Abelian CS-groups

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Abstract

In Birkenmeier's talk to the Conference (30 March, Tulane University) the problem of characterizing the abelian CS resp. FI (fully invariant)-extending groups was stated (as for the FI's giving obvious examples: divisibile, finitely generated, bounded, resp. $\prod_{p\in\mathbf{P}}\mathbf{Z}(p)$ which is not FI).

Prof. Göbel observed (giving a suitable immediate (!) proof) that the problem of the classification of the FI-extending abelian groups is hopeless.

In what follows, using not very popular literature, the characterization of the CS-extending abelian groups is given (as a particular case of much general results).

In what follows we use the terminology of [2].

Definition.- A module M is called *extending (or CS-module:* C*losed* S*ummand)* if every closed submodule is a direct summand.

Remark.- Equivalently, M is extending iff every submodule is essential in a direct summand.

Obvious examples.-semisimple modules (each submodule is a direct summand)
- uniform modules (each non-zero submodule is essential in M).

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1) For a Dedekind domain, denote by $M = \bigoplus_P M(P)$ the decomposition of a torsion module M , P running over the prime ideals of R.

Corollary 23 ([4]): *Let M be a torsion module over a Dedekind domain R. Then M is extending iff for each non-zero prime ideal P of R , either $M(P)$ is injective, or $M(P)$ is a direct sum of copies of R/P^n or R/P^{n+1} for some $n = n(P)$.* □

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Hence, *a torsion abelian group G is extending iff it is divisible, or it is a sum of cocyclic groups, such that for each prime number p there is a $n = n(p) \in N^*$ such that the p-component $G_p \simeq (\bigoplus_s \mathbf{Z}(p^n)) \oplus (\bigoplus_t \mathbf{Z}(p^{n+1}))$ with (possible zero) cardinals s, t.*

- That's why (see examples [2]) *for each prime* p, $\mathbf{Z}(p) \oplus \mathbf{Z}(p^2)$ *is extending and* $\mathbf{Z}(p) \oplus \mathbf{Z}(p^3)$ *is not extending! Generalization:* M *uniserial module with unique composition series* $M \supset U \supset V \supset 0$. *Then* $M \oplus (U/V)$ *is not extending.*

2) Theorem 14 ([5]): *Let* F *be a reduced torsion-free module over a Dedekind domain* R. *Then* F *is extending iff* $F \simeq \bigoplus_{i=1}^{n} NI_i$, *where* N *is a proper submodule of the quotient field* K *and the* I_i *are fractional ideals of* R.

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Hence, *a reduced torsion-free abelian group is extending iff it is homogeneous completely decomposable of finite rank.*

-That's why (see examples [2]) *a free* $\mathbf{Z}$*-module is extending iff it has finite rank.*

3) Corollary 2 + Proposition 3 ([5]): *A module over a commutative domain is extending iff it is torsion extending, or, the direct sum of a torsion-free reduced extending module and an arbitrary injective module.*

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Hence,

an abelian group is extending iff it is torsion extending (see (1)), or the direct sum of a torsion-free reduced extending group and an arbitrary divisible group.

Several remarks

A) Theorem 4 ([9]): *A ring* R *is right noetherian iff every extending right* R*-module is a direct sum of uniform modules.*

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Hence, *the extending abelian groups are direct sums of cocyclics and rank 1 torsion-free groups, that is, subgroups of* $\mathbf{Q}$.

B) It is known ([8]) that *for a subgroup* H *of an abelian group* G *the following are equivalent: (i)* H *is neat [i.e.* $pH = H \cap pG$, *holds for each prime* p*; (ii)* H *is (essentially) closed; (iii)* H *is a complement.*

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Hence, *an abelian group* G *is extending iff each neat subgroup of* G *is a direct summand.*

So the class of all the abelian groups, such that each neat subgroup is a direct summand is exactly the class mentioned (**1-2-3**) above [arbitrary extending groups].

C) Obviously, each pure subgroup of an abelian group is neat. An R-module M will be called *purely extending* (see [1]) if each pure complement submodule is a direct summand. Clearly each extending module is pure-extending, too.

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Hence, an abelian group G is purely extending iff each pure subgroup is a direct summand. But these groups were characterized long time ago (see [3]): *an abelian group* $G = D \oplus R$ *(where* D *is a divisible group and* R *is reduced) is purely extending* *iff R is either the direct sum of cyclic p-groups such that for each prime p, the orders of the cyclic p-groups are bounded, or a homogeneous completely decomposable (torsion-free) group of finite rank. If D is not a torsion (divisible) group, then for R only the second alternative is possible.*

D) The following inclusions of classes of modules are known from [2]:
$\{$ injective$\} \subset \{$ quasi-injective$\} \subset \{$ extending$\} \subset \{$ purely extending$\}$.
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Notice that (see [7]) *an abelian group G is quasi-injective iff if G is injective or, G is a torsion group whose p-components are direct sums of isomorphic cocyclic groups.*
Now all the picture, for abelian groups is completed.

E) Moreover we characterize also the "bold" classes in the following sequence:

$\{$ quasi-injective$\} \subset \{$**continuous** $\} \subset \{$**π-injective**$\} \subset \{$ extending$\}$.

Recall (see [2]) that a module M is called *π-injective* (or *quasi-continuous*) if $f(M) \subseteq M$ for each idempotent endomorphism of $E(M)$, the injective hull of M, and *continuous* if it is π-injective and *direct injective* (for every direct summand X of M, every monomorphism $X \to M$ splits.
Remarks.- (i) a module M is π-injective iff it is extending and satisfies C_3 : has the property of the sum of the direct summands $[M_1 \cap M_2 = 0,\ M_1, M_2$ direct summands $\Rightarrow M_1 \oplus M_2$ direct summand]; (ii) a module M is continuous iff it is extending and satisfies C_2 : each submodule isomorphic with a direct summand is a direct summand.

We use the following results from [10]:
Corollary 3.3.- *Let R be a Dedekind domain. Then an R-module M is quasi-continuous iff either (i) M is quasi-injective, or (ii) $M = K \oplus E$ where E is torsion and injective and $0 \neq K \subset Q$, the quotient field of R.*

Hence, *an abelian group G is π-injective (quasi-continuous) iff it is quasi-injective (see (D) or, if $G = T \oplus K$ where T is torsion divisible and K is a rank one torsion-free group (i.e. a proper subgroup of $\mathbf{Q}$).*
Finally, from
Corollary 3.4.- *Let R be a Dedekind domain. Then a R-module is continuous iff it is quasi-injective.*

Hence, *for abelian groups we have* $\{$ quasi-injective$\} = \{$**continuous** $\}$.

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