Distributively generated lattices

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Abstract

In 1938 [6] Ore proved the following simple caracterization: the lattice of the subgroups of a group is distributive iff every finite set of elements generates a cyclic group. In this paper we study how far this result and other known results for modules given by Camillo in 1975 ([2]) can be generalized for lattices.

1 Preliminaries

As in [3], in a lattice L we denote by 0 (resp. 1) the lowest (resp. greatest) element and for $a, b \in L, a \leq b$ by $b/a = \{x \in L | a \leq x \leq b\}$ also called **quotient sublattice** (usually denoted as an interval $[a, b]$). A lattice L is called **noetherian (resp.artinian)** if it satisfies the DCC (resp.the ACC). An element $x \in L$ is called **noetherian (artinian)** if the sublattice $x/0$ is noetherian (resp. artinian).

An element x in a lattice L with zero is called **distributive** if the sublattice $x/0$ is distributive; a distributive element x is called **cycle** if $x/0$ is also noetherian.

An element $c \in L$ is called **compact** if each cover (i.e. $c \leq \bigvee X$) contains a finite subcover (i.e. $\exists F \subseteq X, F$ finite : $c \leq \bigvee F$) and a lattice L is called **algebraic** (or **compactly generated**) if each element of L is a join of compact elements. In particular, in $S_R(M)$ the compact elements are the finitely generated submodules and the lattice $S_R(M)$ is algebraic.

Another condition that will be used and discussed later is: *the compact elements in L are finite joins of cycles.*

A weaker condition then the algebraicity is: a lattice L is called **upper continuous** if for each (upper) directed set $\{a_i\}_{i \in I}$ and each $b \in L$ the equality $\left(\bigvee_{i\in I} a_i\right) \wedge b = \bigvee_{i\in I} (a_i \wedge b)$ holds. A stronger condition (see [4] is: a lattice L is called **H-noetherian** if the compact elements form an ideal in L (finite unions of compact elements being obviously compact, this is equivalent to $a \leq c, c$ compact $\Rightarrow a$ compact).

2 Remarks and definitions

Remark 2.1 *If G is an abelian group and $L(G)$ the lattice of all the subgroups of G then the cycles are exactly the cyclic subgroups.*

Indeed, in this particular case noetherian is equivalent to finitely generated and distributive is equivalent to locally cyclic (i.e. each finite set of elements generates a cyclic group) - exactly the result of Ore (see abstract).

If ${}_RM$ is a R-module over a ring with identity R we have to compare the cycles from $S_R(M)$ with the cyclic submodules. Hence ${}_RM$ is called a C-**module** if the cycles in the sublattice $S_R(M)$ of all the submodules are cyclic submodules. Using results from [5], one can characterize the rings R such that the cyclic submodules of ${}_RM$ are cycles in $S_R(M)$ (see lemma 5.1).

A module ${}_RM$ is called CC-**module** if its cycles coincide with the cyclic submodules.

We say that a lattice L has **ED** [repectivelly **EC**] (**enough distributive** elements [resp.**enough cycles**]) if for every $b, c \in L$ and every distributive element [resp.cycle] $x \leq b \vee c$ there are distributive elements [resp. cycles] $y, z \in L$ such that $y \leq b, z \leq c$ and $x \leq y \vee z$. A lattice L is called ***ED** (or ***EC**) if in L each interval has **ED** (resp. **EC**). A lattice is called **distributively generated** [resp. **cycle generated**] if every element is a join of distributive elements [resp. cycles].

Remark 2.2 *If ${}_RM$ is a CC-module, then $S_R(M)$ the lattice of all the submodules of M has* **EC** *and* ***EC**.

Indeed, for $B, C \in S_R(M)$ and $\langle x\rangle = Rx \leq B + C$ we have $x = b + c$ for suitable elements $b \in B, c \in C$. Hence obviously $\langle x\rangle \leq \langle b\rangle + \langle c\rangle$. We remark that this is true for each algebraic structure based on a (subjacent) abelian group if the latticial join is the ordinary sum and if the cycles are cyclic substructures. The condition **EC** being preserved by submodules and factor modules, for a C-module ${}_RM$ the lattice $S_R(M)$ has ***EC** too.

3 General Results

Theorem 3.1 *A lattice L is distributive iff L is distributively generated, has* **ED** *and is closed for (finite) joins of distributive elements.*

Proof. The condition is obviously necessary because all the elements in a distributive lattice are distributive (for **ED** we can choose $y = b$ and $z = c$).

Conversely, let $a, b, c \in L$. If L is distributively generated in order to prove the inequality $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$ it suffices to verify that for each distributive element $x \leq a \wedge (b \vee c)$ we also have $x \leq (a \wedge b) \vee (a \wedge c)$.

Now if $x \leq a \wedge (b \vee c)$ then $x \leq a$ and $x \leq b \vee c$. Hence there are distributive elements $y, z \in L$ such that $y \leq b, z \leq c$ and $x \leq y \vee z$. The lattice being closed for finite joins of distributive elements $y \vee z = u$ is a distributive element and $x, y, z \in u/0$ a distributive sublattice. Hence $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \leq (x \wedge b) \vee (x \wedge c) \leq (a \wedge b) \vee (a \wedge c)$.□

Lemma 3.1 *In a distributive lattice the union of two noetherian elements is a noetherian element.*

Proof. Equivalently, we have to verify that in a distributive lattice L, $a, b \in L$ noetherian elements and $a \vee b = 1$ imply L noetherian. Indeed if $b_1 \leq b_2 \leq .. \leq b_n \leq ..$ then there are $n, m \in \mathbf{N}^*$ such that $b_n \wedge a = b_{n+1} \wedge a = ..$ and $b_m \wedge b = b_{m+1} \wedge b = ..$ Hence for $k = \max(n, m)$ the following takes place $b_{k+1} = b_{k+1} \wedge 1 = b_{k+1} \wedge (a \vee b) = (b_{k+1} \wedge a) \vee (b_{k+1} \wedge b) = (b_k \wedge a) \vee (b_k \wedge b) = b_k \wedge (a \vee b) = b_k \vee 1 = b_k$.

So the chain is finite and L is noetherian□

If L is upper continuous then $\{cycles\} \subseteq \{noetherians\} \subseteq \{compacts\}$ ([3] indeed, in an upper continuous lattice each noetherian element is compact). If L is H-noetherian (i.e. the compact elements form an ideal [4]) then the noetherian and compact elements coincide (see [1]).

The following theorem shows when all these classes coincide and generalizes the theorem of Ore.

Theorem 3.2 *Let L be an algebraic lattice which has* **EC,** *such that each compact element is a finite union of cycles. The following conditions are equivalent:* **(a)** *L is distributive;* **(b)** *all the compact elements are cycles (we call such a lattice a Bezout lattice);* **(c)** *L is closed for finite joins of cycles.*

Proof. **(a)**$\Leftrightarrow$**(c)** The hypothesis assure that L is cycle generated. The proof is similar to the one of our previous theorem. For **(a)**$\Rightarrow$**(c)** one uses also the previous lemma.

(b)$\Leftrightarrow$**(c)** Each lattice is closed for finite joins of compact elements ([3]). The rest is an easy exercise of induction.$\square$

Remark 3.1 *The name of Bezout lattice has an obvious explanation: a ring is left Bezout if each finitely generated left ideal is principal.*

4 Lattices with square-free socle

In this section we use a new notion in order to recover also results from [2].

A lattice L has **square-free socle** if every finite join of atoms is a cycle (see [2]; indeed, there, a module $_RM$ has *square-free socle* if its socle has at most one copy of each simple module - a module is called *D-module* if $S_R(M)$ is a distributive lattice and it is proven that *$_RM$ is a D-module iff for each submodule $N \in S_R(M)$, M/N has a square-free socle*). Here our choice (for a suitable lattice condition) is explained as follows: in the lattice $L(G)$ of all the subgroups of an abelian group G, the atoms are the simple subgroups, that is, the cyclic subgroups of prime order. Two such atoms are not isomorphic iff their direct sum is again a cyclic subgroup. So its socle contains at most one copy of each simple subgroups if each sum of atoms is a cycle (this leads also to a **classification of the atoms** in a lattice).

Lemma 4.1 *Let $a \in L$. If L is algebraic then $1/a$ is also algebraic.(see [4]).*
$\square$

Lemma 4.2 *Let $a \in L$ a modular lattice. If in L each compact element is a finite join of cycles then the sublattice $1/a$ has this property too.*

Proof. An element k is compact in $1/a$ iff there is a compact c in L such that $k = c \vee a$ (see [4]). Now, if $c = \bigvee_{i=1}^{n} x_i$ is the finite cycle decomposition then, using the modularity, $(x_i \vee a)/a \cong x_i/(x_i \wedge a) \subseteq x_i/0$ so that $\{x_i \vee a\}_{i=1}^{n}$ are cycles in $1/a$ and hence $k = c \vee a = \bigvee_{i=1}^{n} (x_i \vee a)$ is the required decomposition.$\square$

Theorem 4.1 *Let L be an algebraic lattice with* ***EC** *such that each compact element is a finite join of cycles. If L is distributive then for every $a \in L$ the sublattice $1/a$ has square-free socle.*

Proof. If L is distributive (and hence modular), obviously the sublattice $1/a$ is distributive too. The previous lemmas and the stronger hypothesis ***EC** show that Theorem 3.2 is applicable in $1/a$. Hence in $1/a$ every finite join of cycles is a cycle. But each atom is clearly a cycle so that $1/a$ has square-free socle.$\Box$

Reasonable conditions which assure that the converse of the previous theorem is also true seem to be difficult to find. If all the sublattices $1/a$ have square-free socles and (i) L is H-noetherian, or (ii) L has only cycles of finite length, we can prove that in L unions of cycles are noetherian elements. The difficult problem is to assure that these are also distributive.

5 Applications

As we have already anticipated

Lemma 5.1 *For each R-module ${}_RM$ the cyclic submodules in $S_R(M)$ are cycles iff R is left noetherian and left arithmetic.*$\Box$

Here a ring is called **left arithmetic** (Fuchs,1949) if the lattice of all its left ideals is distributive. The proof in an exercise.

We saw that if${}_RM$ is a R-module, the lattice of all the submodules $S_R(M)$ is modular, algebraic (compact generated) and if it is a CC-module, it has ***EC.** Then, the compact elements being the finitely generated submodules, the compact elements are finite joins of cycles so that all the hypothesis in the previous theorem are fulfilled and half of the result of Camillo could be deduced if the condition "each sum of simple submodules is a cyclic submodule" implies "the module has at most one copy of each simple module in its socle".

We call a ring R a C-**ring** if all the R-modules are C-modules respectively a CC-**ring** if all the R-modules are CC-modules.

From theorem 3.2 we immediately derive

Theorem 5.1 *The lattice of all the submodules $S_R(M)$ of a R-module ${}_RM$ over a CC-ring R is distributive (${}_RM$ is a D-module) iff the sum of two arbitrary cyclic submodules is also a cyclic submodule.*$\square$

We finish with a list of module and ring open problems:

1) For an arbitrary ring R with identity caracterize the C-modules ${}_RM$.

2) Determine the C-rings resp. the CC-rings.

3) Find the rings R, such that for each left R-module ${}_RM$, if each sum of simple submodules is a cyclic submodule then ${}_RM$ has at most one copy of each simple module in its socle.

Added in proof: the implication in theorem 4.1 holds without the use of the condition ***EC**. For this remark I am indebted to Prof.Laszlo Fuchs.

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