UNIT-REGULAR ELEMENTS HAVE STABLE RANGE 1: A DIRECT PROOF

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Abstract. Using a result by Chen and Nicholson, that is, finite products of stable range one elements have stable range one, a simple direct proof is given for the statement in the title.

1. Introduction

In [6], using results from [3], a proof is given for

Corollary 2.6 If R is a ring and $e^2 = e \in R$ then R is exchange (suitable) iff eRe and $(1-e)R(1-e)$ are both exchange.

From this paper we quote: "It would be of interest to see a direct ring-theoretic proof of the fact that if R is a ring, $e \in R$ is an idempotent and eRe and $(1-e)R(1-e)$ are both exchange rings then R is also an exchange ring".

After 31 years, such a proof was given in [1].

Also in [6], a proof is given for "the (finite) exchange property is a left-right symmetric condition". It took 20 years until Nicholson himself gave this direct proof (see [7]).

Definition (see [4]). An element a in a unital ring is called *unit-regular* if there exists a unit u such that $a = aua$. It follows easily that a is unit-regular iff it is a left (or right) unit multiple of an idempotent.

In this note we start from

Theorem 3.2 ([5]) In any ring, unit-regular elements have stable range one. In particular, all idempotents have stable range 1.

In [5], a proof is given using

Lemma 3.3. *If a and a' are unit-regular in a ring R, then $aR = a'R$ iff $a' = au$ for some $u \in U(R)$.*

Nicholson's Lemma 3.4. *Let P be a projective right module over any ring R, and let A, B be submodules of P such that $A + B = P$. If A is a direct summand of P, then there exists a submodule $C \subseteq B$ such that $P = A \oplus C$.*

In section two we provide a (very simple) direct ring-theoretic proof (i.e., without aid of Module Theory) for the theorem mentioned above.

2. The direct proof

Definition. *An element* a in a ring R is said to have *right stable range 1* (for short *rsr1*) if whenever $aR + bR = R$ for some $b \in R$, there is an element y such that $a + by \in U(R)$. Equivalently, a has rsr1 iff whenever $ax + b = 1$ for some $a, x, b \in R$, there exists $y \in R$ such that $a + by \in U(R)$, or else (we can eliminate b), a has rsr1 iff for every x there exists y such that $a + (1 - ax)y$ is a unit.

To simplify the wording, the element y will be called a *unitizer* for a, depending on x.

A symmetric definition is given for *left stable range 1* elements. An element has *stable range 1* if it has right and left stable range 1.

By taking the unitizer $y = 0$, one sees that units have stable range 1.

First recall from [2], the following easy

Lemma 1. *(i) If a has rsr1 and $u \in U(R)$ then au has rsr1.*
(iii) Right sr1 elements are invariant to conjugations.
(iv) If a has rsr1 and $u \in U(R)$ then au has rsr1.

Similar statements hold for the left stable range condition.

Next, recall from [8] (Lemma 17)

Proposition 2. *Any finite product of left (or right) stable range 1 elements has left (resp. right) stable range 1.*

Further, we have

Lemma 3. *In any ring, idempotents have stable range one.*

Proof. For any given idempotent e, and an arbitrary $x \in R$, we are searching for a unitizer y, such that $e + (1 - ex)y$ is a unit.

It suffices to take the unitizer $y = \overline{e} = 1 - e$ the complementary idempotent: indeed, $e + (1 - ex)\overline{e} = 1 - ex\overline{e} \in 1 + N(R) \subseteq U(R)$.

Hence, idempotents have right stable range 1. The same unitizer suits well for the left stable range condition. □

Finally

Theorem 4. *In any ring, unit-regular elements have stable range one.*

Proof. The statement now follows from Lemma 1 and Lemma 3, since the unit-regular are precisely the products eu for $e^2 = e$ and $u \in U(R)$. □

Added in proof. As direct as possible ! (but written for left stable range one).

Suppose $a = aua$ with $u \in U(R)$ and x arbitrary in R. We show that $a + (u^{-1} - a)(1 - xa) \in U(R)$, that is, a unitizer for a is (as simple as) $u^{-1} - a$.

It suffices to replace a with auu^{-1}:

$a + (u^{-1} - a)(1 - xa) = auu^{-1} + (u^{-1} - auu^{-1})(1 - xauu^{-1}) =$

$= auu^{-1} + (1 - au)u^{-1}(1 - xauu^{-1}) = u^{-1} - (1 - au)u^{-1}xauu^{-1} =$

$= [1-(1-au)u^{-1}x(au)]u^{-1} \in U(R)$, since au is idempotent and $(1-au)u^{-1}x(au)$ is square-zero.

References

[1] G. Călugăreanu *Split-extensions of exchange rings: A direct proof.* J. of Algebra and Its Appl. **8** (5) (2009), 629-632.

[2] G. Călugăreanu, H. F. Pop *On stable range one matrices.* to appear in Bull. Soc. Math. Roumanie, 2022.

[3] P. Crawley, B. Jónsson *Refinements for infinite direct decompositions of algebraic systems.* Pacific J. Math. **14** (1964), 797-855.

[4] G. Ehrlich *Unit regular rings.* Portugaliae Math. **27** (4) (1968), 209-212.

[5] D. Khurana, T. Y. Lam *Rings with internal cancellation.* J. of Algebra **284** (2005), 203-235.

[6] W. K. Nicholson *Lifting idempotents and exchange rings.* Trans. Amer. Math. Soc. 229 (1977) 269-278.
[7] W. K. Nicholson *On exchange rings.* Comm. Algebra **25** (6) (1997), 1917-1918.
[8] Huanyin Chen,W.K.Nicholson *Stable modules and a theorem of Camillo and Yu.* J. of Pure and Applied Algebra **218** (2014), 1431-1442.