STRONGLY CLEAN DECOMPOSITIONS MAY NOT BE UNIQUE

In [1], it is proved

Corollary 3.8. *If an element of a ring is strongly nil clean, then it has precisely one strongly nil clean decomposition.*

This follows from the fact that this is true for strongly π-regular elements [**Proposition 2.6**] and strongly nil-clean elements are strongly π-regular [**Proposition 3.5**].

On the other hand [**Proposition 2.5**], an element $a \in R$ is strongly π-regular iff a is strongly clean, $a = e + u$ and $eue \in N(R)$.

Hence, elementwise

$$strongly\ nil - clean \Rightarrow strongly\ \pi - regular \Rightarrow strongly\ clean.$$

True or false ?

If an element of a ring is strongly clean, then it has precisely one strongly clean decomposition.

FALSE (T. Y. Lam) Let R be a nonzero ring with $2 \in U(R)$. Then $0 + (-1) = 1 + (-2)$ are two different strongly clean decompositions for -1.

Remarks. 1) More general, for a nonzero ring R, whenever $u \in U(R) \cap (1 + U(R)$, $0 + u = 1 + v$ are two different strongly clean decompositions for u.

2) Owing to above, one can consider a subclass of strongly clean rings, those in which elements are uniquely the sum of an idempotent and a unit that commute. Such rings were considered and studied in [2].

An element a of a ring R is called *uniquely strongly clean* (or USC for short) if a has a unique strongly clean expression in R as stated above. The ring R is called *uniquely strongly clean* (or USC for short) if every element of R is uniquely strongly clean.

Since uniquely clean rings are Abelian, these are strongly clean and so uniquely strongly clean. Hence these rings form a class that lies properly between the class of uniquely clean rings and the class of strongly clean rings.

3) *An element in a ring can be USC but not uniquely clean.* For example, in a ring R such that 3 is not a zero divisor, take the strongly clean matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It is easy to see that $A = \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -y \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ z & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -z & -1 \end{bmatrix}$ are all the clean decompositions for every y (or every z), so A is not uniquely clean. Computation gives $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -y \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2y \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & -y \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$, so the factors commute iff $3y = 0$. Similarly (just transpose) for the factors which include z. Hence, A is USC.

References

[1] A. J. Diesl *Nil clean rings.* Journal of Algebra **383** (2013), 197-211.

[2] J. Chen, Z. Wang, Y. Zhou *Rings in which elements are uniquely the sum of an idempotent and a unit that commute.* J. Pure and Appl. Algebra, **213** (2009), 215-223.