basis coincides to the rank of $H$. Moreover, in this hypothesis every $v$-basis is linearly independent and it generates $H$. Therefore, it is enough to prove that $\{h_i : i = 1, \dots, n\}$ is a $v$-basis for $H$.

Since $K = \bigoplus_{i=1}^{n-1} H_i$ is a direct sum of cyclic valuated groups, it follows that the set $\{h_i : i = 1, \dots, n-1\}$ is a $v$-basis for $K$.

Let $V$ be the Ulm sequence of $h_n$. If $U < V$ then $H(U) = H(U)^* = H$, so $H_U = 0$. If $U$ and $V$ are not comparable then $H(U) = H(U)^*$ since $V$ is minimal as Ulm sequence of an element of $H$, so $H_U = 0$.

It is easy to see that $H(V) = H$, and $H(V)^* = (\bigoplus_{i<k} H_i) \oplus (\bigoplus_{i=k}^{n} pH_i)$, so $h_k, \dots, h_n$ represent a basis in $H_V$.

If $V < U$ then

$$H(U) + pH = \Big( \bigoplus_{U(h_i) \ge U} H_i \Big) + pH = K(U) \oplus pH_n$$

and

$$H(U)^* + pH = \Big(\bigoplus_{U(h_i)>U} H_i\Big) + pH = K(U)^* \oplus pH_n.$$

Therefore every set in $K = \bigoplus_{i=1}^{n-1} H_i$ which represents a basis in $K_U$ is also a representative set for a basis in $H_U$.

It follows that $\{h_1, \ldots, h_n\}$ is a $v$-basis, and an application of the proof of [9, Theorem 3] will complete the proof. □

**Lemma 4.3.** *Let $G$ be a $p$-group and $x, y \in G$. If $U(x+y) = U(x)$ then $U(x) \leq U(y)$.*

*Proof.* Suppose that $U(x) \nleq U(y)$. Then there exists a positive integer $k$ such that $h(p^k y) < h(p^k x)$. It follows that $h(p^k(x+y)) = h(p^k y) \neq h(p^k x)$, and this contradicts our hypothesis. □

**Theorem 4.4.** *Let $G$ be a group. Then $\mathcal{P}_f(G) = \mathcal{Q}_f(G)$.*

*Proof.* Since only the inclusion $\mathcal{P}_f(G) \subseteq \mathcal{Q}_f(G)$ requires a proof, we start with a finitely generated subgroup $H \in \mathcal{P}(G)$.

If $G$ is a $p$-group, we write $H = \bigoplus_{i=1}^{n} H_i$ such that all $H_i = \langle h_i \rangle$ are cyclic groups with

$$\exp(h_1) \leq \exp(h_2) \leq \cdots \leq \exp(h_n).$$

We will prove that this decomposition satisfies the conditions (ii) and (iii) from Lemma 4.2.

Let $m \in \{1, \ldots, n\}$ and $j < m$. Then $H = \bigoplus_{i=1}^{n} H_i'$, where $H_i' = \langle h_i \rangle$ for all $i \neq m$ and $H_i' = \langle h_m + h_j \rangle$. Since $H \in \mathcal{P}(G)$, the isomorphism $\varphi : H \to H$ defined by $\varphi(h_i) = h_i$ for all $i \neq m$ and $\varphi(h_m) = h_m + h_j$ can be extended to an endomorphism of $G$. Then $U(h_m) \leq U(h_m + h_j)$. But $\varphi^{-1}$ also can be extended to an endomorphism of $G$, hence $U(h_m) \geq U(h_m + h_j)$. Therefore, $U(h_m) = U(h_m + h_j)$, and applying Lemma 4.3 we obtain that the condition (ii) is satisfied.

In order to prove (iii), it is enough to observe that if $\exp(h_i) = \exp(h_j)$ with $i < j$ we can replace in the direct decomposition of $H$ as direct sum of cyclic groups either of the two direct summands $\langle h_i \rangle$ or $\langle h_j \rangle$ by $\langle h_i + h_j \rangle$. By what we just proved for (ii) we have $U(h_i) = U(h_j)$.

Therefore, we proved that $\mathcal{Q}_f(G) = \mathcal{P}_f(G)$ for all $p$-groups. It is not hard to extend this property to all torsion groups and we now show that result can be extended to all abelian groups.

Let $G$ be a group and $H \in \mathcal{P}_f(G)$. Since $H$ is finitely generated, $H = F \oplus K$, with $F$ a free subgroup of finite rank and $K$ a finite subgroup. We claim that $F$

and $K$ are in $\mathcal{Q}(G)$ and every homomorphism $\varphi : F \to G$ can be extended to an endomorphism $\overline{\varphi}$ of $G$ such that $\overline{\varphi}(K) = 0$.

Since $K \leq T(G)$, every monomorphism $\varphi : K \to G$ can be extended to a monomorphism $\varphi' : H \to G$ such that $\varphi'(x) = x$ for all $x \in F$. Then there is an endomorphism $\overline{\varphi}$ of $G$ which extends $\varphi'$. Since $T(G)$ is fully invariant, $\overline{\varphi}$ induces an endomorphism of $G$ which extends $\varphi$. Therefore $K \in \mathcal{P}_f(T(G)) = \mathcal{Q}_f(T(G))$, and it follows that we can embed $K$ in a finite direct summand $L$ of $G$.

In order to prove $F \in \mathcal{Q}(G)$, let $\varphi : F \to G$ be a homomorphism. For every positive integer $i$ we consider the subgroup

$$U_i = \{x \in F : \varphi(x) = ix\} \leq F,$$

and we will prove by induction on $n$ that

$$\sum_{i=1}^{n} U_i = \bigoplus_{i=1}^{n} U_i$$

for all $n > 0$. Since the case $n = 1$ is obvious, suppose that $\sum_{i=1}^{n} U_i = \bigoplus_{i=1}^{n} U_i$. Let $x \in (\sum_{i=1}^{n} U_i) \cap U_{n+1}$. Then $x = \sum_{i=1}^{n} x_i$ with $x_i \in U_i$, hence

$$(n+1)\sum_{i=1}^{n} x_i = \varphi(x) = \sum_{i=1}^{n} \varphi(x_i) = \sum_{i=1}^{n} i x_i.$$

Then $\sum_{i=1}^{n}(n+1-i)x_i = 0$, and by the induction hypothesis $(n+1-i)x_i = 0$ for all $i = 1, \dots, n$. Since $F$ is torsion-free, it follows that $x = 0$. Then $\sum_{i>0} U_i = \bigoplus_{i>0} U_i \leq F$. But $F$ is of finite rank, hence we can find an integer $N > 0$ such that $U_n = 0$ for all $n \geq N$. Let $q > N$ be a prime such that $\gcd(q, |K|) = 0$. Therefore the homomorphism $\psi : F \to G$, $\psi(x) = \varphi(x) - qx$ is a monomorphism. Then $\psi(F)$ is torsion-free, and as in the first part of the proof it can be extended to a monomorphism $\psi' : H \to G$ such that $\psi'(x) = -qx$ for all $x \in K$. Since $H \in \mathcal{P}(G)$, there is an endomorphism $\overline{\psi}$ of $G$ which extends $\psi'$. Then $\overline{\varphi} = \overline{\psi} + q1_G$ is an endomorphism of $G$ which extends $\varphi$ and $\overline{\varphi}(K) = 0$.

Now we will prove that every homomorphism $\varphi : K \to G$ can be extended to an endomorphism $\overline{\varphi}$ of $G$ such that $\overline{\varphi}(F) = 0$. Let $\varphi : K \to G$ be a homomorphism. If $\varphi' : G \to G$ extends $\varphi$ then the restriction $\varphi'|_F : F \to G$ can be extended to an endomorphism $\psi$ of $G$ such that $\psi(K) = 0$. Then $\overline{\varphi} = \varphi' - \psi$ has the required properties.

In order to complete the proof, let $\varphi : H \to G$ be a homomorphism. Then it induces by restrictions two homomorphisms $\varphi_1 : F \to G$ and $\varphi_2 : K \to G$. But by what we just proved we can extend these homomorphisms to two endomorphisms $\overline{\varphi}_1$ and $\overline{\varphi}_2$ of $G$ such that $\overline{\varphi}_1$ extends $\varphi_1$ and $\overline{\varphi}_1(K) = 0$, and $\overline{\varphi}_2$ extends $\varphi_2$ and $\overline{\varphi}_2(F) = 0$. Then $\overline{\varphi} = \overline{\varphi}_1 + \overline{\varphi}_2$ extends $\varphi$ and the proof is complete. □

**Remark 4.5.** In the case $G$ is a $p$-group with $p \geq 3$ there is a simpler proof. In fact, in order to prove $H \in \mathcal{Q}(G)$ for all finite subgroups $H \in \mathcal{Q}(G)$ it is enough to prove that Lemma 3.1 is valid for $K \oplus L \in \mathcal{P}(G)$. For the case $p \geq 3$ it is easy to see that the homomorphism $\varphi : K \to L \to G$, $\varphi(k,\ell) = 2k + \ell$ is a monomorphism, hence it can be extended to an endomorphism $\overline{\varphi}$ of $G$. Then $\overline{\varphi} - 1_G$ extends the canonical projection $K \oplus L \to K$ and the proof presented for Lemma 3.1 also works for $\mathcal{P}(G)$.

A careful analysis of the previous proof reveals the fact that $F$ can be replaced by any finite rank torsion-free group. Therefore we obtain:

**Corollary 4.6.** *Let $G$ be a group and $H$ a subgroup of $G$ of finite rank. Then $H \in \mathcal{Q}(G)$ if and only if $H \in \mathcal{P}(G)$.*

We have been unable to determine whether Theorem 4.4 can be extended to all subgroups in $\mathcal{P}(G)$, i.e., if there exists an abelian group $G$ such that $\mathcal{Q}(G) \neq \mathcal{P}(G)$.

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**Author information**

Simion Breaz, Faculty of Mathematics and Computer Science,
“Babeş-Bolyai” University, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania.
E-mail: bodo@math.ubbcluj.ro

Grigore Călugăreanu, Faculty of Mathematics and Computer Science,
“Babeş-Bolyai” University, Str. Mihail Kogălniceanu 1, 400084 Cluj-Napoca, Romania.
E-mail: calu@math.ubbcluj.ro

Phill Schultz, School of Mathematics and Statistics,
The University of Western Australia, Nedlands, 6009, Australia.
E-mail: phill.schultz@uwa.edu.au