Equivalent nilpotents may not be conjugate.

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Definitions. If R is a ring with identity, $a, b \in R$, we say that a is *equivalent* to b, denoted by $a \approx b$, if there exist units $u, v \in R$ such that $uav = b$, and, a is called *conjugate* to b, denoted by $a \sim b$, if there exists a unit $u \in R$ such that $u^{-1}au = b$.

Obviously, both (binary) relations are equivalences on R and $\sim$ implies $\approx$.

In [1], it was proved that *equivalent idempotents are conjugate* (see another 2019 note, here).

We may ask whether equivalent *units* or *nilpotents* are conjugate, respectively.

The answer is obviously *no* for units.

Take any unit $u \in U(R)$ such that $u \neq 1$. Then (the unit) $u = u \cdot 1 \cdot 1$ is equivalent to 1 but not conjugate to 1. Indeed, only 1 is conjugate to itself.

As for nilpotents, 0 is clearly conjugated (and so equivalent) only with itself. Therefore in the sequel we consider only nonzero nilpotents.

Before giving an *example of equivalent (nonzero) nilpotents which are not conjugate*, we mention rings in which equivalent nilpotents are conjugate.

A commutative domain is called *GCD* if every two nonzero elements have a gcd.

Lemma 1 *In a GCD commutative domain,* $\gcd(a; b) = 1$ *implies* $\gcd(a^2; b) = 1$.

Proof. First recall that every nonzero element of a GCD commutative domain is *primal* ($x|yz$ implies $x = x_1x_2$ with $x_1|y$, $x_2|z$). Then suppose $1 \neq d = \gcd(a^2; b)$. Since $d|a^2$, $d = d_1^2$ with $d_1|a$. Hence $1 \neq d_1$ divides both a and b and so $\gcd(a; b) \neq 1$. ■

Proposition 2 *Every nonzero nilpotent* 2×2 *matrix over a commutative GCD domain* R *is similar to* rE_{12}*, for some* $r \in R$.

Proof. We are looking for an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and $x^2 + yz = 0$.

Let $d=\gcd(x;y)$ and denote $x=dx_1$, $y=dy_1$ with $\gcd(x_1;y_1)=1$. Then $d^2x_1^2=-dy_1z$ and since $\gcd(x_1;y_1)=1$ implies $\gcd(x_1^2;y_1)=1$, it follows y_1 divides d. Set $d=y_1y_2$ and so $T=\begin{bmatrix} x_1y_1y_2 & y_1^2y_2 \\ -x_1^2y_2 & -x_1y_1y_2 \end{bmatrix}=$ $y_2\begin{bmatrix} x_1y_1 & y_1^2 \\ -x_1^2 & -x_1y_1 \end{bmatrix}=y_2T'$.

Since $\gcd(x_1;y_1)=1$ there exist $s,t\in R$ such that $sx_1+ty_1=1$. Take $U=\begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}$ which is invertible (indeed, $U^{-1}=\begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}$). One can check $T'U=\begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix}=UE_{12}$, so $r=y_2$. ■

Remark. 1) In any ring R, $\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & -r \\ 0 & 0 \end{bmatrix}$: indeed, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -r \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -r \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2) In any ring R, $\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}$: indeed, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}=\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Hence, for $R=\mathbb{Z}$, to have the non-similar representatives of classes of nilpotents, it suffices to take $r\in\mathbb{N}$.

Theorem 3 *In $\mathbb{M}_2(\mathbb{Z})$ equivalent nilpotents are conjugate.*

Proof. According to the above, nilpotents of $\mathbb{M}_2(\mathbb{Z})$ belong to disjoint (conjugation) classes, whose representatives are nE_{12} for all nonnegative integers. By the way of contradiction, we first show that if m,n are different positive integers, mE_{12} is not equivalent to nE_{12}.

Suppose there are units $U,V\in GL_2(\mathbb{Z})$, such that $U(nE_{12})=(mE_{12})V$.

Denote $U=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $V=\begin{bmatrix} x & y \\ z & t \end{bmatrix}$ and assume $ad-bc=\pm1=xt-yz$.

The equality $U(nE_{12})=(mE_{12})V$ amounts to $\begin{bmatrix} 0 & na \\ 0 & nc \end{bmatrix}=\begin{bmatrix} mz & mt \\ 0 & 0 \end{bmatrix}$ and so $c=z=0$ and $na=mt$, $ad=\pm1=xt$. Therefore, $a,d,x,t\in\{\pm1\}$.

Let $\delta=\gcd(m;n)$ and let $m=\delta m'$, $n=\delta n'$. Then $n'a=m't$ and since m',n' are coprime, m' divides a. Hence $a=m'a'$ and so $t=n'a'$.

Finally, $xt=xn'a'=1$, and so all $x,n',a'\in\{\pm1\}$. If $n'=1$ then $\delta=n$ and so $m=nm'$. Moreover, $m'\neq1$; otherwise $m=n$. Finally, since $a=m'a'$, $a\neq1$, a contradiction.

Now let $T^2=S^2=0_2$ be not conjugated (nonzero) nilpotents. Then T and S belong to two different, and so disjoint, conjugacy classes, represented by (say), mE_{12} and nE_{12}, with different positive integers m,n. Each conjugacy class is included in an equivalence class and two such classes are (also) disjoint or coincide. The conjugacy classes which include T and S respectively cannot be included in the same equivalence class because $mE_{12}\not\approx nE_{12}$. Hence these are included into disjoint equivalence classes and so $T\not\approx S$, indeed. ■

As for equivalent nilpotents which are not conjugate, here is an

Example. Consider $T = 3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_{12})$. Then $T^2 = 6\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $T^3 = 0_2$ so T is an index 3 nilpotent.

$S = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$ is an index 2 nilpotent, and since conjugation preserves the index of nilpotency, S and T are not conjugate.

However, these are equivalent since $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} T \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = S$.

Added in proof. Professor T. Y. Lam *commutative* example: let R be any not reduced commutative ring such that $2a = 0$ implies $a = 0$. For any nonzero nilpotent element a, a is equivalent to $-a$, but clearly not similar to $-a$.

References

[1] G. Song, X. Guo *Diagonability of idempotent matrices over non commutative rings*. Linear Algebra and its Applications297(1999), 1-7.