The rings all whose isomorphic idempotents are equal are precisely the Abelian rings

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We show that the rings which have only equal isomorphic idempotents (i.e., if $e = e^2$, $f = f^2$ and $e \cong f$, then $e = f$) are precisely the Abelian rings (i.e., rings with only trivial idempotents). Hereafter, just to have a name a ring is temporary called *EII* if it has only equal isomorphic idempotents.

First recall the following

Definition. Two idempotents e, f are *isomorphic* (written $e \cong f$) if $eR \cong fR$ as right R-modules, or equivalently, $Re \cong Rf$ as left R-modules.

In the next lemma we recall some well-know facts about isomorphic and conjugated idempotents, respectively (see [3] for exercises and [2] for solutions). For a summary and some completions see [1]. As customarily, $Z(R)$ denotes the center of the ring R.

Lemma 1 *(1) Two idempotents e, f of a ring R are isomorphic iff there exist $a, b \in R$ such that $e = ab$ and $f = ba$.*

(2) Two idempotents e, f are conjugate iff $e \cong f$ and $\overline{e} \cong \overline{f}$.

(3) Two central idempotents e, f in a ring are isomorphic iff if these are equal.

(4) The following conditions are equivalent:

(i) $e \in Z(R)$,

(ii) $eR = Re$,

(iii) e commutes with all the idempotents,

(iv) e commutes with all the idempotents of R that are isomorphic to e.

(v) e is subcommutative (i.e., $Re \subseteq eR$ and $eR \subseteq Re$).

(vi) eR is uniquely complemented.

(vii) $eR\overline{e} = \overline{e}Re = 0$.

(viii) e commutes with all nilpotents.

(ix) e commutes with all units.

Using (1) we can easily provide proofs for (2) and (3), avoiding R-modules isomorphisms.

For example (3): assume $e = ab$ and $f = ba$ are idempotents, and so by hypothesis, central. Then $e = ab = a(ba)b = (ba)ab = b(ab)a = ba = f$.

A proof for (2) is given among the "Elementary" on my Homepage.

According to (3), Abelian rings are EII.

From (2), the previous lemma, it follows that *conjugate idempotents are isomorphic*. Therefore, as binary (equivalence) relations on $Id(R)$, "equality" $\subseteq$ "conjugation" $\subseteq$ "isomorphism".

Conversely, assume R is EII. As such, any two isomorphic idempotents are equal. In particular, conjugate idempotents must be equal, that is, idempotents commute with units. Hence, by (4), (ix) the lemma above, R is Abelian.

As a detail for (4), (ix), just note that $e \in Z(R) \Rightarrow e$ commutes with all units $\overset{unipotents}{\Rightarrow}$ e commutes with all nilpotents $\overset{(viii)}{\Rightarrow}$ $e \in Z(R)$.

Here (viii) actually is **Ex.12.7** from [2].

References

[1] G. Călugăreanu, P. Schulz *Modules with Abelian endomorphism rings*. Bull. Austral. Math. Soc. **82** (1) (2010), 99-112.

[2] T. Y. Lam *Exercises in Classical Ring Theory*. Second Edition, Problem Books in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[3] T. Y. Lam *A First Course in Noncommutative Rings*. Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.