STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXIV. 2, 1979

ON AN ENRICHED THEORY OF MODULES (I)

GRIGORE CĂLUGĂREANU

0. Introduction. This paper uses entirely the terminology from the Eilenberg-Kelly's ample monography [3]. Various results from this monography will also be used. Therefore, in order to simplify the references, we shall denote by [III, 2.4] (square brackets) the theorem (lemma, proposition or corollary) 2.4. from the chapter III of [3].

In the third chapter of the well-known Mac Lane's paper *Categorical Algebra* [4], the author is bounded by a symmetric monoidal category $\underline{V}$ (a category with a multiplication) adding some restrictively enough conditions: V_0 is an abelian category and the bifunctor $\otimes : V_0 \times V_0 \to V_0$ is additive and right exact in each argument separately. In such a category (called tensored) he defines the notion of $\underline{V}$-algebra (in the present paper this notion will be called a monoidal monoid over $\underline{V}$) in a natural way (such as this is done by Bénabou [1]), showing that $\underline{V}$-algebras also form a symmetric monoidal category, the tensor product of two $\underline{V}$-algebras being essentially determined by the „middle four interchange" isomorphism.

Further on, for a fixed $\underline{V}$-algebra R, he defines the notion of left R-module and the corresponding notion of morphism. The results obtained are the following: if $\underline{V}$ is a tensored category, so is the category of modules over a commutative $\underline{V}$-algebra R, and, if $\underline{V}$ is a tensored category and R a $\underline{V}$-algebra, the functor $F : V_0 \to {}_RMV$ with $F(A) = R \otimes A$ is an adjoint to the forgetfull functor $G : {}_RMV \to V_0$; both functors are additive, G is exact and F is right exact.

Stimulated by the excellent monograph [3], the author of the present paper works out the closed and monoidal closed part of the theory of modules over a fixed monoid, theory, which for MacLane only worked out the monoidal part. Secondly, the author elaborates this theory in considerably weaker assumptions, assumptions present in almost all concrete categories (the abelianity condition evidently being not of this kind).

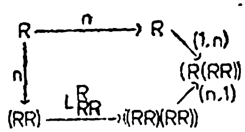
In the first section, the basic notions: closed and monoidal monoids and the corresponding morphisms, left and right R-modules over a fixed closed (monoidal) monoid R, are defined, and the basic situations: in a monoidal closed category, closed and monoidal monoids may be identified, etc., are studied.

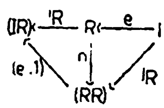
In the second section we successively prove the "enriched" versions of the following classical results: every ring can be viewed as a category with a single object, the category of modules can be identified with a suitable category of functors from the category with a single object mentioned above, the category of modules inherits properties, such as completeness, from the category Ab. Assuming that the basic category V_0 has equalizers, the monoid of the biendomorphisms is constructed and the classical results about it, proved.

In the third and final section we prove several results that are leading us to the following result: if $\underline{V}$ is a symmetric monoidal closed category with equalizers and the functor of subjacency V preserves equalizers, then the category ${}_RMV$ of the left modules over a commutative monoid R, is closed.

After recovering , in much more weaker conditions, the results of MacLane concerning the monoidal structure of ${}_RMV$, we prove our principal result: *if $\underline{V}$ is a symmetric monoidal closed category, V_0 has equalizers and coequalizers, R is a commutative monoid over $\underline{V}$, V preserves equalizers and $R \otimes -$ preserves coequalizers, then ${}_RMV$ is a symmetric monoidal closed category.*

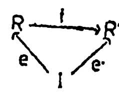
1. **Basic notions and results.** DEFINITION 1.1. Let $\underline{V}$ be a closed category. A *closed monoid* (R, e, n) *over* $\underline{V}$ consists of the following data: an object R in V_0 and two morphisms $e: I \to R$, $n: R \to (RR)$ in V_0. These data are to satisfy the following axioms:

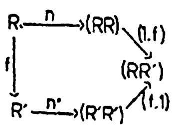
CM 1 

CM 2 

are commutative diagrams.

DEFINITION 1.2. If (R, e, n) and (R', e', n') are two closed monoids over $\underline{V}$, a *morphism* $f: (R, e, n) \to (R', e', n')$ *of closed monoids* is a morphism $f: R \to R'$ in V_0 which satisfies the following axioms:

MCM 1 

*MCM*2 

are commutative diagrams.

PROPOSITION 1.1. *For each object A from V_0, $((AA), j_A, L^A_{AA})$ is a closed monoid.*

Proof. Straightforward from axioms [CC1—3] for the closed structure of $\underline{V}$.

We shall denote by $MonV$ the category of the closed monoids over $\underline{V}$ and of the morphisms of closed monoids.

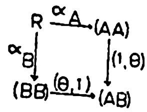
Remark. A closed structure on $MonV$ seems to be difficult to be found. For this purpose one could expect to use ends.

— Let (R, e, n) be a closed monoid over $\underline{V}$.

DEFINITION 1.3. An object A in V_0 together with a morphism $\alpha_A: (R, e, n) \to ((AA), j_A, L^A_{AA})$ of closed monoids is called a *closed left R-module over* $\underline{V}$.

Definition 1.4. If (A, α_A) and (B, α_B) are two left R-modules over $\underline{V}$, $\theta : (A, \alpha_A) \to (B, \alpha_B)$ is a *morphism of closed left R-modules* if $\theta : A \to B$ is a morphism in V_0 which satisfies the axiom

$CR1$



We shall denote by ${}^c_R MV$ the category of the closed left R-modules with the morphisms of closed left R-modules.

Proposition 1.2. *R admits a canonic structure of closed left R-module ovcr $\underline{V}$.*

Proof. Obviously, $n : (R, e, n) \to ((RR), j_R, L^R_{RR})$ is a morphism of closed monoids.

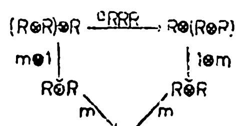
Proposition 1.3. *Each object A in V_0 has a canonic structure of (AA)-module, over the closed monoid defined in proposition 1.1.*

Proof. The identity of (AA) gives the required structure.

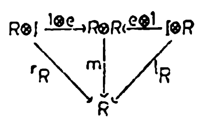
— Let $\underline{V}$ be a monoidal category.

Definition 1.5. (B é n a b o u) A *monoidal monoid* (R, e, m) *over* $\underline{V}$ consists of the following data: an object R in V_0 and two morphisms $e : I \to R$, $m : R \otimes R \to R$ in V_0. These data are to satisfy the following axioms

$MM1$



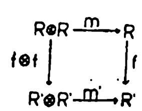
$MM2$



are commutative diagrams.

Definition 1.6. If (R, e, m) and (R', e', m') are two monoidal monoids over $\underline{V}$, a *morphism* $f : (R, e, m) \to (R', e', m')$ *of monoidal monoids* is a morphism $f : R \to R'$ in V_0 which satisfies the axiom $MMM1 = MCM1$ and the axiom

$MMM2$



Proposition 1.4. *$((AA), j_A, M^A_{AA})$ is a monoidal monoid over $\underline{V}$, for each object A of V_0.*

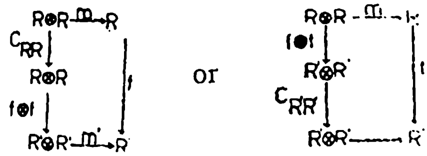
Proof. Obvious by axioms $[VC1']$, $[VC2']$ and $[VC3']$ for the special case $\underline{A} = \underline{V}$.

We shall denote by ${}^{m}MonV$ the category of the monoidal monoids over $\underline{V}$ and of the morphisms of monoidal monoids.

PROPOSITION 1.5 (MacLane) — *If $\underline{V}$ is a symmetric monoidal category then so is ${}^{m}MonV$.*

We refer to [4] for the proof of this result.

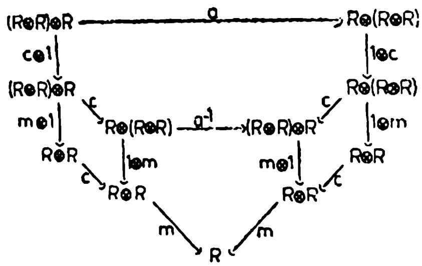
DEFINITION 1.7. Let $\underline{V}$ be a symmetric monoidal category. An *antimorphism $f: (R, e, m) \to (R', e', m')$ of monoidal monoids* is a morphism $f: R \to R'$ in V_0 which satisfies the axiom $AMM1 = MMM1 = MCM1$ and the axiom

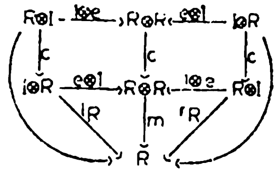
$AMM2$ 

these conditions being equivalent by the naturality of the isomorphisms c

PROPOSITION 1.6. *If (R, e, m) is a monoidal monoid over $\underline{V}$, then $(R, e, m.c_{RR})$ is also a monoidal monoid over $\underline{V}$.*

Proof. The commutativity of the following diagrams





follows by coherence, naturality of the isomorphisms c and by axioms $MM1$ and $MM2$ for the monoidal monoid (R, c, m).

DEFINITION 1.8. If (R, e, m) is a monoidal monoid over $\underline{V}$, the monoidal monoid defined above will be called *the opposite monoid of (R, e, m).*

Obviously we then have

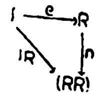
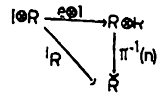
COROLLARY 1.7. *$f: (R, e, m) \to (R', e', m')$ is an antimorphism of monoidal monoids iff f is a morphism from the opposite monoid of (R, e, m) to (R', e', m') or iff f is a morphism of monoidal monoids from (R, e, m) to the opposite monoid of (R', e', m').*

As for the following, we shall consider a (symmetric) monoidal closed category $\underline{V}$. We shall make full use of the bijections

$$\pi_{ABC} : V_0(A \otimes B, C) \to V_0(A(BC)).$$

PROPOSITION 1.8. *If (R, e, n) is a closed monoid over $\underline{V}$ then $(R, e, \pi^{-1}_{RRR}(n))$ is a monoidal monoid over $\underline{V}$. Conversely, if (R, e, m) is a monoidal monoid over $\underline{V}$, then $(R, e, \pi_{RRR}(m))$ is a closed monoid over $\underline{V}$.*

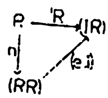
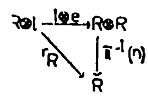
Proof. We shall show that

(a)  implies 

π_{IRR} being a bijection, the commutativity of the right triangle is equivalent to $\pi_{IRR}(l_R) = \pi_{IRR}(\pi^{-1}(n) \cdot e \otimes 1)$. In proving the last equality we shall use equalities from [chap. II]. Indeed

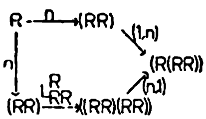
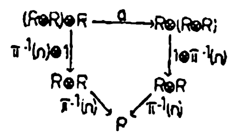
$$\pi(\pi^{-1}(n) \cdot e \otimes 1)^{[(3.1)]} = \pi(\pi^{-1}(n) \cdot e) = n \cdot e = j_R^{[(3.15)]} = \pi(l_R).$$

Next, we have to show that

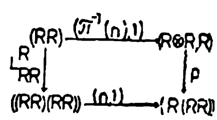
(b)  implies 

Indeed, $\pi(\pi^{-1}(n) \cdot 1 \otimes e)^{[(3.1)]} = (e, 1) \cdot \pi(\pi^{-1}(n)) = (e, 1) \cdot n = i_R^{[(3.17)]} = \pi(r_R)$.

Similarly,

(c)  implies 

In this case, we shall prove that $\pi_{R,R,(RR)}\pi_{R\otimes R,R,R}$ applied to the right diagram is commutative. We shall use [(3.22)] in the following form



Indeed, $\pi(\pi(\pi^{-1}(n) \cdot 1 \otimes \pi^{-1}(n) \cdot a))_{[(3.19)]} = p \cdot \pi(\pi^{-1}(n) \cdot 1 \otimes \pi^{-1}(n))_{[(3.1)]} =$

$$= p \cdot (\pi^{-1}(n), 1) \cdot n^{[(3.22)]} = (n, 1) \cdot L_{RR}^{R} \cdot n = (1, n) \cdot n^{[(3.1)]} =$$

$$= \pi(n \cdot \pi^{-1}(n))^{[(3.1)]} = \pi(\pi(\pi^{-1}(n) \cdot \pi^{-1}(n))^{[(3.1)]} = \pi(\pi(\pi^{-1}(n) \cdot \pi^{-1}(n) \otimes 1)).$$

Similarly, for the second part of the proposition, we only have to use the following equalities

(a′) $\quad \pi(m) \cdot e^{[(3.1)]} = \pi(m \cdot e \otimes 1) = \pi(l_R)^{[(3.15)]} = j_R$

(b′) $\quad (e, 1) \cdot \pi(m)^{[(3.1)]} = \pi(m \cdot e \otimes 1) = \pi(r_R)^{[(3.17)]} = i_R$

(c′) $\quad (1, \pi(m)) \cdot \pi(m)^{[(3.1)]} = \pi(\pi(m) \cdot m)^{[(3.1)]} = \pi\pi(m \cdot m \otimes 1) =$

$$= \pi\pi(m \cdot 1 \otimes m \cdot a)^{[(3.19)]} = p \cdot \pi(m \cdot 1 \otimes m)^{[(3.1)]} = p \cdot (m, 1) \cdot \pi(m)^{[(3.22)]} =$$

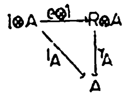
$$= (\pi(m), 1) \cdot L^{R}_{RR} \cdot \pi(m).$$

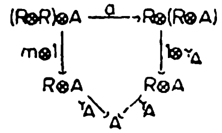
By a simple application of [(3.1)] we obtain

PROPOSITION 1.9. *If $f: (R, e, n) \to (R', e', n')$ is a morphism of closed monoids over $\underline{V}$ then $f: (R, e, \pi^{-1}(n)) \to (R', e', \pi^{-1}(n'))$ is a morphism of monoidal monoids over $\underline{V}$ and conversely.*

According to these two last propositions, in a (symmetric) monoidal closed category $\underline{V}$, we shall identify closed and monoidal monoids, speaking roughly of *monoids over* $\underline{V}$. In doing so, (R, e, n, m) will denote a monoid over $\underline{V}$ such that $\pi(m) = n$. Similarly, we shall use morphisms and antimorphisms of monoids.

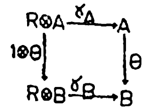
DEFINITION 1.9. (MacLane) — An object A in V_0 together with a morphism $\gamma_A: R \otimes A \to A$ in V_0 is called a *monoidal left R-module over* $\underline{V}$ if the following diagrams are commutative





Remark. A monoidal left R-module may also be defined as an object A in V_0 together with a morphism $\varepsilon_A : (R, e, m) \to ((AA), j_A, M^{A}_{AA})$ of monoidal monoids (using proposition 1.4). One simply gets the above definition applying π^{-1}.

DEFINITION 1.10 (MacLane) — A morphism $\theta : (A, \gamma_A) \to (B, \gamma_B)$ is called a *morphism of monoidal left R-modules* if $f: A \to B$ is a morphism in V_0 and

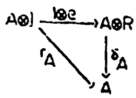


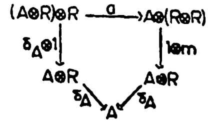
is a commutative diagram.

One can easily show (in a similar way to the last two propositions), that monoidal left R-modules over a symmetric monoidal closed category $\underline{V}$ can be identified with the closed ones. According to this, in what follows, we shall use only the notion of left R-module and morphism of left R-modules. As for monoids, by (A, α_A, γ_A) we shall denote a left R-module with $\alpha_A = \pi(\gamma_A)$; we shall denote by ${}_R M V$ the corresponding category.

Let $\underline{V}$ be a symmetric monoidal closed category and (R, e, n, m) a fixed monoid over $\underline{V}$.

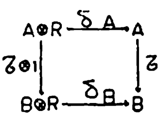
DEFINITION 1.11. (MacLane) An object A in V_0 together with a morphism $\delta_A : A \otimes R \to A$ in V_0 is called a *right R-module over* $\underline{V}$ if the following diagrams are commutative



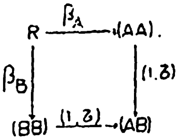


Remark. A right R-module over $\underline{V}$ may also be defined as an object A in V_0 together with an antimorphism of monoids $\beta_A : (R, e, n, m) \to ((AA), j_A, L^A_{AA}, M^A_{AA})$. In proving that these two definitions are equivalent one has to make use of the following natural isomorphisms $s_{RAA} : (R(AA)) \to (A(RA))$ which exist, the V-functor $L^A : \underline{V} \to \underline{V}$ being a self $\underline{V}$-adjoint to the left.

DEFINITION 1.12 (MacLane) — A morphism $\tau : (A, \delta_A) \to (B, \delta_B)$ is called a *morphism of right R-modules* if $\tau : A \to B$ is a morphism in V_0 making commutative the following diagram



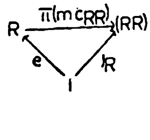
In a similar manner, $\tau : (A, \beta_A) \to (B, \beta_B)$ is a morphism of right R-modules if the following diagram commutes

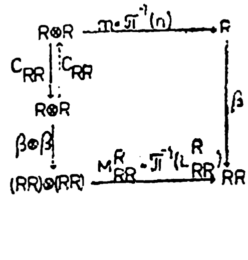


We shall denote this category with MV_R.

PROPOSITION 1.10. — *R admits a canonic structure of right R-module over* $\underline{V}$.

Proof. We have to verify that $\beta_R = \pi(\pi^{-1}(n) \cdot c_{RR}) : (R, e, n, m) \to ((RR), j_R, L^R_{RR}, M^R_{RR})$ actually is an antimorphism of monoids, i.e. the following diagrams are commutative





Using π^{-1}, the first is equivalent to $l_R = m \cdot c_{RR} \cdot c \otimes 1 = m \cdot 1 \otimes e \cdot c_{IR}$ which is $MM2$ for (R, e, m) by c's naturality and coherence. As for the second, using $[MC6]$ we have $c_{RR} \cdot c_{RR} = 1$ and so we have to check $\beta \cdot \pi^{-1}(n) \cdot c_{RR} = \pi^{-1}(L^R_{RR}) \cdot \beta \otimes \beta$ or $\beta \cdot \pi^{-1}(\beta) = \pi^{-1}(L^R_{RR}) \cdot \beta \otimes \beta$; using [(3.1)] in the application of π we get $CM1$ for β. From proposition 1.8 this is equivalent to $MM1$ for $\pi^{-1}(\beta)$. But this is true by proposition 1.6.

— Let (R', e', n', m') be a monoid over $\underline{V}$ and $f : R' \to R$ a morphism of monoids over $\underline{V}$. If (A, α_A) is a left R-module then using $\alpha_A \cdot f$, f induces on A a structure of left R'-module. $(A, \alpha_A \cdot f)$ is called *the R'-ification of A*. In this way, a morphism $f: R' \to R$ of monoids over $\underline{V}$ induces (details are straightforward) a covariant faithful functor $f: {}_R M V \to {}_{R'} M V$ (identical on morphisms). Now, if $f: R' \to R$ is an antimorphism of monoids over $\underline{V}$, it induces a covariant faithful functor ${}_R M V \to M V_R$. The identity $1_R : (R, e, m) \to (R, e, m \cdot c_{RR})$ is a canonic antimorphism from the monoid (R, e, m) to its opposite. This antimorphism induces two category equivalences ${}_{R^{op}} M V \to M V_R$ and ${}_R M V \to M V_{R^{op}}$, equivalences which permit, in a symmetric monoidal closed category $\underline{V}$, the well-known identifications.

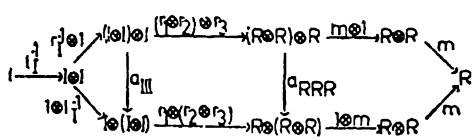
One can now define and recover the basic situations for bimodules. Details are now straightforward.

2. **"Enriched" versions of classical results.** Let $\underline{V}$ be a monoidal closed category and (R, e, n, m) a fixed monoid over $\underline{V}$.

PROPOSITION 2.1. *The following data form a category $[R]$ with a single object R:*

(i) $[R](R, R) = V_0(I, R)$; (ii) *for each* $r_1, r_2 \in [R](R, R)$ *the "composition" is* $r_1 * r_2 = m \cdot r_1 \otimes r_2 \cdot l_I^{-1} = m \cdot r_1 \otimes r_2 \cdot r_I^{-1} : I \to R$; (iii) *the morphism* $e : I \to R$ *is "the identity" in* $[R](R, R)$.

Proof. The associativity of the composition $(r_1 * r_2) * r_3 = r_1 * (r_2 * r_3)$ can be deduced from the following diagram



where regions commute by $[MC2, MC5]$, naturality of a and $MM1$ for the monoid (R, e, m). Using $r_I = l_I$ and the naturality of the isomorphisms r and l, one easily checks that e actually is an identity.

Using this result and the well-known representation of the subjacency functor $V : \underline{V} \to Ens$ given by [I, 2.1] one gets

COROLLARY 2.2. *The following data form a category $[R]'$ with a single object* R:

(i) $[R]'(R, R) = V(R)$; (ii) *for each* $r_1, r_2 \in V(R)$ *"the composition" is* $r_1 \square r_2 = Vi_R^{-1}(Vi_R^{-1}(r_1) * Vi_R^{-1}(r_2))$; (iii) *the element* $Vi_R^{-1}(e)$ *is "the identity" in* $[R]'(R, R)$. *The categories $[R]$ and $[R]'$ are isomorphic.*

PROPOSITION 2.3. *The following data form a $\underline{V}$-category* $\{R\}$: (i) *obj* $\{R\}$: *only* R; (ii) $\{R\}(R, R) = R$, *object in* V_0; (iii) *a morphism* $j_R : I \to \{R\}(R, R)$, *namely* $e : I \to R$; (iv) *a morphism* $M^R_{RR} : \{R\}(R, R) \otimes \{R\}(R, R) \to \{R\}(R, R)$ *namely* $m : R \otimes R \to R$.

Proof. Axioms of $\underline{V}$-category are easily seen to coincide with the monoid axioms.

More important for what follows is now

THEOREM 2.4. *There is a canonic identification between the category* $_RMV$ *of left R-modules and morphisms of left R-modules and the category of the* $\underline{V}$-*functors and the* $\underline{V}$-*natural transformations from the* $\underline{V}$-*category* $\{R\}$ *to* $\underline{V}$ *(as* $\underline{V}$-*category over itself).*

Proof. A $\underline{V}$-functor $T : \{R\} \to \underline{V}$ consists of a function obj $\{R\} \to$ obj$\underline{V}$, that is, an object $T(R) = A$ in V_0 and a morphism $T_{RR} : \{R\}(R, R) \to (AA)$, that is, $T_{RR} = \alpha_A : R \to (AA)$ a morphism in V_0, such that

are commutative. But this proves that $\alpha_A : (R, e, m) \to ((AA), j_A, M^A_{AA})$ is a morphism of monoids. The rest of this proof goes similarly.

COROLLARY 2.5. *There is a canonic identification between the category* MV_R *of the right R-modules and the morphisms of right R-modules and the category of the* $\underline{V}$-*functors and the* $\underline{V}$-*natural transformations from the* $\underline{V}$-*category* $\{R^{op}\}$ *to* $\underline{V}$.

— Before giving the next result, we shall introduce here an obvious functor of subjacency $W : {_RMV} \to V_0$ defined by $W(A, \alpha_A) = A$, $W(f) = f$. W is obviously a faithful functor.

PROPOSITION 2.6. *If* V_0 *is complete, so is* $_RMV$.

Proof. This result can be derived from a more general one from [2].

We also derive

COROLLARY 2.7. *The functor* W *reflects monomorphisms, epimorphisms and limits.*

Remark. At this point two directions could also be followed: conditions on $\underline{V}$ in which some properties of V_0 are inherited by $_RMV$; the second, the construction of a left adjoint for the functor W, or, the construction of the "free" left R-module over an arbitrary object of V_0. Satisfactory results can be found in [4].

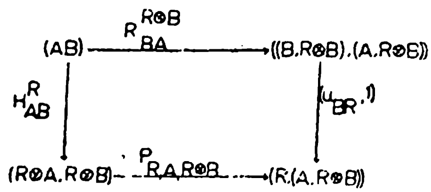
In what follows, we suppose that $\underline{V}$ is a symmetric monoidal closed category and that V_0 *has equalizers.*

LEMMA 2.8. *Let* (A, α_A, γ_A) *and* (B, α_B, γ_B) *be two left R-modules. We have*

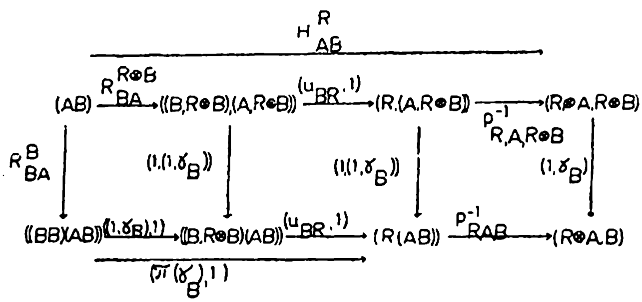
$$\mathrm{equ}\,((1_{R\otimes A}, \gamma_B) \cdot H^R_{AB}, (\gamma_A, 1_B)) \cong \mathrm{equ}((\alpha_B, 1_{(AB)}) \cdot R^B_{BA}, (\alpha_A, 1_{(AB)}) \cdot L^A_{AB})$$

Proof. First, since $\alpha_A = \pi(\gamma_A)$, an easy application of [II, (3.22)] shows that $(\gamma_A, 1_B) = p^{-1}_{RAB} \cdot (\alpha_A, 1_{(AB)}) \cdot L^A_{AB}$. We show that $p^{-1}_{RAB} \cdot (\alpha_B, 1_{(AB)}) \cdot R^B_{BA} = (1_{R\otimes A}, \gamma_B) \cdot H^R_{AB}$ and this will prove our lemma modulo the isomorphism p_{RAB}.

From [p. 545] we already have the following commutative diagram



Further, the following diagram commutes by the naturality of R and p^{-1}



using for the bottom triangle [II, (3.4)]. But the exterior is just the required equality.

We shall denote by $\{AB\}$ and call *the object of the morphisms of R-modules between* (A, α_A, γ_A) *and* (B, α_B, γ_B), the object defined by the previous lemma.

Remark. If V preserves equalizers, and $f \in V(\{AB\})$, one has for the first member in lemma 3.1

$$V_0(\gamma_A, 1_A)(f) = V_0(1_{R\otimes A}, \gamma_B)(V(H^R_{AB})(f)) = V_0(1_{R\otimes A}, \gamma_B)(1_R \otimes f)$$

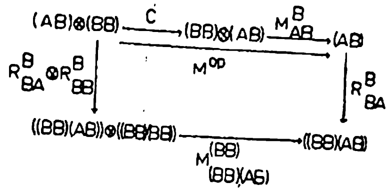
which is the definition of the morphism of monoidal left R-modules. Similarly, using the second member, one finds the definition of the morphism of closed left R-modules.

Hence, if V preserves equalizers, $V(\{AB\}) = {}_R MV((A, \alpha_A), (B, \alpha_B))$.

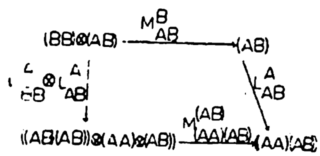
— We then have a subobject (in V_0) $\{AB\}$ of (AB). The corresponding monomorphism will be denoted by $\mathrm{equ}_{AB} : \{AB\} \to (AB)$.

LEMMA 2.9. *We have* $(\alpha_B, 1_{(AB)}) \cdot R^B_{BA} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} = (\alpha_A, 1_{(AB)}) \cdot L^A_{AB} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB}$, *i.e.,* $M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB}$ *factors through* $\{AB\}$.

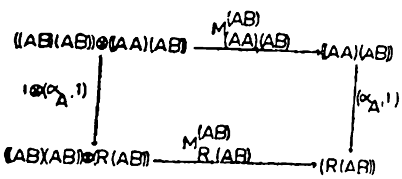
Proof. We shall use, among others, the following five facts: the $\underline{V}$-functoriality of $R^B : \underline{V}^{op} \to \underline{V}$, that is, the commutativity of

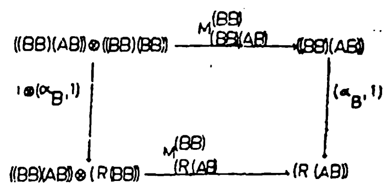


the $\underline{V}$-functoriality of $L^A : \underline{V} \to \underline{V}$, that is, the commutativity of

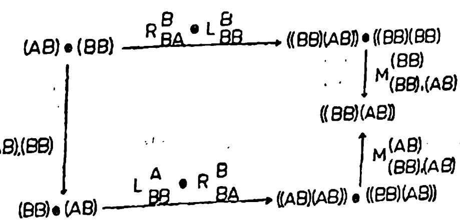


the naturality of M, that is, the commutativity of the next diagrams





and the commutativity of the following diagram, derived from [III, (4.4)]



Using all these, we have $(\alpha_B, 1) \cdot R^B_{BA} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= (\alpha_B, 1) \cdot M^{(BB)}_{(BB),(AB)} \cdot R^B_{BA} \otimes R^B_{BB} \cdot c_{(BB),(AB)} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$

$= M^{(RR)}_{R,(AB)} \cdot 1 \otimes (\alpha_B, 1) \cdot R^B_{AB} \otimes R^B_{BB} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$= (\alpha_B, 1) \cdot M^{(BB)}_{(BB),(AB)} \cdot R^B_{BA} \otimes L^B_{BB} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$= (\alpha_B, 1) \cdot M^{(AB)}_{(BB),(AB)} \cdot L^A_{BB} \otimes R^B_{BA} \cdot c_{(AB),(BB)} \cdot \mathrm{equ}_{AB} \otimes \mathrm{equ}_{BB} \cdot c_{(AB),(BB)} =$

$$= (\alpha_B, 1) \cdot M^{(AB)}_{(BB),(AB)} \cdot L^A_{BB} \otimes R^B_{BA} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$$

$$= M^{(AB)}_{R,(AB)} \cdot 1 \otimes (\alpha_B, 1) \cdot L^A_{BB} \otimes R^B_{BA} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$$

$$= M^{(AB)}_{R,(AB)} \cdot 1 \otimes (\alpha_A, 1) \cdot L^A_{BB} \otimes L^A_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$$

$$= (\alpha_A, 1) \cdot M^{(AB)}_{(AA),(AB)} \cdot L^A_{BB} \otimes L^A_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB} =$$

$$= (\alpha_A, 1) \cdot L^A_{AB} \cdot M^B_{AB} \cdot \mathrm{equ}_{BB} \otimes \mathrm{equ}_{AB}.$$

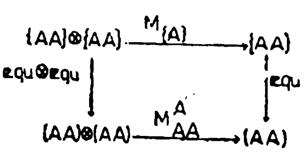
We now prove our main theorem

THEOREM 2.10. *Let* (R, c, n, m) *be a monoid over* $\underline{V}$ *and* (A, α_A, γ_A) *be a left R-module.* $\{AA\}$ *admits a structure of monoid such that* equ_{AA} *is a morphism of monoids over* $\underline{V}$.

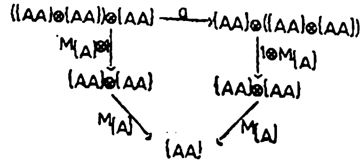
Proof. The components of this monoid are denoted by $j_{\{A\}} : I \to \{AA\}$ and $M_{\{A\}} : \{AA\} \otimes \{AA\} \to \{AA\}$.

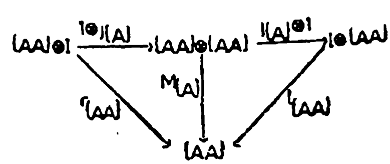
We shall prove that $(\gamma_A, 1) \cdot j_A = (1_{R\otimes A}, \gamma_A) \cdot H^R_{AA} \cdot j_A$ and derive from here a factorization of $j_A : I \to (AA)$ through $\{AA\}$ which yields our $j_{\{A\}}$. Indeed, one reads from the diagram of [p. 483] that $K^R_{AA} \cdot j_A = j_{A\otimes R}$ and hence $H^R_{AA} \cdot j_A = j_{R\otimes A}$ using the naturality of j. The stated equality now follows using again the naturality of j. Thus, we have $j_A = \mathrm{equ}_{AA} \cdot j_{\{A\}}$.

Next, using the previous lemma for $A = B$, we derive the existence of $M_{\{A\}}$ on the commutative diagram



So, if we show that $(\{AA\}, j_{\{A\}}, M_{\{A\}})$ is a monoid over $\underline{V}$, the above equality and commutative diagram will show that $\mathrm{equ}_{AA} : (\{AA\}, j_{\{A\}}, M_{\{A\}}) \to ((AA), j_A, M^A_{AA})$ actually is a morphism of monoids. We must check $MM1$ and $MM2$, that is, the commutativity of the following diagrams





As for the first, the equality to be checked is equivalent to the one obtained by composition to the left with equ_{AA}. We then have to verify $M^A_{AA} \cdot (M^A_{AA} \cdot \mathrm{equ} \otimes \mathrm{equ}) \otimes \mathrm{equ} = M^A_{AA} \cdot \mathrm{equ} \otimes (M^A_{AA} \cdot \mathrm{equ} \otimes \mathrm{equ}) \cdot a$. Applying $\pi\pi$ to both members, we successively have (using the naturality of L, [CC3], [II, (3.19) and (3.22)])

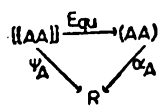
$$\pi\pi(M^A_{AA}\cdot(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ})\otimes \mathrm{equ}) = \pi((\mathrm{equ},1)\cdot L^A_{AA}\cdot M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ}) =$$
$$= (\mathrm{equ},(\mathrm{equ},1)\cdot L^A_{AA})\cdot L^A_{AA}\cdot \mathrm{equ} = (\mathrm{equ},(\mathrm{equ},1))\cdot(1,L^A_{AA})\cdot L^A_{AA}\cdot \mathrm{equ} =$$
$$= (\mathrm{equ},(\mathrm{equ},1))\cdot(L^A_{AA},1)\cdot L^{(AA)}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =$$
$$= (\mathrm{equ},1)\cdot(L^A_{AA},1)\cdot(1,(\mathrm{equ},1))\cdot L^{(AA)}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =$$
$$= (\mathrm{equ},1)\cdot(L^A_{AA},1)\cdot((\mathrm{equ},1),1)\cdot L^{\{AA\}}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =$$
$$= (\pi(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ}),1)\cdot L^{\{AA\}}_{(AA),(AA)}\cdot L^A_{AA}\cdot \mathrm{equ} =$$
$$= p\cdot(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ},1)\cdot L^A_{AA}\cdot \mathrm{equ} = p\cdot\pi(M^A_{AA}\cdot \mathrm{equ}\otimes(M\cdot \mathrm{equ}\otimes \mathrm{equ})) =$$
$$= \pi\pi(M^A_{AA}\cdot \mathrm{equ}\otimes(M^A_{AA}\cdot \mathrm{equ}\otimes \mathrm{equ})\cdot a).$$

Finally, the last two equalities are equivalent (by left composition with equ_{AA}) to $\mathrm{equ}\cdot r_{\{AA\}} = M^A_{AA}\cdot \mathrm{equ}\otimes j_A$, $\mathrm{equ}\cdot l_{\{AA\}} = M^A_{AA}\cdot j_A\otimes \mathrm{equ}$. Applying π to these, [II, (3.1), (3.15), (3.17) and (6.2)] and also axioms [$CC1$] and [$CC2$], one easily gets $(1,\mathrm{equ})\cdot i_{\{AA\}} = i_{(AA)}\cdot \mathrm{equ}$, $(1,\mathrm{equ})\cdot j_{\{AA\}} = (\mathrm{equ},1)\cdot j_{(AA)}$, true by naturality of i and j.

COROLLARY 2.11. *Each left R-module (A,α_A) has a canonic structure of left $\{AA\}$-module, namely (A,equ_{AA}).*

DEFINITION 2.1. The monoid $(\{AA\}, j_{\{A\}}, M_{\{A\}})$ is called the *monoid of the R-endomorphisms of* (A,α_A). One can now iterate this construction getting the *monoid of the biendomorphisms of the left R-module* (A,α_A). More exactly, this monoid is $\{\{AA\}\} = \mathrm{Equ}((\mathrm{equ},1)\cdot R^A_{AA}, (\mathrm{equ},1)\cdot L^A_{AA})$ where $\mathrm{Equ}:\{\{AA\}\}\to(AA)$ is a morphism of monoids over $\underline{V}$.

THEOREM 2.12. *If (A,α_A) is a left R-module, then there exists a canonic morphism of monoids over $\underline{V}$, $\psi_A : R\to\{\{AA\}\}$, such that the diagram*



is commutative.

Proof. By the equ_{AA}'s definition we have $(\alpha_A,1)\cdot R^A_{AA}\cdot \mathrm{equ} = (\alpha_A,1)\cdot L^A_{AA}\cdot \mathrm{equ}$. In fact, we must prove that α_A factors through Equ, i.e., $(\mathrm{equ},1)\cdot L^A_{AA}\cdot\alpha_A = (\mathrm{equ},1)\cdot R^A_{AA}\cdot\alpha_A$.

Using the following analogous of [II, (3.1)]: $\pi^{-1}((g,h)xf) = h\cdot\pi^{-1}(x)\cdot f\otimes g$ and applying π^{-1} to the equalities above we get the following equivalent ones

(1) $\quad M^A_{AA}\cdot c_{(AA),(AA)}\cdot \mathrm{equ}\otimes\alpha_A = M^A_{AA}\cdot \mathrm{equ}\otimes\alpha_A$, respectively

(2) $\quad M^A_{AA}\cdot\alpha_A\otimes \mathrm{equ} = M^A_{AA}\cdot c_{(AA),(AA)}\cdot\alpha_A\otimes \mathrm{equ}$. Using $c^{-1}_{(AA),(AA)} = c_{(AA),(AA)}$ and the naturality of c, one obtains (2) by a right composition of (1) with $c_{R,\{AA\}}$.

Finally, we show that $\psi_A : R \to \{\{AA\}\}$ is a morphism of monoids over $\underline{V}$. First, $\psi_A \cdot e = j_{\{\{A\}\}}$ is easily seen to be equivalent to $\alpha_A \cdot e = j_A$ by a simple left composition with Equ. Next, $\psi_A \cdot m = M_{\{\{A\}\}} \cdot \psi_A \otimes \psi_A$ combined with Equ $\cdot M_{\{\{A\}\}} = M^A_{AA} \cdot$ Equ $\otimes$ Equ (true, Equ being morphism of monoids over $\underline{V}$) is seen to be equivalent with $M^A_{AA} \cdot \alpha_A \otimes \alpha_A = \alpha_A \cdot m$ which is true, α_A being a morphism of monoids over $\underline{V}$.

— We shall call $\psi_A : R \to \{\{AA\}\}$ the *canonic morphism* from R to the monoid of the biendomorphisms of (A, α_A).

Remark. Using ψ_A one can define faithful and balanced left R-modules.

We conclude this section with the enriched version of a wellknown result:

PROPOSITION 2.13. $\{\{\{AA\}\}\} = \{AA\}$.

Proof. We have $\{\{\{AA\}\}\} = \mathcal{E}$qu ((Equ, 1) $\cdot R^A_{AA}$, (Equ, 1) $\cdot L^A_{AA}$) and $\{AA\}$=equ $((\alpha_A, 1) \cdot R^A_{AA}, (\alpha_A, 1) \cdot L^A_{AA})$, these being subobjects of (AA).

We only have to check the following two equalities $(\alpha_A, 1) \cdot R^A_{AA} \cdot \mathcal{E}$qu $=$ $= (\alpha_A, 1) \cdot L^A_{AA} \cdot \mathcal{E}$qu, (Equ, 1) $\cdot R^A_{AA} \cdot$ equ $=$ (Equ, 1) $\cdot L^A_{AA} \cdot$ equ. Using $\alpha_A =$ $=$ Equ $\cdot \psi_A$, the first one obviously follows from the definition of $\mathcal{E}$qu. The second can be deduced from (equ, 1) $\cdot R^A_{AA} \cdot$ Equ $=$ (equ, 1) $\cdot L^A_{AA} \cdot$ Equ (which is true by Equ's definition) in a very analogous manner to the first part of the proof of theorem 2.12.

(Received July 6, 1978)

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ASUPRA UNEI TEORII ÎMBOGĂŢITE A MODULELOR (I)

(Rezumat)

Stimulat de excelenta monografie despre categoriile închise, monoidale şi monoidal închise a lui Eilenberg şi Kelly [3], autorul elaborează partea închisă şi monoidal închisă a teoriei modulelor peste un monoid fixat, teorie pentru care MacLane a elaborat în [4] partea monoidală în condiţii mai restrictive.