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Some matrix completions over integral domains

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ABSTRACT

We characterize 3×3 nilpotent matrices which are completions of 2×2 arbitrary matrices and 3×3 idempotent matrices which are completions of 2×2 arbitrary matrices over integral domains. As an application we show that a nil-clean element of a ring which belongs to a corner of the ring, may not be nil-clean in this corner.

1. Introduction

Throughout the last decades, numerous results have been published in the area of the so-called Matrix Completion Problems (see [1] for a recent survey).

In this paper we discuss two such completions over arbitrary (commutative) integral domains. While nilpotents and idempotents can be easily characterized in $\mathcal{M}_2(R)$ for any commutative ring R, it is much harder to do this in $\mathcal{M}_3(R)$. In this short note we characterize nilpotent 3×3 matrices obtained by completing an arbitrary 2×2 matrix

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and idempotent 3×3 matrices obtained by completing an arbitrary 2×2 matrix. As an application we give *a negative example related to* a long lasting question on nil-clean rings, stated by Diesl already in his Ph.D. thesis (2006), and restated in [2]: are corners of nil-clean rings also nil-clean?

More precisely, since so far, this question turns out to be much harder to answer, one may begin by asking more generally (for a ring R and an idempotent $e \in R$) how the nil-clean elements of eRe (denoted in the sequel $NC(eRe)$) are related to those of R (denoted $NC(R)$). If it were true that $eRe \cap NC(R) \subseteq NC(eRe)$ (for any full idempotent $e \in R$), then certainly the question above would have a "yes" answer. However, this inclusion relation does not hold in general, as our example shows.

In this section we present a method of constructing 3×3 completions of 2×2 matrices which are nilpotent respectively idempotent. We describe this construction for matrices over any (commutative) integral domain.

First recall the following formula (folklore): let A and B be square matrices of the same size. Then the trace

$$\mathrm{Tr}(AB) = \sum A * B^T$$

where the RHS is obtained by adding the elements of the elementwise product (*) of the matrices (B^T denotes the transpose).

Next note that for an arbitrary 2×2 matrix M, $\mathrm{Tr}(M^2) = \mathrm{Tr}(M)^2 - 2\det(M)$.

Finally, the characteristic polynomial of a 3×3 matrix is $p_A(X) = \det(X.I_3 - A) = X^3 - \mathrm{Tr}(A)X^2 + \frac{1}{2}(\mathrm{Tr}(A)^2 - \mathrm{Tr}(A^2))X - \det(A)$. Hence a 3×3 matrix A is nilpotent iff $p_A(X) = X^3$ iff $\det(A) = \mathrm{Tr}(A) = \mathrm{Tr}(A^2) = 0$ in any (commutative) integral domain.

In the sequel, for any given matrix U, u_{ij} denotes the (i, j) entry of U.

Proposition 1. *Let R be a (commutative) integral domain and let U be an arbitrary matrix in $\mathcal{M}_2(R)$. There is a nilpotent matrix $N \in \mathcal{M}_3(R)$ which has U as the northwest 2×2 corner, whenever there exist elements $a, b, x, y \in R$ such that $ax + by = \det(U) - \mathrm{Tr}(U)^2$ and $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U)$. Such a matrix exists if (e.g.) u_{12} or u_{21} is a unit.*

Conversely, if N is a 3×3 nilpotent matrix which has U as the northwest 2×2 corner, the previous relations hold for $a = n_{13}, b = n_{23}, x = n_{31}$ and $y = n_{32}$.

Proof. To simplify the writing we use block multiplication. We search for $N = \begin{bmatrix} U & \alpha \\ \beta & -t \end{bmatrix}$ where $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$ is a column, $\beta = \begin{bmatrix} x & y \end{bmatrix}$ is a row and $t = \mathrm{Tr}(U) = u_{11} + u_{22}$. Notice that already $\mathrm{Tr}(N) = 0$.

Then $N^2 = \begin{bmatrix} U^2+\alpha\beta & U\alpha - t\alpha \\ \beta U - t\beta & \beta\alpha + t^2 \end{bmatrix}$ where $\beta\alpha = ax+by$. Here $\mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\alpha) = \beta\alpha$ and $\mathrm{Tr}(U^2) = \mathrm{Tr}(U)^2 - 2\det(U)$. Hence $0 = \mathrm{Tr}(N^2) = 2\mathrm{Tr}(U)^2 - 2\det(U) + 2\beta\alpha$ implies

$$\beta\alpha = \det(U) - \mathrm{Tr}(U)^2 \tag{1}$$

Further, we need $\det(N) = 0 = bxu_{12} + ayu_{21} - axu_{22} - byu_{11} - t\det(U)$ that is

$$bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = \mathrm{Tr}(U)\det(U) \tag{2}$$

This way conditions (1)–(2) form the linear system of two equations with coefficients in R and with four integer unknowns, namely a, b, x and y, which is stated above. The example is obvious: denoting $m = \det(U) - \mathrm{Tr}(U)^2$ and $l = \mathrm{Tr}(U)\det(U)$, if $u_{12} \in U(R)$, take $a = 0, y = m, b = 1$ and $x = (l + mu_{22})u_{12}^{-1}$, respectively $x = 0, b = m, y = 1$ and $a = (l + mu_{22})u_{21}^{-1}$ if $u_{21} \in U(R)$.

The converse follows since $\det(N) = \mathrm{Tr}(N) = \mathrm{Tr}(N^2) = 0$ were exactly the conditions equivalent with (1) and (2), together with $n_{33} = -\mathrm{Tr}(U)$. □

Remarks. 1) The system has the trivial solution (i.e. $a = b = x = y = 0$) iff $\det(U) = \mathrm{Tr}(U) = 0$, that is iff U is nilpotent.

2) Condition (2) can be equivalently written as $\det(U) - \det(\alpha\beta + U) = t\det(U)$.

As for idempotent 3×3 matrices we prove the following

Proposition 2. *A 3×3 matrix $E = \begin{bmatrix} F & -\alpha \\ -\beta & t \end{bmatrix}$ where F is a 2×2 matrix, $\alpha = \begin{bmatrix} a \\ b \end{bmatrix}$ is a column, $\beta = \begin{bmatrix} x & y \end{bmatrix}$ is a row and $t \in R$, is idempotent iff (3) $F^2 + \alpha\beta = F$; (4) $(F + (t-1)I_2)\alpha = 0$; (5) $\beta(F + (t-1)I_2) = 0$ and $\beta\alpha = t - t^2$.*

Further suppose U is a 2×2 matrix which satisfies (1) such that $t = \mathrm{Tr}(U)$. Then (6) $\det(U) = \mathrm{Tr}(U)$.

Proof. Since by block multiplication $E^2 = \begin{bmatrix} F^2+\alpha\beta & -F\alpha - t\alpha \\ -\beta F - t\beta & \beta\alpha + t^2 \end{bmatrix}$, the conditions result just by equalizing the entries of E^2 and E. As for (6), since (1) is $\beta\alpha = \det(U) - \mathrm{Tr}(U)^2 = \det(U) - t^2$, (6) follows using $\beta\alpha = t - t^2$. □

2. An example

In his 2006 Ph.D. thesis, Diesl stated the following question: *If R is a nil-clean ring and $e \in R$ is a full idempotent* (that is, an idempotent such that $ReR = R$), *is the corner ring eRe necessarily a nil-clean ring?*

Denote by $Id(R)$ the idempotents, by $N(R)$ the nilpotent elements and by $NC(R) = Id(R) + N(R)$, the set of all nil-clean elements in a ring R. If $e \in Id(R)$ then $Id(eRe) = (eRe) \cap Id(R)$ and $N(eRe) = (eRe) \cap N(R)$.

While these equalities do provide a relation between the two sets $NC(eRe)$ and $NC(R)$, these are far from sufficient for answering the above question.

In what follows we show that for a general element $a \in eRe$, $a \in NC(R)$ may not imply that $a \in NC(eRe)$, that is, $(eRe) \cap NC(R) \subseteq NC(eRe)$ does not hold in general (even) for full idempotents $e \in R$.

The example is found in $\mathcal{M}_3(\mathbf{Z})$, that is 3×3 integral matrices, using the full idempotent $e = \mathrm{diag}(1; 1; 0) \in \mathcal{M}_3(\mathbf{Z})$. This way we identify eRe with $\mathcal{M}_2(\mathbf{Z})$ (which corresponds to the 2×2 north–west "corner" of $\mathcal{M}_3(\mathbf{Z})$). Thus, we are looking for a 2×2 integral matrix A, *which is not* nil-clean in eRe. As an element of eRe, A is identified with the 3×3 matrix $\mathrm{diag}(A; 0) \in \mathcal{M}_3(\mathbf{Z})$, matrix *which should be* nil-clean in $\mathcal{M}_3(\mathbf{Z})$.

The example. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$. It is easy to see that nil-clean integral matrices have trace 0, 1 or 2, depending on the idempotent which appears in their decomposition (nilpotent matrices have zero trace). The trace is $= 1$ if the idempotent is not trivial, it is $= 0$ if the idempotent is 0_2 and it is $= 2$ *only* if the idempotent is I_2. Now, since $\mathrm{Tr}(A) = 2$, this matrix would be nil-clean only if $A - I_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ is nilpotent, which fails since $\det(A - I_2) = -2$. Hence A is *not* nil-clean.

However

$$\mathrm{diag}(A;0) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = E + N = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

is a nil-clean decomposition with $E^2 = E$ and $N^3 = 0_3$.

3. How this example was found

As mentioned in the previous section, we are looking for a 2×2 (integral) matrix A, which is not nil-clean such that $\mathrm{diag}(A; 0)$ is nil-clean.

According to the completion results obtained in Section 2, for a 2×2 (integral) matrix A, $\mathrm{diag}(A; 0)$ is nil-clean iff $A = F + U$ with two matrices $U, F \in \mathcal{M}_2(\mathbf{Z})$, a 2×1 column α and a 1×2 row β such that

(1) $\beta\alpha = ax + by = t - t^2$,
(2) $bxu_{12} + ayu_{21} - axu_{22} - byu_{11} = t^2$,
(3) $F^2 + \alpha\beta = F$,
(4) $(F + (t-1)I_2)\alpha = 0$,

(5) $\beta(F + (t-1)I_2) = 0$,
(6) $t = \det(U) = \mathrm{Tr}(U)$.

Since both $(F + (t-1)I_2)\alpha = 0$ and $\beta(F + (t-1)I_2) = 0$, are homogeneous linear systems in (a, b) respectively (x, y), if $\det(F+(t-1)I_2) = \det(F)+(t-1)\mathrm{Tr}(F)+(t-1)^2 \neq 0$ then α and β are zero column respectively row. In this case, by Remark 1, U is nilpotent and (by (3)) F idempotent and so the sum $A = F + U$ is nil-clean.

Thus, in the sequel we assume

(7) $\det(F) + (t-1)\mathrm{Tr}(F) + (t-1)^2 = 0$.

We start by inspecting the equation (3). Since $\alpha\beta = \begin{bmatrix} ax & ay \\ bx & by \end{bmatrix}$, necessary conditions for equation (3) $F^2 + \alpha\beta = F$ are: (i) $F^2 - F$ has equal products of entries on diagonals, and (ii) $\mathrm{Tr}(F^2 - F) = -\mathrm{Tr}(\alpha\beta) = -\beta\alpha = t^2 - t$.

By computation, we find

(i) $\det(F)[\det(F) - \mathrm{Tr}(F) + 1] = 0$, i.e., $\det(F) = 0$ or $\det(F) = \mathrm{Tr}(F) - 1$, and
(ii) $\mathrm{Tr}(F)^2 - 2\det(F) - \mathrm{Tr}(F) = t^2 - t$.

We distinguish two cases: $\det(F) = 0$ or $\det(F) \neq 0$.

In the first case we show that either (integral) matrices F, U cannot be constructed, or else $A = F + U$ is nil-clean.

The construction of the example given in the previous section will follow from subcase (b) of Case 2.

Case 1. If $\det(F) = 0$, then (from (ii)) $\mathrm{Tr}(F) \in \{t, 1-t\}$. By (7) we obtain $(t-1)[\mathrm{Tr}(F) + t - 1] = 0$ and so $t = 1$ or $\mathrm{Tr}(F) = 1 - t$.

If $t = 1$, then by (ii), $\mathrm{Tr}(F) \in \{0, 1\}$.

First we discard the case $\det(F) = 0$ *and* $\mathrm{Tr}(F) = 1$.

Cayley–Hamilton theorem shows that F is idempotent. It is not hard to show that (4) and (5) combined with (6), contradict (2).

Next, if $\det(F) = \mathrm{Tr}(F) = 0$ *and* $\det(U) = \mathrm{Tr}(U) = 1$ *then* $\mathrm{Tr}(F) = 1 - \mathrm{Tr}(U)$ and this case may be included in the next case.

Finally if $\det(F) = 0$ *and* $\mathrm{Tr}(F) = 1-t$, notice that $\mathrm{Tr}(A) = \mathrm{Tr}(F+U) = 1-t+t = 1$ so that Cayley–Hamilton theorem gives $A^2 - A + \det(A).I_2 = 0_2$. Since the case $t = 0$ was already covered in Remark 1, for $t \neq 0$ we show that $\det(A) = 0$ and so A is idempotent (and so nil-clean).

Cayley–Hamilton theorem for F gives $F^2 = (1-t)F$ and then (3) gives $\alpha\beta = tF$. (Notice that this implies (1): $\beta\alpha = \mathrm{Tr}(\beta\alpha) = \mathrm{Tr}(\alpha\beta) = t\mathrm{Tr}(F) = t(1-t) = t - t^2$). Finally

$$\begin{aligned}\det(A) &= \det(F+U) = \det(F) + \det(U) + f_{11}u_{22} + f_{22}u_{11} - f_{12}u_{21} - f_{21}u_{12} \\ &= 0 + t - t = 0.\end{aligned}$$

Indeed, since $\alpha\beta = \begin{bmatrix} ax & ay \\ bx & by \end{bmatrix} = tF$, we have $ax = tf_{11}$, $ay = tf_{12}$, $bx = tf_{21}$ and $by = tf_{22}$. Replacement in (2) gives $t(f_{11}u_{22} + f_{22}u_{11} - f_{12}u_{21} - f_{21}u_{12}) = t^2$ and so (here $t \neq 0$) $f_{11}u_{22} + f_{22}u_{11} - f_{12}u_{21} - f_{21}u_{12} = t$, as claimed.

Case 2. If $\det F \neq 0$ then (by (i)), $\det(F) = \mathrm{Tr}(F) - 1$, and replacing in (ii), $\mathrm{Tr}(F)^2 - 3\mathrm{Tr}(F) + 2 = t^2 - t$, a degree two equation in $\mathrm{Tr}(F)$. Here $\Delta = (2t-1)^2$, so $\mathrm{Tr}(F) \in \{t+1, 2-t\}$ and accordingly $\det(F) = \{t, 1-t\}$.

(a) $\mathrm{Tr}(F) = t+1$, $\det(F) = t$. Replacing in (7), we get $2t^2 - t = 0$ with only $t = 0$ integer solution, that is $\mathrm{Tr}(F) = 1$ and $\det(F) = 0$. From Cayley–Hamilton theorem $F^2 - F = 0_2$ and so F is idempotent. Since by (6), $\det(U) = \mathrm{Tr}(U) = 0$, by Remark 1, U is nilpotent and so $A = F + U$ is nil-clean.

(b) $\mathrm{Tr}(F) = 2 - t$, $\det(F) = 1 - t$ (with $t \neq 1$). Now (7) holds for every $t \in \mathbf{Z}$.

Cayley–Hamilton theorem for F gives $F^2 - (2-t)F + (1-t)I_2 = 0_2$, or $F^2 - F = (1-t)(F - I_2) = -\alpha\beta$. By (1), $\beta\alpha = t(1-t)$, and we get $(1-t)^2(F-I_2)^2 = \alpha\beta\alpha\beta = t(1-t)\alpha\beta$.

Summarizing, an example of a 2×2 (integral) matrix A, which is not nil-clean such that $\mathrm{diag}(A;0)$ is nil-clean, must necessarily satisfy the conditions in this last subcase. Since $t \in \mathbf{Z}$, it is reasonable to check some small (positive) values for t.

To keep this exposition short, one can check that the matrices A obtained for $t = 0$ are also nil-clean and so (since $t \neq 1$) the next case to investigate is $t = 2$. The matrix U was chosen not to be nilpotent nor a unit (otherwise we obtain again nil-clean A), and for $U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, (1) and (2) give a linear system to be solved in α and β (only integer solutions are suitable). Finally, from the relations above $F^2 - F = -(F - I_2) = -\alpha\beta$, we obtain $F = I_2 + \alpha\beta$, and this gives our desired example.

References

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