On a problem of Prof.A.Orsatti *

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Abstract

In this communication we give partial results concerning the determination of the abelian W_0-groups.

1 Introduction

Definition. Let A, R be two rings, $\mathcal{D}_A$ and $\mathcal{G}_R$ be full subcategories of $Mod - A$ respectively $Mod - R$.

An equivalence $\mathcal{G}_R \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}_A$ is called **representable** by the bimodule ${}_AP_R$ if $G \sim H = Hom_R(P_R, -)|_{\mathcal{G}_R}$ and $F \sim T = (- \otimes_A P)|\mathcal{D}_A$.

Definition. $Gen(P_R)$ denotes the subcategory of $Mod - R$ **generated** by P_R [$M \in Gen(P_R)$ iff there is an exact sequence $P_R^{(X)} \to M \to 0$ with a set X].

Definition. Let $P_R \in Mod - R$ and $A = End(P_R)$. We call P_R a W_0**-module** (after Ricardo, W is for Wisbauer) if the bimodule ${}_AP_R$ represents an equivalence between $Gen(P_R)$ and $Im(H)$ [subcategory in $Mod - A$].

The interest for representable equivalences between categories of modules is considerable and comes back to the classical results of Morita (1958) and Fuller (1974) but is continued also now-a-days by the intense study of particular W_0-modules named *-modules and tilting modules.

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In 1995, Prof. Adalberto Orsatti suggested me the determination of the W_0-abelian groups, that is, the study of the W_0-modules in the case $R = \mathbf{Z}$.

Not being at all "at home" in this domain, I simply put this problem in a drawer and forgot it.

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Having the wonderful opportunity of being in Padova, 1 month and a half before this special Conference I decided that results concerning the problem he stated for me should naturally frame here.

The results I want to present today are respectfully dedicated to the 60-th anniversary of the birthday of the "maestro".

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In what follows G will always denote an arbitrary abelian group, Ab will denote the category of all the abelian groups and group homomorphisms and $Gen(G)$ denotes the full subcategory of Ab generated by G (i.e. direct sums of G and factor groups of such direct sums).

G is called a **W_0-group** if the bimodule ${}_EG_{\mathbf{Z}}$ represents an equivalence between $Gen(G)$ (right $\mathbf{Z}$-modules) and $Im(\mathcal{H})$ (right E-modules) where $E = End_{\mathbf{Z}}(G)$ and $\mathcal{H} = Hom_{\mathbf{Z}}(G_{\mathbf{Z}}, -)$, in fact an adjoint equivalence together with the corresponding restriction of $- \otimes_E G$.

Hence, G is a W_0-group iff the restriction of the functor $\mathcal{H}$ to $Gen(G)$ is full and faithful, that is, for each $H, K \in Gen(G)$ the group homomorphisms

$\mathcal{H}_G^{H,K} : Hom_{\mathbf{Z}}(H, K) \to Hom_E(Hom_{\mathbf{Z}}(G, H), Hom_{\mathbf{Z}}(G, K))$

are isomorphisms. Indeed if $\mathcal{H}|_{Gen(G)} : Gen(G) \to Im(\mathcal{H})$ is full and faithfull (and obviously surjective on objects) it induces an adjoint equivalence with identical unity.

The following characterization of the W_0-groups from [2] will also be used: G is a W_0-group iff for each group H the canonical group homomorphism (arising from the counity of the above adjoint equivalence)

$\rho_H : Hom_{\mathbf{Z}}(G, H) \otimes_E G \to H$ is injective (here $\rho_H(f \otimes x) = f(x)$).

2 Examples and reductions

First let us provide some easy examples and counterexamples of W_0-groups:

All the cyclic and cocyclic groups are W_0-goups.

1) Indeed: for $\mathbf{Z}$, for each group H, $\rho_H : Hom_{\mathbf{Z}}(\mathbf{Z}, H) \to H$ is essentially $1_{\mathbf{Z}}$ (modulo some canonical isomorphisms) and for $\mathbf{Z}_n$ in a similar way ρ_H is essentially the inclusion of $H[n]$ into H. *More generally*, $\mathbf{Z}^{(n)}$ is surely a finitely generated projective generator in $Mod - \mathbf{Z}$ and so this *is also a W_0-group*, using the classical result of Morita (see [4]).

2) A classical result of [3] shows (using homological methods) that $\mathbf{Q}/\mathbf{Z}$ *is a W_0-group.*

3) Using results of Onodera , one can show that *finitely generated torsion groups such that each p-component is a direct sum of cyclic groups of the same order p^n are W_0-groups.*

4) $\mathbf{Q}$ *is not a W_0-group.*

Using the definition one has to choose $H, K \in Gen(\mathbf{Q})$ in order to find a non-isomorphism of groups

$\mathcal{H}_{\mathbf{Q}}^{H,K} : Hom(H, K) \to Hom_E(Hom(\mathbf{Q}, H), Hom(\mathbf{Q}, K))$ for $E = End(\mathbf{Q}) \cong \mathbf{Q}$.

Indeed, for $H=\mathbf{Z}(p^\infty), K=\mathbf{Q}$ surely $Hom(H,K)=0$ but
$Hom_E(Hom(\mathbf{Q},\mathbf{Z}(p^\infty)),Hom(\mathbf{Q},\mathbf{Q}))\cong Hom_{\mathbf{Q}}(\bigoplus_\lambda \mathbf{Q},\mathbf{Q})\cong \prod_\lambda \mathbf{Q}\neq 0$.

3 Results

Being confronted with an entire new class of abelian groups, I first tried to obtain the well-known reductions for abelian groups.

That is I first proved the following

Theorem 3.1 *Let G be a torsion group. G is a W_0-group iff for all prime number p the p-components G_p are W_0–groups.*

The proof is based on an elementary

Lemma 3.1 *Let R_i be a family of rings with identity, A_i a family of corresponding right R_i-modules, B_i a family of corresponding left R_i-modules and let $\prod_{i\in I} R_i$ be the direct product of rings. Then the map*

$\sigma:(\prod_{i\in I}A_i)\otimes_{\prod_{i\in I}R_i}(\bigoplus_{i\in I}B_i)\to\bigoplus_{i\in I}(A_i\otimes_{R_i}B_i)$ *defined as* $\sigma((a_i)_{i\in I}\otimes(b_i)_{i\in I})=\sum_{i\in I}(a_i\otimes b_i)$ *is a canonical group isomorphism.*$\square$

Indeed this result enables us to prove the following reduction

Theorem 3.2 *Let $G=\bigoplus_{i\in I}H_i$ and $E_i=End(H_i)$. If for each $H\in Ab$*

$\bar\rho_H : Hom_{\mathbf{Z}}(\bigoplus_{i\in I}H_i,H)\otimes_{\prod_{i\in I}E_i}(\bigoplus_{i\in I}H_i)\to H$ *defined as* $\overline{\rho}_H(f\otimes x)=f(x)$ *are monomorphisms, then H_i are W_0-groups.*$\square$

Consequence 3.1 *Let $G=\bigoplus_{i\in I}H_i$ and H_i be fully invariant subgroups of G. If G is W_0-group then all the H_i are W_0-groups too.*

Indeed, one has only to remaind that for fully invariant direct summands H_i of G, there is a canonical ring isomorphism $End(G)\cong\prod_{i\in I}End(H_i)$ and so the condition in the previous theorem is equivalent to $\rho_H: Hom_{\mathbf{Z}}(G,H)\otimes_E G\to H$ defined by $\rho_H(f\otimes x)=f(x)$, are monomorphisms.$\square$

Remark 3.1 *Unfortunately, even for fully invariant subgroups the converse of this consequence in not true.*

Indeed, if all the $\rho_H^{(i)}: Hom_{\mathbf{Z}}(H_i,H)\otimes_{E_i}H_i\to H$ are monomorphisms, the corresponding factorizations through the direct sums $\overline{\rho}_H:\bigoplus_{i\in I}(Hom_{\mathbf{Z}}(H_i,H)\otimes_{E_i}H_i)\to H$ need not to be monomorphisms (e.g. if $i\neq j$ does not imply $im(\rho_H^{(i)})\cap im(\rho_H^{(j)})=0$).

Remark 3.2 *In fact this is essential for a direct sum: how* $\prod_{i\in I} End(H_i)$ *embeds (as a subring) in* $End(\bigoplus_{i\in I} H_i)$.

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However if I is the set of all the prime numbers and $G = \bigoplus_{p\in\mathbf{P}} G_p$ (the p-components) then $\rho_H^{(p)} : Hom_{\mathbf{Z}}(G_p, H) \otimes_{E_p} G_p \to H$ being defined by $\rho_H^{(p)}(f \otimes x) = f(x), \forall f \in Hom_{\mathbf{Z}}(G_p, H), \forall x \in G_p$ and so $im(\rho_H^{(p)}) \subseteq H_p$ so that for each $H \in Ab$ the above condition holds. Hence $\overline{\rho}_H : \bigoplus_{p\in\mathbf{P}}(Hom_{\mathbf{Z}}(G_p, H) \otimes_{E_p} G_p) \to H$ are monomorphisms.□

Consequence 3.2 $\mathbf{Z}(p^\infty)$ *is a* W_0*-group.*

Indeed, $\mathbf{Q}/\mathbf{Z} = \bigoplus_{p\in\mathbf{P}} \mathbf{Z}(p^\infty)$ and one uses example 2.

Consequence 3.3 *For each fixed* $n \in \mathbf{N}^*$, $\bigoplus_{p\in\mathbf{P}} \mathbf{Z}_{p^n}$ *are other examples of non-finitely generated* W_0*-group. [each *-group is finitely generated, see [8]]*

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One sees immediately that the class of the W_0-groups has not a good behaviour towards direct sums. [see also bellow].

Using only the definition of the W_0-groups [but the proofs contain very long computations] I finally proved the following three results

Proposition 3.1 *Let* $n \in \mathbf{N}^*$. *A group* G *is a* W_0*-group iff* $G^{(n)}$ *is a* W_0*-group too.*

Using the above consequence and example **5** we immediately obtain

Consequence 3.4 *An finite direct sum of* $\mathbf{Z}(p^\infty)$ *is a* W_0*-group.*□

Consequence 3.5 *A finite direct sum* $\bigoplus_n \mathbf{Q}$ *is not a* W_0*-group*

The above results can be rephrased as

Consequence 3.6 *(a) A divisible torsion group of finite rank* D *is always a* W_0*-group. (b) A divisible torsion-free group of finite rank is not a* W_0*-group.*

(a) Indeed, if D is a divisible torsion group of finite rank each p-component has the form $\bigoplus_n \mathbf{Z}(p^\infty)$ and hence is a W_0-group. But then D is a W_0-group too (by Theorem 2.2).□

I thought that *a divisible group of finite rank is a* W_0*-group iff it is torsion*, but the tilting example $\mathbf{Q}/\mathbf{Z} \oplus \mathbf{Q}$ [or $\mathbf{Z}(p^\infty) \oplus \mathbf{Q}$] in the conference of Prof.Wiesbauer showed me that this is not true.

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The following result gives us a large class of abelian W_0-groups

Theorem 3.3 *For an arbitrary group G the group $G \oplus \mathbf{Z}$ is always a W_0-group.*$\square$

Consequence 3.7 $\mathbf{Q}^{(n)} \oplus \mathbf{Z}$ *is a W_0-group.*

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Theorem 3.4 *If G is a group bounded by $n \in \mathbf{N}^*$ then $G \oplus \mathbf{Z}_n$ is always a W_0-group.*$\square$

Consequence 3.8 $\mathbf{Q}^{(m)} \oplus \mathbf{Z}_n$ *is a W_0-group.*

Remark 3.3 *So the direct sum of a W_0-group with a non-W_0-group can be a W_0-group! [already seen in Wiesbauer example].*

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Finally we have

Consequence 3.9 *Each direct sum $\sum$ of cyclic groups with bounded p-components is a W_0-group.*

Indeed, if $\sum$ is torsion we apply the previous theorem to p-components and theorem 3.1. If $\sum$ is not torsion, it has a $\mathbf{Z}$ direct summand and we apply theorem 3.3.$\square$

Consequence 3.10 *Each finitely generated group is a W_0-group.*$\square$

Remark 3.4 *Using as start the example of Prof. Wiesbauer perhaps a result of the same kind for direct sums $G \oplus \mathbf{Z}(p^\infty)$ could also hold.*

4 Final Comments

Surely, to determine the structure of all the W_0-groups seems not a simple problem. But there is obviously another problem naturally connected with: to determine for each W_0-group the corresponding category of modules, i.e. $Im(\mathcal{H})$ (that is, to point out the precise equivalences obtained in this way).

Finally we list some easy examples.

If $G = \mathbf{Z}$ then surely $Gen(\mathbf{Z}) = Ab$ and so is $Im(\mathcal{H})$ (here $E = End(\mathbf{Z}) \cong \mathbf{Z}$) (indeed, $\mathbf{Z}$ is a projective generator in Ab).

If $G = \mathbf{Z}_n$ then $Gen(\mathbf{Z}_n) = \{G \in Ab | nG = 0\}$ and so is $Im(\mathcal{H})$ (and moreover, both are varieties) and these n-bounded groups are naturally considered as $\mathbf{Z}_n$-modules.

If $G = \mathbf{Z}(p^\infty)$ a result already mentioned of [3] shows that $\mathcal{H}$ provides an equivalence between $Gen(\mathbf{Z}(p^\infty))$ which is the subcategory of Ab of all the divisible p-groups and $Im(\mathcal{H})$ which is exactly the category of all the cotorsion torsion-free J_p-modules.

If $G = \mathbf{Q}$ then $Gen(\mathbf{Q})$ contains all the torsion-free divisible groups and their factor groups (hence contains also a lot of torsion divisible groups) and $Im(\mathcal{H})$ contains $\mathbf{Q}$-modules (here $E = End(\mathbf{Q}) \cong \mathbf{Q}$) which are - as groups - torsion-free. This clarifies somewhat the counterexample from section 2.

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