COCOMPACT LATTICES

Grigore Călugăreanu

AMS Subject Classification 06 C 05, 06 C 20; keywords: cocompact, algebraic, inductive, reducible lattices, compact, essential, pseudocomplement, superfluous elements, socle and radical of a lattice

Abstract

A lattice L is called cocompact if its dual L^0 is compact. If M is a R-module the lattice $S_R(M)$ of all the submodules of M is cocompact iff M is finitely cogenerated. Most of the properties of these modules are proved in the latticial general setting.

1 Introduction

A complete lattice L is called **cocompact** if each discover of 0 has a finite subdiscover, i.e. for every subset X of L such that $\bigwedge X = 0$ there is a finite subset F of X such that $\bigwedge F = 0$. Obviously, L is cocompact iff the dual L^0 is compact. An element $a \in L$ is called **cocompact** if the sublattice $a/0$ is cocompact.

The following characterization is well-known: a complete lattice L is artinian iff for each subset A of L there is a finite subset F of A such that $\bigwedge F = \bigwedge A$. Hence

Remark 1.1 *Every artinian lattice is cocompact.*

Remark 1.2 *If L is a cocompact lattice, for each $0 \neq a \in L$ the sublattice $a/0$ is also cocompact.*

Our main result is the theorem 2.2: *let L be an algebraic lattice. L is cocompact iff the socle $s(L)$ is compact and essential in L.*

In the sequel we will use only complete lattices L and the following definitions: a non-zero element e is called **essential** if for every element $a \in L$, $a \wedge e = 0$ implies $a = 0$ and **superfluous** dually; the **socle** $s(L)$ of a lattice is defined as the join of all the atoms of L and, dually, the **radical** $r(L)$ as the meet of all the maximal elements (dual atoms) of L; a lattice L is called **atomic** if for every $0 \neq a \in L$ the sublattice $a/0$ contains atoms, **inductive** if for each $a \in L$ and every chain $\{b_i\}_{i \in I}$, $\forall i \in I$, $a \wedge b_i = 0 \Rightarrow a \wedge (\bigvee_{i \in I} b_i) = 0$ and every sublattice (interval) of L has this property, **(R3)** if for every $a \neq 1$, a essential in L, $1/a$ contains atoms, **reducible** if the socle $s(L) = 1$, and **torsion** if for each $a \neq 1$, $1/a$ contains atoms (see [1], [2] and [3]). As in [1] we use the following definitions: we say that a set $\{a_i\}_{i \in I}$ of elements of a lattice is **independent** if $a_i \wedge \left(\bigvee_{j \neq i} a_j\right) = 0$ for all $i \in I$; in this case we denote the join $\bigvee_{i \in I} a_i$ by $\bigoplus_{i \in I} a_i$ and call it the **direct sum (join)**. For all the notions (such as: compact element, pseudocomplement in a lattice and algebraic, artinian, pseudocomplemented or upper continuous lattice) and notation we refer to [4],[5] and [6].

2 Results

Lemma 2.1 *Let a be an essential element of a lattice L. If $a/0$ is cocompact then L is also cocompact.*

Proof. Let $\{a_i\}_{i \in I}$ be a family of non-zero elements of L such that $\bigwedge_{i \in I} a_i = 0$. The element a being essential in L, we have $a \wedge a_i \neq 0$ and $0 = a \wedge \left(\bigwedge_{i \in I} a_i\right) = \bigwedge_{i \in I} (a \wedge a_i)$. Hence $\{a \wedge a_i\}_{i \in I}$ is a discover of 0 in $a/0$, and $a/0$ being cocompact there is a finite subset $F \subseteq I$ such that $0 = \bigwedge_{i \in F} (a \wedge a_i) = a \wedge \left(\bigwedge_{i \in F} a_i\right)$. Finally, a being essential , $\bigwedge_{i \in F} a_i = 0$ and L is cocompact.□

Lemma 2.2 *In an algebraic, modular, reducible lattice the radical $r(L) = 0$.*

Proof. We verify that for each atom s, $s \wedge r(L) = 0$ (this suffices in a reducible lattice, which is also atomic). Reducible, inductive lattices being complemented (each algebraic lattice is upper continuous, each upper continuous lattice is inductive), let m be a complement of s. Using modularity, one easily proves that m is maximal in L. Hence $s \wedge m = 0$ implies $s \wedge r(L) = 0$.□

Lemma 2.3 *In an algebraic cocompact lattice L the socle $s(L)$ is essential in L (more can be proved; see the last theorem).*

Proof. Let $a \in L$ be such that $s(L) \wedge a = 0$ or, equivalently, $s(a/0) = 0$. The sublattice $a/0$ being algebraic, the socle is also the join of all the essential elements (of $a/0$) and so, being cocompact $0 = \bigwedge_{i \in F} e_i$ for a finite family of essential elements $\{e_i\}_{i \in F}$ of $a/0$. Hence 0 is essential in $a/0$ and so $a = 0$.□

Remark 2.1 *In every atomic lattice the socle is essential. If the lattice L is inductive then the converse is also true.*

Indeed, if $a \neq 0$ then $0 \neq s(L) \wedge a \in s(L)/0$ an inductive and reducible lattice. Using Theorem 9.2 from [1], each element of L is a direct sum of atoms. Hence $a/0$ contains atoms.

So, *cocompact algebraic lattices are atomic.* Moreover, one can prove that *algebraic cocompact* **(R3)** *lattices are torsion lattices* (cf.[2]).

Proposition 2.1 *A lattice L is artinian iff for every $a \neq 1$ the sublattice $1/a$ is cocompact.*

Proof. Each sublattice of an artinian lattice is clearly artinian and so, by the Remark 1.1, is cocompact. Conversely, let $... \leq a_n \leq ... \leq a_2 \leq a_1$ be an ascending chain of elements in L. If $a = \bigwedge_{n \in \mathbf{N}} a_n$ then $\{a_n\}_{n \in \mathbf{N}}$ is surely a discover of a in $1/a$. The sublattice $1/a$ being cocompact there is a finite subset $F \subset \mathbf{N}$ such that $a = \bigwedge_{n \in F} a_n$. Hence $a = a_m$ where $m = \min(F)$ and $a_{m+l} = a_m$ for each $l \in \mathbf{N}$, so the chain is finite and L is artinian.$\square$

Proposition 2.2 *If for an element a of an modular inductive lattice L the sublattices $a/0$ and $1/a$ are cocompact then the lattice L is cocompact.*

Proof. If $a = 0$ nothing remains to be proved. If $a \neq 0$ let $0 = \bigwedge_{i \in I} b_i$ a discover of 0 in L. Then $\bigwedge_{i \in I}(a \wedge b_i) = a \wedge (\bigwedge_{i \in I} b_i) = a \wedge 0 = 0$, is a discover of 0 in $a/0$. By cocompacity, there is a finite subset F of I such that $0 = \bigwedge_{i \in F}(a \wedge b_i) = a \wedge (\bigwedge_{i \in F} b_i)$. If $\bigwedge_{i \in F} b_i = 0$ (e.g. if a is essential in L) the proof is complete. If $\bigwedge_{i \in F} b_i \neq 0$ then let c be a pseudocomplement of a which contains $\bigwedge_{i \in F} b_i$. We have $\bigwedge_{i \in F} b_i \in c/0 = c/(a \wedge c) \cong (a \vee c)/a \subseteq 1/a$ (the isomorphism is given by modularity). The sublattice $1/a$ being cocompact , $(a \vee c)/a$ and hence $c/0$ are also cocompact. $0 = \bigwedge_{i \in I}(c \wedge b_i)$ being a discover of 0 in $c/0$ there is a finite subset G of I such that $0 = \bigwedge_{i \in G}(c \wedge b_i) = c \wedge (\bigwedge_{i \in G} b_i)$. Now, for $b = \bigwedge_{i \in F \cup G} b_i$ we have $b \leq \bigwedge_{i \in F} b_i \leq c$ and $c \wedge b \leq c \wedge (\bigwedge_{i \in G} b_i) = 0$ so that $b = 0$, and we have the required finite discover of 0.$\square$

This is a purely laticial proof which avoids the injective hull , a non-latticial notion (see [7]).

Consequence 2.1 *A direct sum of cocompact elements in an inductive modular lattice is cocompact.*

Proof. If $a/0$,$b/0$ are cocompact and $a \oplus b = 1$ (b is a complement of a) then by modularity $b/0 = b/(a \wedge b) \cong (a \vee b)/a = 1/a$ and we use the previous Proposition.$\square$

Proposition 2.3 *Let L be an algebraic cocompact lattice with the radical $r(L) = 0$. Then L is reducible and compact.*

Proof. From the third lemma we already know that L is atomic. The lattice L being algebraic the radical is also the union of all the superfluous elements. Hence the condition $r(L) = 0$ implies that the only superfluous element of L is 0. Equivalently, for each $0 \neq a \in L$ there is an $x \neq 1$ such that $a \vee x = 1$. In particular, each atom has a complement (maximal if L is also modular). Indeed, if s is an atom, as mentioned, there is an $m \neq 1$ such that $s \vee m = 1$. But $s \wedge m \in \{0, s\}$ and $s \wedge m = s$ implies $s \leq m$ or $m = 1$. Hence $s \wedge m = 0$ and s has a complement.

Now if the socle $s(L) \neq 1$ then let $x \neq 1$ be such that $s(L) \vee x = 1$ ($L \neq 0$ atomic implies $s(L) \neq 0$). One gets an atom which would not be contained in $s(L)$, contradiction. Hence L is reducible.

Finally, L being cocompact, the radical $r(L)$, which is the intersection of the maximal elements, and so is a discover of 0, must give a finite subdiscover of 0 by , say n maximal elements. The compacity of L follows now by induction on n. One verifies that each cover of 1 has a finite subcover . The dual analogon of this proof is detailed in the proof of the next theorem.□

Theorem 2.1 *Let L be an algebraic, reducible and modular lattice. Then the following conditions are equivalent: (a) L is compact; (b) L is cocompact; (c) 1 is a finite direct sum of atoms.*

Proof. $(a) \Rightarrow (c)$ L being reducible and inductive we have $1 = \bigoplus_{i \in I} s_i$, with s_i atoms (see [1]). But $\{s_i\}_{i \in I}$ is a cover for 1 , compact element , so a finite subset $F \subseteq I$ exists such that $1 = \bigoplus_{i \in F} s_i$.

$(c) \Rightarrow (b)$ If $\overset{n}{\bigoplus}_{i=1} s_i = 1$ we prove that every discover of $0 = \bigwedge_{i \in I} a_i$ has a finite subdiscover by induction on n. If $n = 1$ the assertion is obvious. We assume that the assertion is true for each lattice such that 1 is a direct sum of at most $n - 1$ atoms.

First, observe that there is a $k \in I$ such that $a_k \wedge s_n = 0$. Indeed, otherwise $a_i \wedge s_n = s_n$ for every $i \in I$ or $s_n \leq \bigwedge_{i \in I} a_i$, contradiction. The element a_k is also a direct sum of at most $n - 1$ atoms (the modularity is needed for the use of the Jordan-Hölder theorem). By the induction hypothesis a finite subset of the family $\{a_i \wedge a_k\}_{i \in I}$ has the intersection 0. Hence L is cocompact.

$(b) \Rightarrow (a)$ follows from the second lemma (which assures $r(L) = 0$) and the previous Proposition.□

Remark 2.2 *The implication $(c) \Rightarrow (a)$ follows easily:*

in an upper continuous lattice every atom is compact and finite unions of compact elements are compact.

Theorem 2.2 *Let L be an algebraic lattice. Then L is cocompact iff the socle $s(L)$ is compact and essential in L.*

Proof. If L is cocompact and $a \neq 0$ then clearly $a/0$ is also cocompact. Hence the sublattice $s(L)/0$ is cocompact and reducible. By the above theorem $a/0$ is also compact, i.e. $s(L)$ is compact in L. The essentialness follows from the third lemma.

Conversely, if $s(L)$ is compact then $s(L)/0$ is reducible and compact and hence cocompact, again by the above theorem. The socle $s(L)$ being also essential in L, L is cocompact by the first lemma.□

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