”Some matrix completions over integral domains” : corrections and addendum

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Abstract

We correct a typo and an incorect remark (which are not essential in the paper), and develop over $\mathbb{Z}$ the main result.

The main result in [1] is the following

Proposition 1 *Let R be a (commutative) integral domain and let U be an arbitrary matrix in $\mathcal{M}_2(R)$. There is a nilpotent matrix $N \in \mathcal{M}_3(R)$ which has U as the northwest 2×2 corner, whenever there exist elements $a, b, x, y \in R$ such that $ax+by = \det(U)-\mathrm{Tr}(U)^2$ and $bxu_{12}+ayu_{21}-axu_{22}-byu_{11} = \mathrm{Tr}(U)\det(U)$. Such a matrix exists if (e.g.) u_{12} or u_{21} is a unit.*

Conversely, if N is a 3×3 nilpotent matrix which has U as the northwest 2×2 corner, the previous relations hold for $a = n_{13}, b = n_{23}, x = n_{31}$ and $y = n_{32}$.

The correction of a typo: in the proof, commenting the special case u_{12} is a unit, a completion is indicated, namely, $a = 0$, $b = 1$, $y = m$, $x = (l + mu_{22})u_{12}^{-1}$, and if u_{21} is a unit, $x = 0$, $y = 1$, $b = m$, $a = (l + mu_{22})u_{21}^{-1}$.

In both formulas, u_{22} must be replaced by u_{11}, that is $x = (l + mu_{11})u_{12}^{-1}$ and $a = (l + mu_{11})u_{21}^{-1}$, respectively.

The correction of remark 1, p. 3: if $a = b = x = y = 0$ then clearly $\det(U) = \mathrm{Tr}(U) = 0$ (i.e., U is nilpotent) from the conditions in the Proposition 1.

However, the converse fails: if U is a 2×2 nilpotent, the completion with $a = b = x = y = 0$ clearly gives a 3×3 nilpotent N, but *this is not the only possible completion.* For example, $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}^3 = 0_3$, and the nilpotent U is completed with $a = x = y = 1$, $b = -1$.

Rephrasing, $ax + by = 0 = bxu_{12} + ayu_{21} - axu_{22} - byu_{11}$, $u_{11} + u_{22} = 0 = u_{11}u_{22} - u_{12}u_{21}$ do not necessarily imply $a = b = x = y = 0$.

We can also obtain an index 2 nilpotent by completion of the same U:

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^2 = 0_3$$

Addendum. From now on we take $R = \mathbb{Z}$, the integers and continue to use the notations $m = \det(U) - \mathrm{Tr}(U)^2$, $l = \mathrm{Tr}(U)\det(U)$.

What we intend to discuss here in the generation of all the 3×3 nilpotents.

For any given pair $(a, b) \in \mathbb{Z}^2$, the completion equations, i.e.

$$\begin{array}{rcl} ax + by & = & m \\ (bu_{12} - au_{22})x + (au_{21} - bu_{11})y & = & l \end{array}$$

give a system of two linear Diophantine equations with unknowns x, y.

The following is well-known:

Proposition 2 *The Diophantine equation $ax + by = c$ has an integer solution iff $\gcd(a, b)$ divides c. If we denote $a = u \cdot \gcd(a, b)$, $b = v \cdot \gcd(a, b)$ and (x_0, y_0) is a solution then the other solutions have the form $(x_0 + kv, y_0 - ku)$, where k is an arbitrary integer.*

Remark. If $c = w \cdot \gcd(a, b)$ the equation is equivalent to $ux + vy = w$ with coprime u, v. Then a solution is given by reversing the Euclid's algorithm for u and v. If $us + vt = 1$ for integers s, t then (sw, tw) is a solution for the initial equation $ax + by = c$ (here $w = \dfrac{c}{\gcd(a, b)}$).

Therefore, for possible solutions of each equation above (separately), it is necessary that $\gcd(a, b)$ divides m and $\gcd(bu_{12} - au_{22}, au_{21} - bu_{11})$ divides l.

The system has solutions if these two conditions are fulfilled and there are common solutions.

Such a system may be written in a matrix form $AX = C$, i.e.

$$\begin{bmatrix} a & b \\ bu_{12} - au_{22} & au_{21} - bu_{11} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m \\ l \end{bmatrix}$$

and may be solved by computing the *Smith normal form* of its matrix, in a way that is similar to the use of the reduced row echelon form to solve a system of linear equations over a field.

In the general $n \times n$ case, if $U, V \in GL_n(\mathbb{Z})$ are such that $B = UAV$ is a diagonal $n \times n$ matrix (b_{ii} is not zero for i not greater than some integer k, and all the other entries are zero) then $B\,(V^{-1}X) = UC$ and denoting y_i the entries of $V^{-1}X$ and d_i those of $D = UC$, this leads to the system $b_{ii}\, y_i = d_i$ for $1 \le i \le k$, $0\, y_i = d_i$ for $k < i \le n$.

Finally, the system has a solution iff b_{ii} divides d_i for $i \le k$ and $d_i = 0$ for $i > k$. If this condition is fulfilled, the solutions of the given system are

$$V\begin{bmatrix} \frac{d_1}{b_{11}} \\ ... \\ \frac{d_k}{b_{kk}} \\ h_{k+1} \\ ... \\ h_n \end{bmatrix}$$, where $h_{k+1},...,h_n$ are arbitrary integers.

References

[1] G. Călugăreanu *Some matrix completions over integral domains*. Linear Algebra and its Appl. (2016).