RINGS WHOSE ELEMENTS ARE SUMS OF AN IDEMPOTENT AND A QUASI-REGULAR ELEMENT

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1. The monoid $(R, \circ)$, Quasiregular elements

From [2].

One starts with $p \circ q = p + q - pq$. Then $(R, \circ)$ is a monoid with zero as identity element.

Ex.4.2 (Lam) An element $a \in R$ is left (or right) quasiregular if a has a left (resp. right) inverse in the monoid $(R, \circ)$. If a is both left and right quasiregular then a is called *quasiregular*. Thus, quasiregular elements are the units of the monoid $(R, \circ)$. If R has an identity 1, then a is quasiregular iff $1 - a \in U(R)$ and we denote $Q(R) = \{q \in R \mid 1 - q \in U(R)\}$.

Remark. $Q(R) = 1 + U(R)$.

Examples. 1) $0 \in R$ is quasiregular.

2) Every nilpotent is quasiregular [if $r^{n+1} = 0$ then $(1-r)(1+r+r^2+...+r^n) = 1$].

3) If r^2 is quasiregular, then r is quasiregular [if $1 - r^2 = u \in U(R)$ then $(1 - r)(1 + r) \in U(R)$ implies $1 - r \in U(R)$].

4) A matrix is quasiregular in a matrix ring if it does not possess 1 as an eigenvalue [Let $A = I_n + U$ be quasiregular. Then $p_A(t) = \det((t - 1)I_n - U)$. If $t = 1$ is an eigenvalue then $\det(-U) = 0$, a contradiction; the converse works over any field].

5) The elements in the Jacobson radical are quasiregular (but not conversely) [**Corollary 4.5**. $J(R)$ is the largest ideal I of R such that $1 + I \subseteq U(R)$].

6) Nonzero idempotents are not quasiregular [for $e^2 = e \neq 0$, $1 - e$ is also idempotent $\neq 1$, so not a unit].

7) Quasiregular elements are clean [$1 + u$ with idempotent 1].

8) From [1]: a ring R is UU iff every quasiregular element is nilpotent.

2. Q-clean rings

Definition. An element $a \in R$ is called *q-clean* if $a = e + q$ with idempotent e and quasiregular q. A ring is called (temporary) *q-clean* if all its elements are q-clean. Since $N(R) \subseteq Q(R)$, nil-clean elements (or rings) are q-clean.

However, as for rings, this is not a new notion because

Lemma 1. *A ring is q-clean iff it is clean.*

Proof. Let $a \in R$. If R is q-clean then $a + 1 = e + q = e + 1 + u$ for some $e^2 = e$ and $u \in U(R)$. Hence $a = e + u$ is clean. Conversely, if R is clean then $a - 1 = e + u$ for some $e^2 = e$ and $u \in U(R)$. Hence $a = e + 1 + u$ is q-clean. □

Elementwise, this may not hold. Indeed, for example, since $Q(\mathbb{Z}) = 1 + U(\mathbb{Z}) = \{0, 2\}$ we get $Id(\mathbb{Z}) + Q(\mathbb{Z}) = \{0, 1, 2, 3\}$. However $cn(\mathbb{Z}) = Id(\mathbb{Z}) + U(\mathbb{Z}) = \{-1, 0, 1, 2\}$. Thus -1 is clean but not q-clean and 3 is q-clean but not clean.

References

[1] G. Călugăreanu *UU rings*. Carpathian J. of Math. **31** (2) (2015), 157-163.
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