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ON AN ENRICHED THEORY OF MODULES (II)

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Introduction. Stimulated by the excelent monography [4], the author of the present paper works out the closed and monoidal closed part of the theory of modules over a fixed monoid, a theory for which, in [5], M a c L a n e only worked out the monoidal part.

The reader needs only the first section from [3]-where the basic notions: closed and monoidal monoids and the corresponding morphisms, left and right modules over monoids, are defined, and the basic situations studied, — in order to recover our main definitions.

From section two all the definitions and results following the corrolary 2.7 are needed.

In this way, we shall start this second part of this paper with section three. In what follows, we suppose that $\underline{V}_0$ is a symmetric monoidal closed category with equalizers.

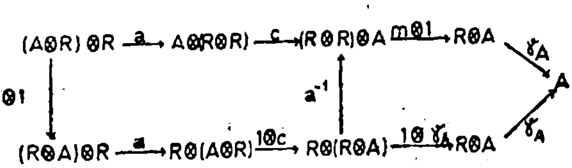
3. The closed and monoidal closed structure of $_R\underline{MV}$. LEMMA 3.1. — *The morphism $z_A = \pi(\gamma_A \cdot c_{AR}) : A \to (RA)$, considered for a left R-module (A, α_A, γ_A) over $\underline{V}$, factors through $\{RA\}$. Moreover, denoting by $\mathrm{i}_A : A \to \{RA\}$ the factorization morphism, this is an isomorphism in $\underline{V}_0$.*

Proof. First, we have to check (3) $(\alpha_A, 1) \cdot R^A_{AR} \cdot z_A = (n, 1) \cdot L^R_{AR} \cdot z_A$. Using [II, (3.1), (3.19), (3.22)] we have

$$(n, 1) \cdot L^R_{RA} \cdot z_A = (\pi(m), 1) \cdot L^R_{RA} \cdot z_A = p \cdot (m, 1) \cdot \pi(\gamma_A \cdot c_{AR}) =$$

$$= p \cdot \pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes m) = \pi\pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes m \cdot a) = \pi\pi(\gamma_A \cdot m \otimes 1 \cdot c_{A,R\otimes R} \cdot a).$$

The following commutative diagram



enables us to continue the equalities above

$$= \pi\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR} \cdot c_{AR} \otimes 1) = \pi(\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a) \cdot c_{AR})$$

using again [II, (3,1)]. At the same time, we have

$$(\alpha_A, 1) \cdot R^R_{AR} \cdot z_A = \pi(M^A_{AR} \cdot c_{(RA),(AA)} \cdot z_A \otimes \alpha_A) = \pi(M^A_{RA} \cdot \alpha_A \otimes z_A \cdot c_{AR}).$$

Thus, the equality (3) is equivalent to the following

$M^A_{RA} \cdot \alpha_A \otimes z_A = \pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR})$ or to the one derived from this applying π, which is proved as follows

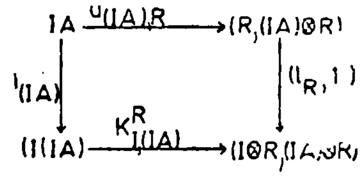
$$\pi(M^A_{RA} \cdot \alpha_A \otimes z_A) = (z_A, 1) \cdot L^R_{AA} \cdot \alpha_A = (\pi(\gamma_A \cdot c_{AR}), 1) \cdot L^R_{AA} \cdot \alpha_A =$$
$$= p \cdot (\gamma_A \cdot c_{AR}, 1) \cdot \alpha_A = p \cdot \pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR})) = \pi\pi(\gamma_A \cdot 1 \otimes (\gamma_A \cdot c_{AR}) \cdot a_{RAR})$$

using [II, (3.1), (3.19), (3.22)].

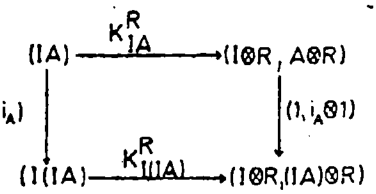
Now, let us prove that $i_A^{-1}: \{RA\} \xrightarrow{equ} (RA) \xrightarrow{(e,1)} (IA) \xrightarrow{i_A^{-1}} A$ is a twosided inverse for i_A, that is, let us check the following two equalities $(e, 1) \cdot z_A = i_A$, $z_A \cdot i_A^{-1} \cdot (e, 1) \cdot equ_{RA} = equ_{RA}$. We simply obtain the first as follows

$$(e, 1) \cdot \pi(\gamma_A \cdot c_{AR}) = \pi(\gamma_A \cdot c_{AR} \cdot 1 \otimes e) = \pi(\gamma_A \cdot e \otimes 1 \cdot c_{AI}) = \pi(l_A \cdot c_{AI}) =$$
$$= \pi(r_A) = i_A.$$

As for the second, we first derive from [II, (7.4)] the following commutative diagram



Using $i_{(IA)} = (1, i_A)$ and the following commutative diagram (by naturality of K)



we have $u_{(IA),R} = (l_R^{-1}, i_A \otimes 1) \cdot K^R_{IA}$. We now prove the second equality required above as follows

$$\gamma_A \cdot i_A^{-1} \cdot (e, 1) \cdot equ_{RA} = \pi(\gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot equ_{RA} =$$
$$= (1, \gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot u_{(IA),R} \cdot (e, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (1, \gamma_A \cdot c_{AR} \cdot i_A^{-1} \otimes 1) \cdot (l_R^{-1}, i_A \otimes 1) \cdot K^R_{IA} \cdot (e, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot K^R_{IA} \cdot (e, 1) \cdot \mathrm{equ}_{RA} = (l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot (e \otimes 1, 1) \cdot K^R_{RA} \cdot \mathrm{equ}_{RA} =$$
$$= (\mathrm{e} \otimes 1 \cdot l_R^{-1}, \gamma_A \cdot c_{AR}) \cdot (c_{RR}, c_{RA}) \cdot H^R_{RA} \cdot \mathrm{equ}_{RA} =$$
$$= (c_{RR} \cdot e \otimes 1 \cdot l_R^{-1}, 1) \cdot (1, \gamma_A) \cdot H^R_{RA} \cdot \mathrm{equ}_{RA} = (c_{RR} \cdot \mathrm{e} \otimes 1 \cdot l_R^{-1}, 1) \cdot (m, 1) \cdot \mathrm{equ}_{RA} =$$
$$= (1 \otimes e \cdot r_R^{-1} \cdot m, 1) \cdot \mathrm{equ}_{RA} = \mathrm{equ}_{RA},$$

using also [II, (3.4)] and lemma 3.1.

LEMMA 3.2. — *For each left R-module* (A, α_A, γ_A) *over* $\underline{V}$, *the morphism* $y_{RA} = M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR}) \otimes \mathrm{equ}_{RA}: R \otimes \{RA\} \to (RA)$ *factors through*

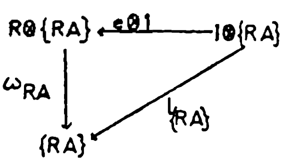
$\{RA\}$. The unique factorization morphism $\omega_{RA}: R\otimes\{RA\} \to (RA)$ provides a left R-module structure over $\underline{V}$ for $\{RA\}$.

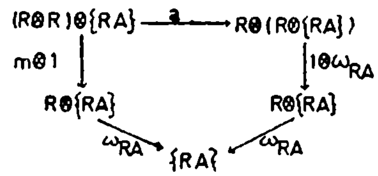
Proof. Using the $\underline{V}$-functoriality of R^A and L^R, the naturality of M, a commutative diagram derived from [III, (4.4)] and (3) from the previous lemma we have

$$(\alpha_A, 1)\cdot R^A_{AR}\cdot M^R_{RA}\cdot c_{(RR),(RA)}\cdot \pi(m\cdot c_{RR})\otimes \mathrm{equ}_{RA} =$$
$$= (n, 1)\cdot L^R_{RA}\cdot M^R_{RA}\cdot c_{(RR),(RA)}\cdot \pi(m\cdot c_{RR})\otimes \mathrm{equ}_{RA},$$

which proves the existence of the required morphism ω_{RA}.

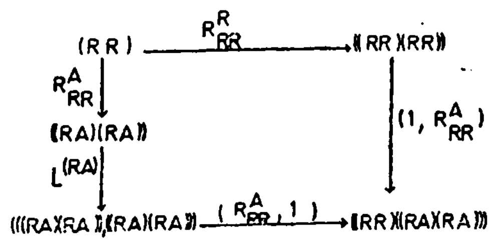
Next, let us check that $(\{RA\}, \omega_{RA})$ actually is a left R-module over $\underline{V}$, that is, the commutativity of the following diagrams





By left composition with equ_{RA}, the first one is equivalent with $y_{RA}\cdot e\otimes 1 = \mathrm{equ}_{RA}\cdot l_{\{RA\}}$ denoting by $y_{RA} = \mathrm{equ}_{RA}\cdot\omega_{RA}$, or, applying π, $\pi(y_{RA})\cdot e = (1, \mathrm{equ}_{RA})\cdot j_{\{RA\}}$. Using $\pi(y_{RA}) = (\mathrm{equ}, 1)\cdot R^A_{RR}\cdot\pi(m\cdot c_{RR})$ and $\pi(m\cdot c_{RR})\cdot e = j_R$ (from proposition 1.10) it is sufficient to verify $R^A_{RR}\cdot j_A = j_{(RA)}$. But this follows easily from [CC2], i.e., $(j_R, 1)\cdot L^R_{RA} = i_{(RA)}$ applying π^{-1} and using [III, (4.4)] for [III, 6.4].

The commutativity of the second diagram is equivalent with $y_{RA}\cdot m\otimes 1 = y_{RA}\cdot 1\otimes\omega_{RA}\cdot a$. We first mention that again from proposition 1.10 we have $\beta\cdot m = M^R_{RR}\cdot\beta\otimes\beta\cdot c_{RR} = M^R_{RR}\cdot c_{(RR),(RR)}\cdot\beta\otimes\beta$, which implies, applying π, $(1,\beta)\cdot\pi(m) = (\beta, 1)\cdot R^R_{RR}\cdot\beta$. Next, using the $\underline{V}$-functoriality of R^A and applying π, we derive an analogous of [CC3]



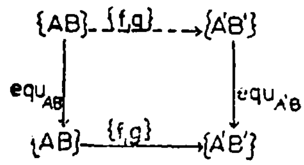
Using all these, the proof is the following

$$\pi\pi(y_{RA}\cdot m\otimes 1) = \pi(\pi(y_{RA})\cdot m) = (1, \pi(y_{RA}))\cdot\pi(m) =$$
$$= (1, (\mathrm{equ}_{RA}, 1))\cdot(1, R^A_{RR})\cdot(1,\beta)\cdot\pi(m) =$$
$$= (1, (\mathrm{equ}_{RA}, 1))\cdot(1, R^A_{RR})\cdot(\beta, 1)\cdot R^R_{RR}\cdot\beta =$$
$$= (1, (\mathrm{equ}_{RA}, 1))\cdot(\beta, 1)\cdot(R^A_{RR}, 1)\cdot L^{(RA)}_{(RA),(RA)}\cdot R^A_{RR}\cdot\beta =$$

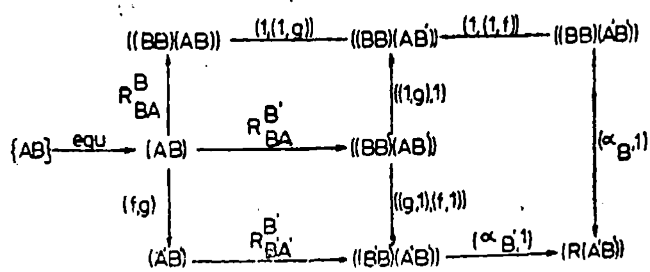
$$= (\beta, 1) \cdot (R^A_{RR}, 1) \cdot ((\mathrm{equ}_{RA}, 1), 1) \cdot L^{\{RA\}}_{\{RA\},(RA)} \cdot R^A_{RR} \cdot \beta =$$
$$= (\pi, (y_{RA}), 1) \cdot L^{\{RA\}}_{\{RA\},(RA)} \cdot R^A_{RR} \cdot \beta = p \cdot (y_{RA}, 1) \cdot R^A_{RR} \cdot \beta =$$
$$= p \cdot \pi(M^R_{RA} \cdot c_{(RR),(RA)} \cdot \pi(m \cdot c_{RR}) \otimes y_{RA}) = p \cdot \pi(y_{RA} \cdot 1 \otimes \omega_{RA}) =$$
$$= \pi\pi(y_{RA} \cdot 1 \otimes \omega_{RA} \cdot a).$$

LEMMA 3.3. — *The construction described in lemma 2.8. defines in a symmetric monoidal closed category $\underline{V}$ with equalizers a bifunctor $\{-, -\}: {}_R\underline{MV}^{op} \times {}_R\underline{MV} \rightarrow \underline{V}_0$.*

Proof. If $f: (A', \alpha_{A'}) \rightarrow (A, \alpha_A)$ and $g: (B, \alpha_{B'}) \rightarrow (B', \alpha_{B'})$ are morphisms of left R-modules over $\underline{V}$, then $\{f, g\}: \{AB\} \rightarrow \{A'B'\}$ is the unique morphism of factorization through the equalizers



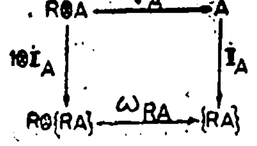
The functoriality is derived from the uniqueness. Thus, in order to prove the existence of the factorization morphism on the above diagram we must check $(\alpha_{B'}, 1) \cdot R^{B'}_{B'A'} \cdot (f, g) \cdot \mathrm{equ}_{AB} = (\alpha_{A'}, 1) \cdot L^{A'}_{A'B'} \cdot (f, g) \cdot \mathrm{equ}_{AB}$. Using the following commutative diagram



the first member of the required equality is $= (1, (f, g)) \cdot (\alpha_B, 1) \cdot R^B_{BA} \cdot \mathrm{equ}_{AB}$. But this is $= (1, (f, g)) \cdot (\alpha_A, 1) \cdot L^A_{AB} \cdot \mathrm{equ}_{AB}$. Finally, a similar diagram for L (like the above one) leads us to the second member of the required equality.

THEOREM 3.4. — *For each left R-module (A, α_A, γ_A) over $\underline{V}$, there is a natural isomorphism in ${}_R\underline{MV}$, $\mathbf{i}_A: A \rightarrow \{RA\}$, where $\{RA\}$ has the left R-module structure given by lemma 3.2.*

Proof. We must show that $\mathbf{i}_A$ actually is a morphism of left R-modules over $\underline{V}$, i.e., that the following diagram commutes



By left composition with equ_{RA} it is sufficient to check $z_A \cdot \gamma_A = y_{RA} \cdot 1 \otimes \mathbf{i}_A$. We derive this from the following equivalent equalities $z_A \cdot \gamma_A \cdot c_{AR} = M^R_{RA} \cdot z_A \otimes z_R$, $(1, z_A) \cdot z_A = (z_R, 1) \cdot L^R_{RA} \cdot z_A$, this last equality being checked analogously, like the one in proposition 1.6.

The naturality (in ${}_R\underline{MV}$!) of the family $\mathbf{i} = \mathbf{i}_A : A \to \{RA\}$ reduces to the naturality of the family $z = z_A$ which is readily checked.

THEOREM 3.5. — $\mathbf{i}_R : R \to \{RR\}$ *is an isomorphism of monoids from the opposite monoid of* R *to the monoid* $\{RR\}$ *of the* R*-endomorphisms of the left* R*-module* R *over itself.*

Proof. Straightforward, using equalities from the proof of proposition 1.10.

Remark. It can be shown, in the usual subjacent way, that if V preserves equalizers and $V \cdot W$ is an epifunctor, then the left R-module (R, n, m) over R is a projective object in ${}_R\underline{MV}$. Analogously one could now define Quasi-Frobenius monoids over $\underline{V}$ in the usual way.

— Let us point out the second bifunctor corresponding to the monoidal structure of ${}_R\underline{MV}$. We assume that $\underline{V}_0$ *has coequalizers* and, for a monoid (R, e, n, m), that (A, γ_A) is a object in ${}_R\underline{MV}$ and (B, δ_B) is an object of $\underline{MV}_R$.

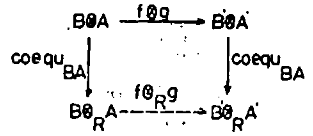
We define the *tensor product* $B \otimes_R A$ as an object in $\underline{V}_0$, namely

$$\text{coequ}\ (((\delta_B \otimes 1_A) \cdot a^{-1}_{BRA},\ 1_B \otimes \gamma_A) : B \otimes (R \otimes A) \to B \otimes A).$$

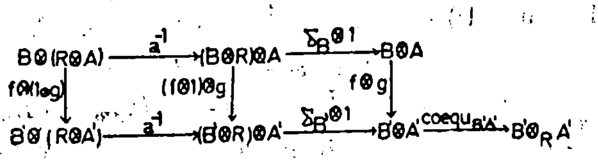
This will be a quotient object of $B \otimes A$. We shall denote by $\text{coequ}_{BA} : B \otimes A \to B \otimes_R A$ the canonic epimorphism to the coequalizer.

PROPOSITION 3.6. *In a monoidal category with coequalizers* $\underline{V}$, *the above construction defines a bifunctor* $\otimes_R : \underline{MV}_R \times {}_R\underline{MV} \to \underline{V}_0$.

Proof. Evidently, $\otimes_R((B, \delta_B), (A, \gamma_A)) = B \otimes_R A$. If $f : (B, \delta_B) \to (B', \delta_{B'})$ is a morphism in $\underline{MV}_R$ and $g : (A, \gamma_A) \to (A', \gamma_{A'})$ is a morphism in ${}_R\underline{MV}$ then $f \otimes_R g : B \otimes_R A \to B' \otimes_R A'$ is the unique morphism of factorization through the coequalizers



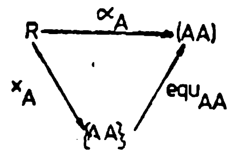
The functioriality is derived from the uniqueness. In order to prove the existence of the factorization morphism on the above diagram we must check $\text{coequ}_{B'A'} \cdot f \otimes g \cdot (\delta_B \otimes 1_A) \cdot a^{-1}_{BRA} = \text{coequ}_{B'A'} \cdot f \otimes g \cdot 1_B \otimes \gamma_A$. But this easily follows using the following commutative diagram



DEFINITION 3.1. *A monoid* (R, e, m) *over* $\underline{V}$ *is called commutative if* $m \cdot c_{RR} = m$.

In what follows we suppose the monoid (R,e,n,m) commutative.

LEMMA 3.7. *If* (R,e,m) *is a commutative monoid over* $\underline{V}$ *and* (A, α_A) *is a left R-module then* $\alpha_A : R \to (AA)$ *factors through* $\{AA\}$ *like in the following commutative diagram*



Proof. α_A being morphism of monoids over $\underline{V}$ we have

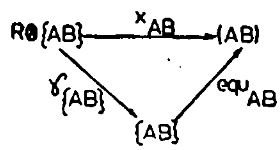
$$M^A_{AA} \cdot \alpha_A \otimes \alpha_A = \alpha_A \cdot m = \alpha_A \cdot m \cdot c_{RR} = M^A_{AA} \cdot \alpha_A \otimes \alpha_A \cdot c_{RR} =$$

$$= M^A_{AA} \cdot c_{(AA),(AA)} \cdot \alpha_A \otimes \alpha_A. \text{ Applying } \pi \text{ we get } (\alpha_A, 1) \cdot L^A_{AA} \cdot \alpha_A =$$

$$= (\alpha_A, 1) \cdot R^A_{AA} \cdot \alpha_A.$$

PROPOSITION 3.8. *If* (A, α_A) *and* (B, α_B) *are left R-modules, there is a morphism* $\gamma_{\{AB\}} : R \otimes \{AB\} \to \{AB\}$ *which gives* $\{AB\}$ *a structure of left R-module over* $\underline{V}$.

Proof. We consider the morphism $x_{AB} = M^B_{AB} \cdot \alpha_B \otimes \mathrm{equ}_{AB} : R \otimes \{AB\} \to (AB)$ From the previous lemma $\alpha_B = \mathrm{equ}_{BB} \cdot x_B$, so that x_{AB} factors through equ_{AB} using lemma 2.9. Hence a morphism $\gamma_{\{AB\}}$ exists and makes the following diagram commutative



Next, let us show that $(\{AB\}, \gamma_{\{AB\}})$ actually is a left R-module over $\underline{V}$. As usual we get equivalent conditions by left composition with equ_{AB}, namely $x_{AB} \cdot e \otimes 1 = \mathrm{equ}_{AB} \cdot l_{\{AB\}}$, $x_{AB} \cdot m \otimes 1 = x_{AB} \cdot 1 \otimes \gamma_{\{AB\}} \cdot a$. For the first, applying π and [II,(3.1)] we have $\pi(x_{AB}) \cdot e = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B \cdot e = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot j_B = (\mathrm{equ}_{AB}, 1) \cdot j_{(AB)} = (1, \mathrm{equ}_{AB}) \cdot j_{\{AB\}}$, also using [CCl], the naturality of j and the fact that B is a left R-module over $\underline{V}$.
For the second, we have

$$\pi\pi(x_{AB} \cdot m \otimes 1) = \pi(\pi(x_{AB}) \cdot m) = (1, \pi(x_{AB})) \cdot \pi(m) =$$

$$= (1, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BB}) \cdot (1, \alpha_B) \cdot \pi(m) = (1, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BB}) \cdot (\alpha_B, 1) \cdot L^B_{BB} \cdot \alpha_B =$$

$$= (1, (\mathrm{equ}_{AB}, 1)) \cdot (\alpha_B, 1) \cdot (L^A_{BB}, 1) \cdot L^{(AB)}_{(AB),(AB)} \cdot L^A_{BB} \cdot \alpha_B =$$

$$= (\alpha_B, 1) \cdot (L^A_{BB}, 1) \cdot ((\mathrm{equ}_{AB}, 1), 1) \cdot L^{\{AB\}}_{\{AB\},(AB)} \cdot L^A_{BB} \cdot \alpha_B =$$

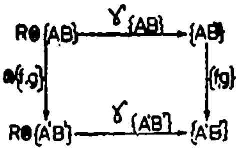
$$= (\pi(x_{AB}), 1) \cdot L^{\{AB\}}_{\{AB\},(AB)} \cdot L^A_{BB} \cdot \alpha_B = p \cdot (x_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B =$$

$$= p \cdot \pi(M^B_{AB} \cdot \alpha_B \otimes x_{AB}) = p \cdot \pi(x_{AB} \cdot 1 \otimes \gamma_{\{AB\}}) = \pi\pi(x_{AB} \cdot 1 \otimes \gamma_{\{AB\}} \cdot a).$$

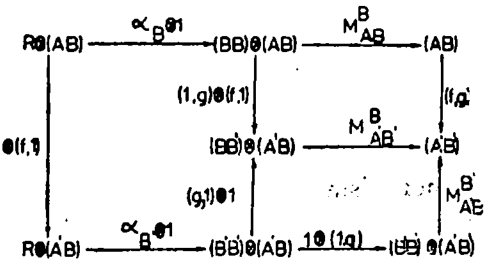
Remark. We must show that in the commutative case the left R-module structures defined on $\{RA\}$, in the lemma 3.2 and in the previous proposition are identical. Using $m \cdot c_{RR} = m$ and applying π^{-1} to the definition of equ_{RA}, one can show that $M^{A}_{RA} \cdot \alpha_A \otimes \mathrm{equ}_{RA} = M^{R}_{RA} \cdot c_{(RR),(RA)} \cdot n \otimes \mathrm{equ}_{RA}$, that is, $x_{RA} = y_{RA}$.

THEOREM 3.9. *Lemma 3.3 defines a bifunctor* $\{-,-\}: {}_R\underline{MV}^{op} \times {}_R\underline{MV} \to {}_R\underline{MV}$.

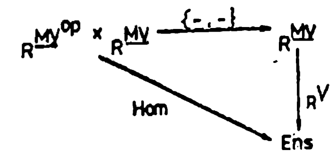
Proof. It only remains to prove that $\{f,g\}: (\{AB\}, \gamma_{\{AB\}}) \to (\{A'B'\}, \gamma_{\{A'B'\}})$ actually is a morphism of left R-modules over $\underline{V}$. The commutativity of the following diagram



follows from the equivalent equality $M^{B'}_{A'B'} \cdot \alpha_B \otimes (f,g) \cdot 1 \otimes \mathrm{equ}_{AB} = (f,g) \cdot M^{B}_{AB} \cdot \alpha_B \otimes 1 \cdot 1 \otimes \mathrm{equ}_{AB}$, which is true using the following commutative diagram



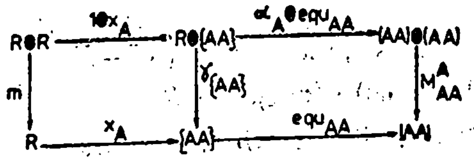
PROPOSITION 3.10. *If* ${}_RV = V \cdot W: {}_R\underline{MV} \to \mathrm{Ens}$ *and* V *preserves equalizers then the following diagram of functors is commutative*



Proof. Straightforward from the remark following lemma 2.8.

PROPOSITION 3.11. *For each left R-module* (A, α_A), *the morphism* $x_A: R \to \{AA\}$ *which appears in lemma 3.7 is a morphism of left R-modules. Further, the family* $\mathbf{j} = \mathbf{j}_{(A,\alpha_A)} = x_A$ *is natural in* ${}_R\underline{MV}$.

Proof. The following commutative diagram shows that x_A is a morphism of left R-modules



The naturality of $\mathbf{j}$ follows from $(1, f) \cdot \alpha_A = (f, 1) \cdot \alpha_{A'}$, true for a morphism of left R-modules $f:(A, \alpha_A) \to (A', \alpha_{A'})$, equ_{AA} being monomorphism.

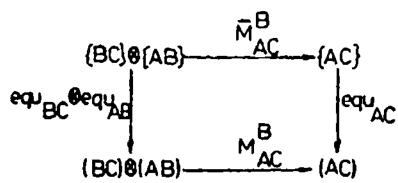
PROPOSITION 3.12. ${}_R V\ (\mathbf{i}_{\{AA\}})(1_A) = \mathbf{j}_A$.

Proof. By left composition with equ_{AA} one can show that we have to show at the subjacent level that $V(z_{(AA)})(1_A) = \alpha_A$. Finally, one reduces this to $V(u_{(AA),R})(1_A) = j_A \otimes 1_A \cdot l_R^{-1}$ using also the equality $z_{(AA)} = (1_R, \gamma_{(AA)} \cdot c_{(AA),R}) \cdot u_{(AA),R}$ and axiom [CC5].

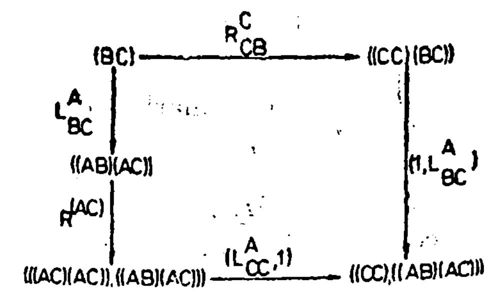
PROPOSITION 3.13. *For each left R-modules* (A, α_A), (B, α_B) *and* (C, α_C) *there is a transformation*

$$\mathbf{L} = \mathbf{L}^{(A,\alpha_A)}_{(B,\alpha_B),(C,\alpha_C)} : \{BC\} \to \{\{AB\}, \{AC\}\} \quad \textit{natural in } {}_R\underline{MV}.$$

Proof. Let us, first, mention the following generalization of lemma 2.9 $(\alpha_C, 1) \cdot R^C_{CA} \cdot M^B_{AC} \cdot \mathrm{equ}_{BC} \otimes \mathrm{equ}_{AB} = (\alpha_A, 1) \cdot L^A_{AC} \cdot M^B_{AC}\ \mathrm{equ}_{BC} \otimes \mathrm{equ}_{AB}$. Hence there is a morphism $\overline{M}^B_{AC}: \{BC\} \otimes \{AB\} \to \{AC\}$ which closes the commutative diagram



Using again π there is a morphism $\overline{L}^A_{BC}: \{BC\} \to (\{AB\}, \{AC\})$ for which $(\mathrm{equ}_{AB}, 1) \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} = (1, \mathrm{equ}_{AC}) \cdot \overline{L}^A_{BC}$ is true. If we want $\overline{L}^A_{BC}$ to give us by factorization a morphism $\mathbf{L}^A_{BC}: \{BC\} \to \{\{AB\}, \{AC\}\}$ we have to check the following equality $(\alpha_{\{AC\}}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \overline{L}^A_{BC} = (\alpha_{\{AB\}}, 1) \cdot L^{\{AB\}}_{\{AB\},\{AC\}} \cdot \overline{L}^A_{BC}$. In doing so, we may use the definitions of the left R-module structure on $\{AB\}$ and $\{AC\}$, i.e., $(1, \mathrm{equ}_{AB}) \cdot \alpha_{\{AB\}} = (\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B$ and the analogous changing B into C. Again, using the $\underline{V}$-functoriality of L^A, composing to the right with $c_{(BC),(CC)}$ and applying π, we get another analogous of [CC3]:



We shall prove the required equality composing to the left with $(1,(1,\mathrm{equ}_{AC}))$ (which still is a monomorphism, the functors $(X, -)$ being monofunctors). Indeed

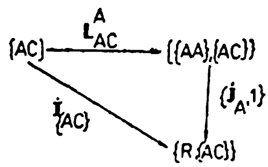
$$(1,(1, \mathrm{equ}_{AC})) \cdot (\alpha_{\{AC\}}, 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \overline{L}^A_{BC} =$$
$$= (\alpha_{\{AC\}}, 1) \cdot ((1, \mathrm{equ}_{AC}), 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \overline{L}^A_{BC} =$$

$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot ((\mathrm{equ}_{AC}, 1), 1) \cdot R^{\{AC\}}_{\{AC\},\{AB\}} \cdot \bar{L}^A_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot R^{(AC)}_{(AC),\{AB\}} \cdot (1, \mathrm{equ}_{AC}) \cdot \bar{L}^A_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot R^{(AC)}_{(AC),\{AB\}} \cdot (\mathrm{equ}_{AB}, 1) \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_C, 1) \cdot (L^A_{CC}, 1) \cdot (1, (\mathrm{equ}_{AB}, 1)) \cdot R^{(AC)}_{(AC)\cdot(AB)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_C, (\mathrm{equ}_{AB}, 1)) \cdot (1, L^A_{BC}) \cdot R^C_{CB} \cdot \mathrm{equ}_{BC} =$$
$$= (1, (\mathrm{equ}_{AB}, 1) \cdot L^A_{BC}) \cdot (\alpha_B, 1) \cdot L^B_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_B, (\mathrm{equ}_{AB}, 1)) \cdot (L^A_{BB}, 1) \cdot L^{(AB)}_{(AB),(AB)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= ((\mathrm{equ}_{AB}, 1) \cdot L^A_{BB} \cdot \alpha_B, 1) \cdot L^{\{AB\}}_{(AB),(AC)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_{\{AB\}}, 1) \cdot ((1, \mathrm{equ}_{AB}), 1) \cdot L^{\{AB\}}_{(AB),(AC)} \cdot L^A_{BC} \cdot \mathrm{equ}_{BC} =$$
$$= (\alpha_{\{AB\}}, 1) \cdot (1, (1, \mathrm{equ}_{AC})) \cdot L^{\{AB\}}_{\{AB\}\,\{AC\}} \cdot \bar{L}^A_{BC} =$$
$$= (1,(1, \mathrm{equ}_{AC})) \cdot (\alpha_{\{AB\}}, 1) \cdot L^{\{AB\}}_{\{AB\},\{AC\}} \cdot \bar{L}^A_{BC}.$$

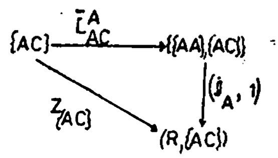
Thus, there is a morphism $\mathbf{L}^A_{BC}\colon \{BC\} \to \{\{AB\}, \{AC\}\}$ in $\underline{V}_0$. The proof of the naturality in ${}_R\underline{MV}$ of the corresponding family is left to the reader.

THEOREM 3.14. *If $\underline{V}$ is a symmetric monoidal closed category with equalizers, (R,e,n,m) is a commutative monoid over $\underline{V}$ and the subjacency functor $V\colon \underline{V}_0 \to$ $\to$ Ens preserves equalizers, then ${}_R\underline{MV}$, the category of the left R-modules over $\underline{V}$, is a closed category.*

Proof. Using theorem 3.4, the remark following proposition 3.8, theorem 3.9, propositions 3.10, 3.11, 3.12, 3.13, all data for the closed structure of ${}_R\underline{MV}$ are constructed, the „unit" object being obviously (R,n,m) as a R-module over (R,e,n,m). Proposition 3.12 is axiom [CC5] for ${}_R\underline{MV}$. Hence, one has only to verify the remaining axioms [CC1—4] for ${}_R\underline{MV}$. In what follows we shall prove, for instance, axiom [CC2]. We have to check the commutativity of the following diagram



By left composition with $\mathrm{equ}_{R,\{AC\}}$ this is reduced to the following commutativity



A new composition with $(1, \mathrm{equ}_{AC})$ gives us the required proof:

$$(1, \mathrm{equ}_{AC}) \cdot z_{\{AC\}} = (1, \mathrm{equ}_{AC}) \cdot \pi(\gamma_{\{AC\}} \cdot c_{\{AC\},R}) =$$

$$= \pi(\mathrm{equ}_{AC} \cdot \gamma_{\{AC\}} \cdot c_{\{AC\},R}) = \pi(x_{AC} \cdot c_{\{AC\},R}) = \pi(M^C_{AC} \cdot \alpha_C \otimes \mathrm{equ}_{AC} \cdot c_{\{AC\},R}) =$$

$$= \pi(M^C_{AC} \cdot c_{(AC),(CC)} \cdot \mathrm{equ}_{AC} \otimes \alpha_C) = (\alpha_C, 1) \cdot R^C_{CA} \cdot \mathrm{equ}_{AC} =$$

$$= (\alpha_A, 1) \cdot L^A_{AC} \cdot \mathrm{equ}_{AC} = (\mathbf{j}_A, 1) \cdot (\mathrm{equ}_{AA}, 1) \cdot L^A_{AC} \cdot \mathrm{equ}_{AC} =$$

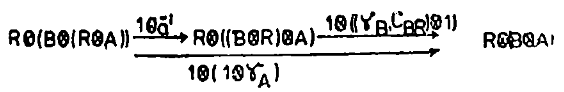
$$= (\mathbf{j}_A, 1) \cdot (1, \mathrm{equ}_{AC}) \cdot \overline{L}^A_{AC} = (1, \mathrm{equ}_{AC}) \cdot (\mathbf{j}_A, 1) \cdot \overline{L}^A_{AC}.$$

Remark. The astute reader has certainly noticed that we constantly use the following fact: the functors $(X, -) : \underline{V}_0 \to \underline{V}_0$ having left adjoints, namely $- \otimes X : \underline{V}_0 \to \underline{V}_0$, preserve limits (equalizers) and monomorphisms.

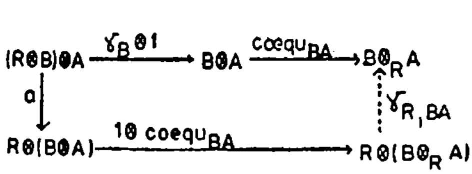
— Let us return now to the monoidal structure of ${}_R\underline{MV}$.

THEOREM 3.15. *For a commutative monoid (R, e, m), if the functor $R\otimes - : \underline{V}_0 \to \underline{V}_0$ preserves coequalizers, proposition 3.6 defines a bifunctor $\otimes_R : {}_R\underline{MV} \times {}_R\mathrm{MV} \to {}_R\underline{MV}$.*

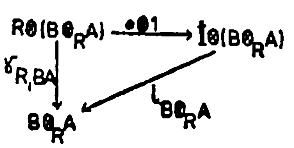
Proof. We first mention that, the basic monoid being commutative, if (A, δ_A) is a right R-module then $(A, \delta_A \cdot c_{RA})$ is a left R-module. Hence, for two left R-modules we shall define $B\otimes_R A = \mathrm{coequ}(((\gamma_B \cdot c_{BR})\otimes 1_A \cdot a^{-1}_{BRA}, 1_B \otimes \gamma_A) : B\otimes(R\otimes A) \to B\otimes A)$. In order to get a left R-module structure on $B\otimes_R A$ we prove that the morphism $\bar{x}_{BA} : R\otimes(B\otimes A) \xrightarrow{a^{-1}} (R\otimes B)\otimes A \xrightarrow{\gamma_B\otimes 1} B\otimes A \xrightarrow{\mathrm{coequ}_{BA}} B\otimes_R A$ coequalizes the following pair

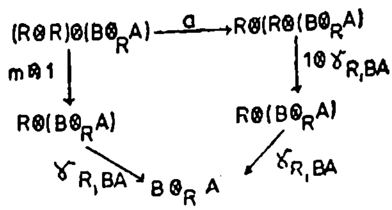


The functor $R\otimes -$ preserving coequalizers, this will prove the existence of a morphism $\gamma_{R,BA} : R\otimes(B\otimes_R A) \to B\otimes_R A$ in V_0 which will provide the left R-module structure on $B\otimes_R A$. We shall avoid this verification which only uses definitions and coherence. So, our $\gamma_{R,BA}$ closes the following commutative diagram



Next, we have to prove that $(B\otimes_R A, \gamma_{R,BA})$ actually is a left R-module, that is, the commutativity of the following diagrams





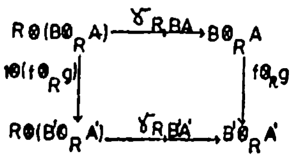
As for the first, we can check the equivalent one obtained by right composition with $1_I \otimes \mathrm{coequ}_{BA}$(this being epimorphism)

$$\gamma_{R,BA} \cdot e\otimes 1 \cdot 1\otimes \mathrm{coequ}_{BA} = \gamma_{R,BA} \cdot 1\otimes \mathrm{coequ}_{BA} \cdot e\otimes 1 =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot a^{-1} \cdot e\otimes 1 = \mathrm{coequ}_{BA}\cdot\gamma_B\otimes 1\cdot(e\otimes 1)\otimes 1 \cdot a^{-1} =$$
$$= \mathrm{coequ}_{BA} \cdot l_B\otimes 1 \cdot a^{-1} = \mathrm{coequ}_{BA} \cdot l_{A\otimes B} = l_{B\otimes_R A} \cdot 1\otimes \mathrm{coequ}_{BA}.$$

As for the second, a right composition with $1_{R\otimes R}\otimes \mathrm{coequ}_{BA}$ gives us the required proof

$$\gamma_{R,BA} \cdot m\otimes 1 \cdot 1\otimes \mathrm{coequ}_{BA} = \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot a^{-1} \cdot m\otimes 1 =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot (1\otimes\gamma_B)\otimes 1 \cdot a\otimes 1 \cdot a^{-1} =$$
$$= \mathrm{coequ}_{BA} \cdot \gamma_B\otimes 1 \cdot a^{-1} \cdot 1\otimes(\gamma_B\otimes 1) \cdot 1\otimes a^{-1} \cdot a =$$
$$= \gamma_{R,BA} \cdot 1\otimes \mathrm{coequ}_{BA} \cdot 1\otimes(\gamma_B\otimes 1) \cdot 1\otimes a^{-1} \cdot a =$$
$$= \gamma_{R,BA} \cdot 1\otimes\gamma_{R,BA} \cdot 1\otimes(1\otimes \mathrm{coequ}_{BA}) \cdot a = \gamma_{R,BA} \cdot 1\otimes\gamma_{R,BA} \cdot a\cdot 1\otimes \mathrm{coequ}_{BA}.$$

Finally, one easily checks that, using notations from proposition 3.6, $f\otimes_R g$ actually is a morphism of left R-modules over $\underline{V}$, i.e., the following diagram commutes



Remark. If we suppose that $\underline{V}_0$ is abelian, $\otimes$ preserves cokernels in both variables and we take cokernels instead of coequalizers we recover the similar MacLane's result.

— Moreover, the following result is true

THEOREM 3.16. *If $\underline{V}$ is a symmetric monoidal category with coequalizers, (R, e, m) is a commutative monoid over $\underline{V}$ and $R\otimes -$ preserves coequalizers, then ${}_R\underline{MV}$ is a symmetric monoidal category.*

Proof. Simple generalization of MacLane's result. For instance, if (A, γ_A) is an object in ${}_R\underline{MV}$, we have from the definition 1.9 $\gamma_A \cdot m\otimes 1 = \gamma_A \cdot 1\otimes\gamma_A\cdot a$; that is, γ_A coequalizes the pair $((m \cdot c_{RR})\otimes 1_A \cdot a^{-1}_{RRA},\ 1_R\otimes\gamma)$. Thus, γ_A factors through coequ_{RA} giving one of our natural isomorphisms $l_A\colon R\otimes_R A \to A$.

We shall end our paper with the principal result which uses all the results obtained above

THEOREM 3.17. *If $\underline{V}$ is a symmetric monoidal closed category, $\underline{V}_0$ has equalizers and coequalizers, V preserves equalizers, (R, e, n, m) is a commutative monoid over $\underline{V}$ and $R\otimes -$ preserves coequalizers, then ${}_R\underline{MV}$, the category of the left R--modules over $\underline{V}$, is a symmetric monoidal closed category.*

Proof. First we must find in $\underline{V}_0$ morphisms $\mathbf{p}_{BAC}: \{B\otimes_R A, C\} \to \{B,\{A,C\}\}$ and prove that this is a natural family of isomorphisms in ${}_R\underline{MV}$. Previously, we prove the existence of morphisms $\bar{p}_{BAC}$ which close commutatively the following diagram

$$\begin{array}{ccc}
\{B\otimes_R A, C\} & \overset{\bar{p}_{BAC}}{\dashrightarrow} & (B,\{AC\}) \\
\Big\downarrow{\scriptstyle \mathrm{equ}_{B\otimes_R A C}} & & \Big\downarrow{\scriptstyle (1,\mathrm{equ}_{AC})} \\
(B\otimes_R A C) & \xrightarrow{(\mathrm{coequ}_{BA},1)} (B\otimes A, C) \xrightarrow{p_{BAC}} & (B,(AC))
\end{array}$$

Because $(B, -)$ preserves equalizers it will suffice to check

$$(1, (\alpha_C, 1)) \cdot (1, R^C_{CA}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{B\otimes_R A, C} =$$

$$= (1,(\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{B\otimes_R A, C}$$

This verification needs the following facts:
(i) denoting by $X = B\otimes_R A$, and applying π to a convenient diagram which expresses the $\underline{V}$-functoriality of R^C, one has $(1, R^C_{CA})\cdot L^A_{XX} = (R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(CC)}\cdot R^C_{CX}$.
(ii) from [III, (4.4)] applying π, we can find the equality

$$(R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(XX)} \cdot L^X_{XC} = (R^X_{XA}, 1)\cdot L^{(XX)}_{(AX),(AC)} \cdot L^A_{XC}$$

(iii) the definition of coequ_{BA} gives by a double application of π, the equality $(1, \pi(\mathrm{coequ}_{BA})) \cdot \pi(\gamma_B \cdot c_{BR}) = (\alpha_A, 1)\cdot L^A_{AX} \cdot \pi(\mathrm{coequ}_{BA})$
(iv) we have $(\alpha_X, 1) \cdot R^X_{XA}\cdot \pi(\mathrm{coequ}_{BA}) = (1, \pi(\mathrm{coequ}_{BA})) \cdot \pi(\gamma_B\cdot c_{BR}) = \pi(\pi(\mathrm{coequ}_{BA}) \cdot \gamma_B\cdot c_{BR})$; this follows also using the result of the forthcoming (vii)

Having this in mind the proof goes like this:

$$(1,(\alpha_C, 1))\cdot(1,R^C_{CA})\cdot\bar{p}_{BAC}\cdot(\mathrm{coequ}_{BA}, 1)\cdot\mathrm{equ}_{XC} =$$

$$= (1,(\alpha_C, 1)) \cdot (1, R^C_{CA}) \cdot (\pi(\mathrm{coequ}_{BA}),1) \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(i)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1,(\alpha_C, 1)) \cdot (R^C_{XA}, 1)\cdot R^{(AC)}_{(XC),(CC)}\cdot R^C_{CX}\cdot\mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1)\cdot (R^C_{XA}, 1) \cdot R^{(AC)}_{(XC),R}\cdot (\alpha_C, 1) \cdot R^C_{CX} \cdot \mathrm{equ}_{XC} = \qquad \text{(ii)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (R^C_{XA}, 1) \cdot R^{(AC)}_{(XC),R}\cdot (\alpha_X, 1) \cdot L^X_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1, (\alpha_X, 1)) \cdot (R^X_{XA}, 1) \cdot L^{(XX)}_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(iv)}$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (R^X_{XA}, 1) \cdot ((\alpha_X, 1), 1) \cdot L^R_{(AX),(AC)}\cdot L^A_{XC} \cdot \mathrm{equ}_{XC} = \qquad \text{(iii)}$$

$$= (\pi(\gamma_B \cdot c_{BR}), 1) \cdot ((1, \pi(\mathrm{coequ}_{BA})), 1)\cdot L^R_{(AX),(AC)}\cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

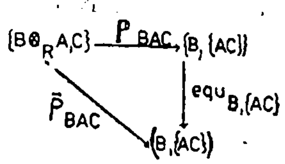
$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (L^A_{AX}, 1) \cdot ((\alpha_A,1),1) \cdot L^R_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (\pi(\mathrm{coequ}_{BA}), 1) \cdot (1, (\alpha_A, 1)) \cdot (L^A_{AX}, 1) \cdot L^{(AA)}_{(AX),(AC)} \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (1,(\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot (\pi(\mathrm{coequ}_{BA}), 1) \cdot L^A_{XC} \cdot \mathrm{equ}_{XC} =$$

$$= (1, (\alpha_A, 1)) \cdot (1, L^A_{AC}) \cdot \bar{p}_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC}.$$

Now, we must show the existence of morphisms $\mathbf{p}_{BAC}$ which commutatively close the diagram



and these will be the required isomorphisms. We have to check the following equality $(\alpha_{\{AC\}},1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot \bar{p}_{BAC}=(\alpha_B,1)\cdot L^{B}_{B,\{AC\}}\cdot \bar{p}_{BAC}$. Again, we need some preliminary results

(v) from proposition 3.13 we take the equality

$$(1,\mathrm{equ}_{AC})\cdot \alpha_{\{AC\}}=(\mathrm{equ}_{AC},1)\cdot L^{A}_{CC}\cdot \alpha_C$$

(vi) the following equality holds

$$(L^{A}_{CC},1)\cdot R^{\{AC\}}_{(AC),B}\cdot p_{BAC}=(1,p_{BAC})\cdot R^{C}_{C,B\otimes A};$$

indeed, this follows applying π to axiom $MCC3$, composing to the left with $c_{(B\otimes A,C),(CC)}$ and applying π^{-1}.

(vii) by a double application of π to the definition of $\gamma_X=\gamma_{R,BA}$ we get $p\cdot(\mathrm{coequ}_{BA},1)\cdot\alpha_X=(1,\pi(\mathrm{coequ}_{BA}))\cdot\alpha_B$

So, the following „enriched diagram chasing" proves the required equality

$$(1,(1,\mathrm{equ}_{AC}))\cdot(\alpha_{\{AC\}},1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}=$$

$$=(\alpha_{\{AC\}},1)\cdot((1,\mathrm{equ}_{AC}),1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}= \qquad \text{(v)}$$

$$=(\alpha_C,1)\cdot(L^{A}_{CC},1)\cdot((\mathrm{equ}_{AC},1),1)\cdot R^{\{AC\}}_{\{AC\},B}\cdot\bar{p}_{BAC}=$$

$$=(\alpha_C,1)\cdot(L^{A}_{CC},1)\cdot R^{(AC)}_{(AC),B}\cdot(1,\mathrm{equ}_{AC})\cdot\bar{p}_{BAC}=$$

$$=(\alpha_C,1)\cdot(L^{A}_{CC},1)\cdot R^{(AC)}_{(AC),B}\cdot p_{BAC}\cdot(\mathrm{coequ}_{BA},1)\cdot\mathrm{equ}_{XC}= \qquad \text{(vi)}$$

$$=(\alpha_C,1)\cdot(1,p_{BAC})\cdot R^{C}_{C,B\otimes A}\cdot(\mathrm{coequ}_{BA},1)\cdot\mathrm{equ}_{XC}=$$

$$=(\alpha_C,1)\cdot(1,p_{BAC})\cdot(1,(\mathrm{coequ}_{BA},1))\cdot R^{C}_{CX}\cdot\mathrm{equ}_{XC}=$$

$$=(1,p_{BAC})\cdot(1,(\mathrm{coequ}_{BA}))\cdot(\alpha_X,1)\cdot L^{X}_{XC}\cdot\mathrm{equ}_{XC}=$$

$$=(\alpha_X,1)\cdot((\mathrm{coequ}_{BA},1),1)\cdot(1,p_{BAC})\cdot L^{B\otimes A}_{XC}\cdot\mathrm{equ}_{XC}= \qquad MCC3$$

$$=(\alpha_X,1)\cdot((\mathrm{coequ}_{BA},1),1)\cdot(p_{BAC},1)\cdot L^{B}_{(AX),(AC)}\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}= \qquad \text{(vii)}$$

$$=(\alpha_B,1)\cdot((1,\pi(\mathrm{coequ}_{BA})),1)\cdot L^{B}_{(AC),(AC)}\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}=$$

$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot(\pi(\mathrm{coequ}_{BA}),1)\cdot L^{A}_{XC}\cdot\mathrm{equ}_{XC}=$$

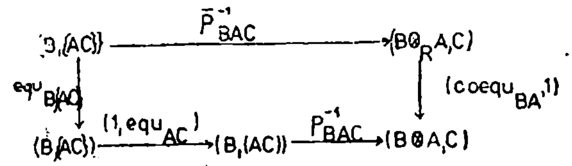
$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot p_{BAC}\cdot(\mathrm{coequ}_{BA},1)\cdot\mathrm{equ}_{XC}=$$

$$=(\alpha_B,1)\cdot L^{B}_{B,(AC)}\cdot(1,\mathrm{equ}_{AC})\cdot\bar{p}_{BAC}=(\alpha_B,1)\cdot(1,(1,\mathrm{equ}_{AC}))\cdot L^{B}_{B,\{AC\}}\cdot p_{BAC}=$$

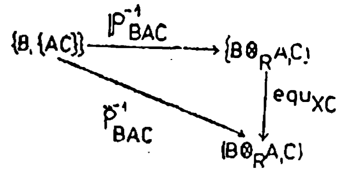
$$=(1,(1,\mathrm{equ}_{AC}))\cdot(\alpha_B,1)\cdot L^{B}_{B,\{AC\}}\cdot\bar{p}_{BAC}$$

We now have, for each three left R-modules (A, α_A, γ_A), (B, α_B, γ_B), (C, α_C, γ_C), a morphism $\mathbf{p}_{BAC}: \{B \otimes A, C\} \to \{B, \{AC\}\}$.

Analogously, we can determine a family of morphisms $\bar{p}^{-1}_{BAC}$ which close the commutative diagrams



and next, a family $\mathbf{p}^{-1}_{BAC}$ which close the commutative triangle



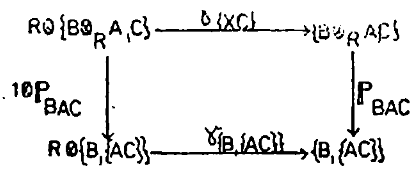
Finally, the following three facts must be checked .

(a) $\mathbf{p}_{BAC}$ and $\mathbf{p}^{-1}_{BAC}$ actually are morphisms of left R-modules over $\underline{V}$;

(b) $\mathbf{p}_{BAC}$ and $\mathbf{p}^{-1}_{BAC}$ are mutually inverse;

(c) the family $\mathbf{p} = \mathbf{p}_{BAC}$ is natural in ${}_R\underline{MV}$.

As for (a), we show for instance that $\mathbf{p}_{BAC}$ actually is a morphism of left R-modules over $\underline{V}$. The commutativity of the diagram



reduces by left composition with $\mathrm{equ}_{B,\{AC\}}$ to $M^{\{AC\}}_{B,\{AC\}} \cdot \alpha_{\{AC\}} \otimes \bar{p}_{BAC} = \bar{p}_{BAC} \cdot \gamma_{\{B \otimes_R A, C\}}$ or, applying π, to $(\bar{p}_{BAC}, 1) \cdot L^{B}_{\{AC\},\{AC\}} \cdot \alpha_{\{AC\}} = (1, \bar{p}_{BAC}) \cdot \alpha_{\{B \otimes_R A, C\}}$, equality which one can verify by left composition with $(1,(1,\mathrm{equ}_{AC}))$ by a new „enriched diagram chasing".

For (b) we choose $\mathbf{p} \cdot \mathbf{p}^{-1} = 1$; indeed we show that

$$(1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} = (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}}.$$

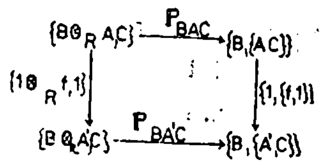
We have

$$(1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} = (1, \mathrm{equ}_{AC}) \cdot \bar{p}_{BAC} \cdot \mathbf{p}^{-1}_{BAC} =$$

$$= p_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} \cdot \mathbf{p}^{-1}_{BAC} = p_{BAC}\,(\mathrm{coequ}_{BA}, 1) \cdot \bar{p}^{-1}_{BAC} =$$

$$= p_{BAC} \cdot p^{-1}_{BAC} \cdot (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}} = (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B,\{AC\}}.$$

For (c) we choose the naturality of $\mathbf{p}_{BAC}$ in (A, α_A, γ_A), that is, the commutativity of the following diagram



Again, we prove an equivalent equality

$$(1, \mathrm{equ}_{A'C}) \cdot \mathrm{equ}_{B,\{AC\}} \cdot \mathbf{p}_{BA'C} \cdot \{1 \otimes_R f, 1\} = (1, \mathrm{equ}_{A'C}) \cdot \bar{p}_{BA'C} \cdot \{1 \otimes_R f, 1\} =$$
$$= p_{BA'C} \cdot (\mathrm{coequ}_{BA'}, 1) \cdot \mathrm{equ}_{B \otimes_R A', C} \cdot \{1 \otimes_R f, 1\} =$$
$$= p_{BA'C} \cdot (\mathrm{coequ}_{BA'}, 1) \cdot (1 \otimes_R f, 1) \cdot \mathrm{equ}_{XC} =$$
$$= p_{BA'C} \cdot (1 \otimes f, 1) \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} = (1, (f, 1)) \cdot p_{BAC} \cdot (\mathrm{coequ}_{BA}, 1) \cdot \mathrm{equ}_{XC} =$$
$$= (1, (f, 1)) \cdot (1, \mathrm{equ}_{AC}) \cdot \mathrm{equ}_{B, \{AC\}} \cdot \mathbf{p}_{BAC} =$$
$$= (1, \mathrm{equ}_{A'C}) \cdot (1, \{f, 1\}) \cdot \mathrm{equ}_{B, \{AC\}} \cdot \mathbf{p}_{BAC} =$$
$$= (1, \mathrm{equ}_{A'C}) \cdot \mathrm{equ}_{B, \{A'C\}} \cdot \{1, \{f, 1\}\} \cdot \mathbf{p}_{BAC}$$

In this way all the symmetric monoidal closed structure for $_R\underline{MV}$ is established. One can complete the proof of our theorem verifying axioms [*MCC*2], [*MCC*3], [*MCC*3′] and [*MCC*4].

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ASUPRA UNEI TEORII ÎMBOGĂŢITE A MODULELOR (II)

(Rezumat)

Utilizînd noţiunile preliminare studiate în partea întîia a aceluiaşi articol, autorul stabileşte rezultatele principale privitoare la partea închisă şi monoidal închisă a teoriei modulelor peste un monoid fixat, rezultate care conduc în final la teorema: dacă $\underline{V}$ este o categorie simetric monoidal închisă, categoria subiacentă V_0 are egalizatori şi coegalizatori, R este un monoid comutativ peste $\underline{V}$, functorul de subiacenţă V păstrează egalizatori şi $R \otimes$ — păstrează coegalizatori, atunci categoria modulelor $_R\underline{MV}$ este simetric monoidal închisă.