A fine element which is not exchange

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March 15, 2019

We start with the example of fine element which is not clean given in our joint paper with T. Y. Lam (see [1]).

Theorem 5.7 *Over a commutative domain S, a diagonal matrix* $A = \mathrm{diag}(a,1)$ *is clean in* $R = \mathcal{M}_2(S)$ *iff* $a \in U(S) \cup (1+U(S))$.

Since $A = \begin{bmatrix} a+1 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ is always fine, for an example which is not clean it suffices to take $a \geq 3$ or $a \leq -2$ in $\mathbb{Z}$.

To the characterization above we can add the exchange property, that is

Theorem 1 *Let* $A = \mathrm{diag}(a,1)$*,* $a \in S$*, be a diagonal matrix over a commutative domain* S*. The following conditions are equivalent:*
(i) A *is clean;*
(ii) A *is exchange;*
(iii) $a \in U(S) \cup (1+U(S))$.

Proof. Only (ii) $\Longrightarrow$ (iii) needs justification. Recall that A is exchange iff $\exists M \in R$ such that $A + M(A - A^2)$ is idempotent.

If $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$, by computation,

$$C = A + M(A - A^2) = \begin{bmatrix} a + x(a-a^2) & 0 \\ z(a-a^2) & 1 \end{bmatrix}.$$

Since $C \neq 0_2$, notice that $C = I_2$ iff $a \in U(S)$.

Further, C is nontrivial idempotent iff $\mathrm{Tr}(C) = 1$, $\det(C) = 0$.

For $a = 0$, the matrix is idempotent, so clean and exchange.

For $a \neq 0$, both equalities reduce to $x(a-1) = 1$ (for arbitrary y, z, t) which implies $a - 1 \in U(S)$. ■

References

[1] G. Călugăreanu, T. Y. Lam *Fine rings: A new class of simple rings*. J. of Algebra and its Appl., **15**, (9) (2016), 18 pages.