Equivalent idempotents are conjugate in any ring

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Definitions. If R is a ring with identity, $a, b \in R$, we say that a is *equivalent* to b, denoted by $a \approx b$, if there exist units $u, v \in R$ such that $uav = b$; a is called *conjugate* to b, denoted by $a \sim b$, if there exists a unit $u \in R$ such that $uau^{-1} = b$.

Obviously, both (binary) relations are equivalences on R.

In [1] we can find the following elementary but useful results.

Lemma 1 *Let R be a ring, $a, b \in R$ with $a^2 = a$ and $aba = a$. Then $a \sim ab \sim ba$.*

Proof. Since $(a - ab)^2 = a - ab - aba + abab = 0$, let $t = 1 - a + ab$. Then t is invertible and $t^{-1} = 1 + a - ab$. So $tabt^{-1} = (1 - a + ab)ab(1 + a - ab) = a$, hence $a \sim ab$. Similarly, it can be proved that $a \sim ba$.

Theorem 2 *Let R be a ring, $a, b \in R$ with $a^2 = a$ and $b^2 = b$, then $a \sim b$ if and only if $a \approx b$.*

Proof. It is only needed to prove that $a \approx b \Rightarrow a \sim b$. Suppose that there exist invertible elements p and q in R such that $paq = b$. Let $s = q^{-1}p^{-1}$. Then $pap^{-1} = paqq^{-1}p^{-1} = bs$, so $a \sim bs$ and $bsb = b$. By Lemma 1, $bs \sim b$, so $a \sim bs \sim b$. ■

Proposition 3 *Let R be a ring and a, b be idempotents of R. If $(a - b)^2 = 0$, then $a \sim ab \sim ba \sim b$.*

Proof. Since $(a - b)^2 = a^2 - ab - ba + b^2$, we have $a + b = ab + ba$ and so $a(a + b) = a^2b + aba$, i.e., $a + ab = ab + aba$ which implies $a = aba$. Similarly, we have $b = bab$. So by Lemma 1, $a \sim ab \sim ba \sim b$. ■

Corollary 4 *Let A be an $n \times n$ idempotent matrix over a ring R. If A is equivalent to a diagonal matrix, then A is (also) similar to a diagonal matrix.*

References

[1] G. Song, X. Guo *Diagonability of idempotent matrices over non commutative rings*. Linear Algebra and its Applications **297** (1999), 1-7.