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CLEAN INTEGRAL 2×2 MATRICES

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Abstract

Clean integral 2×2 matrices are characterized. Up to similarity the strongly clean matrices are completely determined and large classes of uniquely clean matrices are found. In particular, classes of uniquely clean matrices which are not strongly clean are found.

1. Introduction

The important role of idempotents, nilpotent elements and units in Ring Theory was recognized already a century ago. Considering elements which are sums of two such elements is more recent. Sums of an idempotent and a unit (called *clean* elements) were defined by Nicholson (1977) in [7]. Sums of an idempotent and a nilpotent element (called *nil-clean* elements) were considered by Diesl (2006) in his Ph. D. thesis, and finally sums of a unit and a nilpotent element (called *fine* elements) were considered by the author and T. Y. Lam (2015) in [5]. Further, a ring (with identity) is called *clean* if all its elements are clean, *nil-clean* if all its elements are nil-clean and *fine* if all its nonzero elements are fine. An element is called *uniquely* clean (or nil-clean or fine) if it has only one clean (or nil-clean, or fine) decomposition, and *strongly* clean (or nil-clean or fine), if the components of the decompo-

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sition commute. If $a = e + u$ with $e^2 = e$ and unit u, we say that e is *the* idempotent in this clean decomposition.

Analogously, uniquely or strongly clean (or nil-clean or fine) rings were defined and between all these classes, several inclusions were (easily) established.

The following inclusions are well-known: nil-clean rings are clean (see [6]), uniquely clean rings are Abelian (i.e., the idempotents are central; see [8]), uniquely nil-clean rings are Abelian (see [6]) and fine rings are simple (see [5]). Therefore uniquely clean rings are strongly clean and uniquely nil-clean rings are strongly nil-clean.

Despite all these inclusions, when it comes to comparing the corresponding types of elements, everything fails. So far mostly all examples were chosen in $\mathcal{M}_2(\mathbf{Z})$, the ring of all 2×2 integral matrices.

In this paper we investigate this ring as far as cleanness of elements is concerned. Clearly, such an investigation should answer the following questions:

When is a given 2×2 integral matrix clean, strongly or uniquely clean, respectively?

This is the content of sections 3, 4 and 5. Characterizations for clean matrices are given, strongly clean matrices are determined up to similarity and large classes of uniquely clean matrices are found.

2. Similarity and clean 2×2 matrices

Two $n \times n$ matrices A, B over any unital ring R, are *similar* (or *conjugate*) if there is an invertible matrix U such that $B = U^{-1}AU$. Since similarity is an equivalence relation, a partition of $\mathcal{M}_n(R)$ corresponds to it. The subsets in this partition are called *similarity classes*.

The notions of clean, uniquely clean and strongly clean are similarity invariants.

In the sequel $R = \mathbf{Z}$ and $n = 2$, that is, we deal with 2×2 integral matrices. Our goal is to find all the similarity classes of clean matrices. In doing so, it is natural to choose in each similarity class a special representative, namely a representative which uses a special idempotent, the matrix unit E_{11}. Recall that the characteristic polynomial of a 2×2 matrix A is given by $x^2 - \mathrm{Tr}(A)x + \det(A)$, with $\mathrm{Tr}(A)$ denoting the trace of the matrix A.

We first recall from [3] the following

Definition 1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 integral matrix and $D = \mathrm{Tr}(A)^2 - 4\det(A)$. If D is a square (e.g. $\det(A) = 0$), that is, the characteristic polynomial of the matrix factors over the integers, say, $f(x) = (x - \alpha)(x - \delta)$, where $\alpha \geq \delta$, then, for $\alpha \neq \delta$ the matrix $\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}$ is *reduced* if $0 \leq \beta < \alpha - \delta$, and, for $\alpha = \delta$, if $\beta \geq 0$.

EXAMPLE. The matrix unit $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is reduced, I_2 is also (idempotent and) reduced, but E_{22} is *not reduced*. Actually E_{11} and E_{22} are similar: for $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $U^2 = I_2$ and $UE_{11}U = E_{22}$.

Next, also from [3] (Theorem 5.2), recall the following

THEOREM 1. *Let $M \in \mathcal{M}_2(\mathbf{Z})$ and assume that the characteristic polynomial of M factors over $\mathbf{Z}$. Then M is similar to a reduced matrix. Moreover, this class representative is unique thus no two different reduced matrices are similar.*

This theorem has an important consequence.

COROLLARY 2. *Any non trivial 2×2 idempotent integral matrix is similar to E_{11}.*

That is, all nontrivial idempotent matrices belong to the same similarity class and E_{11} is the only reduced representative in this class.

EXAMPLES. 1) If $E = \begin{bmatrix} 1 & 0 \\ s & 0 \end{bmatrix}$ then with $P = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$ we get $E_{11} = P^{-1}EP$.
2) If $E = \begin{bmatrix} s+1 & -s-1 \\ s & -s \end{bmatrix}$ then with $P = \begin{bmatrix} s+1 & 1 \\ s & 1 \end{bmatrix}$ we get $E_{11} = P^{-1}EP$.

Given any clean matrix $A = E + U$, if $E_{11} = P^{-1}EP$ for a suitable invertible matrix P, $P^{-1}AP = E_{11} + P^{-1}UP$ is similar to A and will be called the *E_{11}-reduction* of A.

A clean element (in any ring) will be called *trivial* if its decomposition uses a trivial idempotent. That is, the trivial clean elements are the units and the elements $1 + u$ with $u \in U(R)$. These elements are obviously strongly clean. All the other clean elements will be called *nontrivial*.

Trivial clean elements are easy to characterize in $\mathcal{M}_2(\mathbf{Z})$: the units are precisely the matrices with determinant $= \pm 1$ and the other type is characterized by $\det(A) - \mathrm{Tr}(A) \in \{-2, 0\}$ (i.e. $A - I_2$ is a unit). Since these use trivial idempotents (with trace $= 0$ or $= 2$), which are not similar to E_{11}, for these an E_{11}-reduction cannot be made.

A special case of the following characterization was hidden in [2].

THEOREM 3. *A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nontrivial clean iff the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz + \det(A) - d = \pm 1 & (2) \end{cases}$$

with unknowns x, y, z, has at least one solution over $\mathbf{Z}$. *If $b \neq 0$ and (2) holds, then (1) is equivalent to*

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0. \tag{3}$$

PROOF. Any nontrivial idempotent is characterized by zero determinant and trace $= 1$. The general matrix A is clean iff there is a nontrivial idempotent $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ i.e., $\mathrm{Tr}(E) = 1$ and $-\det(E) = x^2 + x + yz = 0$, that is (1), such that $\det(A - E) = \pm 1$. If (1) holds, the last condition amounts to $(a-d)x + cy + bz + \det(A) - d = \pm 1$, that is (2).

If $b \neq 0$ (the case $c \neq 0$ is symmetric), multiplying (1) by b and eliminating z, we get the Diophantine equation

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0,$$

that is (3). □

This characterization is straightforward, but useful since the solutions of a degree two Diophantine equation in two unknowns (if any) are instantly found using computer aid (see [1]).

REMARKS 1. 1) The Theorem remains true for matrices over any (commutative) integral domain if we replace ± 1 with the set of units (indeed, a square matrix over a commutative ring is invertible iff its determinant is a unit and Cayley-Hamilton's theorem holds).

2) For further use, observe that equations (1) or (3) have always the solutions $(x, y) = (0, 0)$ and $(x, y) = (-1, 0)$ with an arbitrary z. Moreover, (3) has also the solution $(x, y) = (a, b)$ iff $(d \pm 1)b = 0$.

3) There are two equations denoted (± 3) and two equations (± 2). Only solutions for either $(+3)$ with $(+2)$ or else for (-3) with (-2) are suitable.

4) In [2], the matrix $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$ was shown to be nil-clean but not clean. Using computer aid (see [1]) it is readily checked that this matrix is not clean. Indeed, the equations (± 3) are $9x^2 - 5xy + 7y^2 + 9x - 58y = 0$, resp. $9x^2 - 5xy + 7y^2 + 9x - 60y = 0$. The first has only $(0, 0)$ and $(-1, 0)$ as solutions, the second has two more: $(1, 9)$ and $(a, b) = (3, 9)$.

Since (2) is $5x - 7y + 9z + 59 = \pm 1$, clearly $(0, 0)$ and $(-1, 0)$ are not suitable. Since $(d \pm 1)b \neq 0$, (a, b) is also not suitable so that only $(1, 9)$ remains. However, $(1, 9)$ is a solution for (-3) but verifies only $(+2)$. Hence the equations have no integer solutions and the matrix is not clean.

The case $b = c = 0$ (i.e. A is diagonal) was not included in the previous theorem. As in the previous theorem, trivial clean diagonal matrices must be excepted (i.e. those who use an idempotent of trace 1). There are four units, namely $I_2, -I_2, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, or I_2 added to any of these units, that is $2I_2, 0_2, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. The (nontrivial) cleanness of the remaining diagonal matrices is characterized by the following

THEOREM 4. *Let* $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *be a diagonal integral matrix.*

(i) If $a = d \notin \{-1, 0, 1, 2\}$ *the matrix is not clean;*

Suppose $a > d$ *but* $(a, d) \notin \{(1, -1), (2, 0)\}$.

(ii) If $a - d$ *is even then the matrix is not clean;*

(iii) If $a - d$ *is odd then* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *is clean iff* $d(d-1) \equiv \pm 1 (\mathrm{mod}(a - d))$*; a necessary condition is* $a - d = m(m-1) \pm 1$ *for a suitable positive integer* m.

PROOF. Notice that if $b = c = 0$ the cleanness of A is equivalent to (± 2), which is $(a - d)x + \det(A) - d = (a - d)x + (a - 1)d = \pm 1$, because if an integer x satisfying (± 2) exists, we can always chose y, z such that (1) holds. This is equivalent to $(a - 1)d \equiv \pm 1 (\mathrm{mod}(a - d))$.

Therefore, diagonal matrices, if nontrivial clean, are never uniquely clean (indeed, for any integer x, $x(x + 1) = -yz$ has more than one solution for y, z).

(i) If $a = d$ then $A = aI_2$ and the condition above amounts to $a^2 - a \mp 1 = 0$, which has no integer solutions.

(ii) Indeed, in this case a, d have the same parity but $a - 1, d$ are of opposite parity. Therefore $(a - 1)d$ is even and $(a - 1)d \mp 1$ is odd. Hence it is not divisible by $a - d$.

(iii) The condition is obvious for $a - d = 1$, so we can assume $a - d \geq 3$. By adding and subtracting d^2, we write (2) equivalently as $d(d - 1) \equiv \pm 1 \pmod{(a - d)}$.

Finally, for given $a - d$, there exists an integer x such that $x(x - 1) \equiv \pm 1 \pmod{(a - d)}$ only if $a - d = m(m - 1) \pm 1$ for a suitable positive integer m (just view x as a unit in $\mathbf{Z}(a - d)$). □

REMARK 2. The necessary condition is also useful. For some small odd numbers it says that diagonal matrices with $a - d \in \{3, 5, 7, 11, 13, 19, 21\}$ may be clean for some values of d, but diagonal matrices with $a - d \in \{9, 15, 17, 23, 25\}$ are not clean for any value of d.

The following consequence of Theorem 3 will be useful for determining uniquely clean matrices whose idempotent is E_{11} (p. 51).

Corollary 5. *A matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *(is clean and) admits* E_{11} *as idempotent in a (nontrivial) clean decomposition iff* $\det(A) - d = \pm 1$.

Examples. 1) There are clean matrices with any determinant ($k \in \mathbf{Z}$), or any NE entry:

$$\begin{bmatrix} 0 & k \\ -1 & k-1 \end{bmatrix} = E_{11} + \begin{bmatrix} -1 & k \\ -1 & k-1 \end{bmatrix}.$$

Analogous, any SW entry.

2) There are clean matrices with any secondary diagonal (b, c):

$$\begin{bmatrix} 0 & b \\ c & 1-bc \end{bmatrix} = E_{11} + \begin{bmatrix} -1 & b \\ c & 1-bc \end{bmatrix}.$$

3) Any unit with $d = 0$ has a clean E_{11}-decomposition.

3. Strongly clean matrices

Since trivial clean matrices are strongly clean, the determination of strongly clean matrices reduces to nontrivial clean matrices. Up to similarity this is solved by E_{11}-reduction in our next result.

To simplify some of our statements we introduce the following

Definition 2. A clean matrix is called *basic* if it has a decomposition $E + U$ where $U \in \{\pm I_2, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$ (that is, the unit is diagonal).

Remark 3. Basic matrices may not be strongly clean.

For example $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is basic but not strongly clean. However, a basic matrix $E + U$ is strongly clean iff these are either of form $E \pm I_2$ or else of form $E \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, with diagonal idempotent E (i.e. $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b \in \{0, 1\}$).

Theorem 6. *Every nontrivial strongly clean matrix in* $\mathcal{M}_2(\mathbf{Z})$ *is similar to a basic matrix.*

Proof. Suppose A is nontrivial strongly clean in $\mathcal{M}_2(\mathbf{Z})$. Then A is similar to a strongly clean matrix B whose idempotent is E_{11}, that is, $B = E_{11} + U$ with a unit $U = \begin{bmatrix} s & t \\ v & w \end{bmatrix}$ such that E_{11} and U commute. However the equalities

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}s&t\\v&w\end{bmatrix}=\begin{bmatrix}s&t\\0&0\end{bmatrix}=\begin{bmatrix}s&t\\v&w\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}s&0\\v&0\end{bmatrix}$$

hold iff $t = v = 0$, that is, U must be a diagonal unit. Hence $s, w \in \{\pm 1\}$ and $U \in \left\{ I_2, -I_2, \begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix} \right\}$ and so B is basic. $\square$

Rephrasing, *up to similarity, the only (nontrivial) strongly clean matrices are the basic matrices described before the previous theorem.*

4. Uniquely clean matrices

Since we intend to approach uniquely clean matrices by using similarity and E_{11}-reduction, we must deal separately with *trivial clean matrices.*

The determination of uniquely trivial clean matrices consists answering the following two questions:

(a) Which units are uniquely clean? and

(b) Let U be a unit in $\mathcal{M}_2(\mathbf{Z})$. When is $I_2 + U$ uniquely clean?

In dealing with (a), for a unit U, three cases must be considered, corresponding to the discriminant $\Delta = \mathrm{Tr}^2(U) - 4\det(U)$ of the characteristic polynomial associated to U: the elliptic, the parabolic and the hyperbolic cases.

This problem was solved in [4]. It turns out that there are no uniquely clean units in the elliptic and in the parabolic cases and only few uniquely clean matrices in the hyperbolic case.

The following straightforward result will be useful.

Lemma 7. *Let*

$$E \in \{0_2, I_2, E_{11}, E_{22}, E_{11} + E_{12}, E_{11} + E_{21}, E_{21} + E_{22}, E_{12} + E_{22}\}.$$

Then a matrix $A = \begin{bmatrix}a&b\\c&d\end{bmatrix}$ *has a clean* E*-decomposition iff*

$$\begin{aligned}
\det(A) &= \pm 1,\\
\det(A) - \mathrm{Tr}(A) + 1 &= \pm 1,\\
\det(A) - d &= \pm 1,\\
\det(A) - a &= \pm 1,\\
\det(A) + c - d &= \pm 1,\\
\det(A) + b - d &= \pm 1,\\
\det(A) + b - a &= \pm 1,\\
\det(A) + c - a &= \pm 1,
\end{aligned}$$

Proving that clean matrices are uniquely clean may not be easy, even for concrete examples.

In the sequel, here are some infinite classes of (not similar) uniquely clean matrices.

First consider $N_n = \begin{bmatrix} n^2 & -n \\ n & -1 \end{bmatrix}$ for $n \geq 1$.

Proposition 8. *For any positive integer $n \geq 3$, the matrix N_n is uniquely clean.*

Proof. Clearly $N_n = E_{11} + \begin{bmatrix} n^2 - 1 & -n \\ n & -1 \end{bmatrix}$ is a clean decomposition but not strongly clean (by Theorem 6). The proof reduces to the following

Claim. *For $n \geq 3$ equations (± 3), each, have* only $(0, 0)$ *and* $(-1, 0)$ *as solutions.*

Equations (± 3) are now $-nx^2 - (n^2 + 1)xy - ny^2 - nx + (-1 \pm 1)y = 0$ and equations (± 2) are $(n^2 + 1)x + ny - nz + 1 = \pm 1$.

The reduction: The pair $(+3)$ with $(+2)$ has obviously the solution $(0, 0)$ with $z = 0$ which gives the clean decomposition above.

Has not solution $(-1, 0)$: $nz = -(n^2 + 1)$ implies n divides $n^2 + 1$ (impossible from $n \geq 2$: $\gcd(n; n^2 + 1) = 1$).

The pair (-3) with (-2) has not $(0, 0)$: $nz = 2$ has no integer solution for $n \geq 3$.

Has not solution $(-1, 0)$: $nz = -(n^2 - 1)$ implies n divides $n^2 - 1$ (impossible from $n \geq 1$: $\gcd(n; n^2 - 1) = 1$).

Proof of Claim. It suffices to show that $y = 0$ for any solution. Then from (1), $x(x + 1) = 0$ and so $x \in \{-1, 0\}$.

As for $(+3)$, consider the degree two equation in x, $nx^2 + (n^2 + 1)xy + ny^2 + nx = 0$. We show that for $y > 0$ the discriminant $D = [(n^2 + 1)y + n]^2 - 4n^2y^2$ cannot be a square. By easy computations we get $D = [(n-1)^2y + n][(n+1)^2y + n] = [(n^2 - 1)y + n]^2 + 4ny$. Since $n \geq 3$, we have $4ny < 2[(n^2 - 1)y + n] + 1$, and so $[(n^2 - 1)y + n]^2 < D < [(n^2 - 1)y + n + 1]^2$. Since D is strictly between two consecutive squares, D cannot be a square.

The case $y < 0$ is analogous and so is the proof for (-3). □

Remark 4. The *clean index* (i.e. the number of different clean decompositions) shows to some extent, how "far" a given matrix is from being uniquely clean.

N_1 is a *nilpotent* which has clean index ∞.

Indeed,

$$N_1 = \begin{bmatrix} 1 - t^2 & t^2 - t \\ -t^2 - t & t^2 \end{bmatrix} + \begin{bmatrix} t^2 & -1 - t^2 + t \\ 1 + t^2 + t & -1 - t^2 \end{bmatrix}$$

and also

$$N_1 = \begin{bmatrix} (t-1)^2 & -t(t-1) \\ (t-1)(t-2) & -t(t-2) \end{bmatrix} + \begin{bmatrix} 1-(t-1)^2 & -1+t(t-1) \\ 1-(t-1)(t-2) & -1+t(t-2) \end{bmatrix}.$$

N_2 has index 4.

Next, consider the matrices $M_n = \begin{bmatrix} n^2 & n \\ n & 1 \end{bmatrix}$ for $n \geq 1$.

PROPOSITION 9. *For every positive integer $n \geq 3$, the matrix M_n is uniquely clean.*

PROOF. The clean decomposition is $M_n = E_{11} + \begin{bmatrix} n^2-1 & n \\ n & 1 \end{bmatrix}$ (for (-3) and (-2)). Again the proof reduces to the following

CLAIM. *For $n \geq 3$ equations* (± 3)*, each, have* only $(0,0)$ *and* $(-1,0)$ *solutions.*

Equations (± 3) are now $nx^2 - (n^2-1)xy - ny^2 + nx + (1 \pm 1)y = 0$ and equations (± 2) are $(n^2-1)x + ny + nz - 1 = \pm 1$.

The reduction: $(0,0)$ is not a solution for the pair $(+3)$ with $(+2)$. Indeed, $nz = -2$ has no integer solution for $n \geq 3$.

Has not solution $(-1,0)$: $nz = n^2+1$ implies n divides n^2+1 (impossible from $n \geq 2$: $\gcd(n; n^2+1) = 1$).

The pair (-3) with (-2) has solution $(0,0)$: $z = 0$ which gives the E_{11}-decomposition.

Has not solution $(-1,0)$: $nz = n^2-1$ implies n divides n^2-1 (impossible from $n \geq 1$: $\gcd(n; n^2-1) = 1$).

PROOF OF CLAIM. This is analogous to the proof of the previous proposition. With similar notations, the proof (for $(+3)$ and $y > 0$) reduces to $[(n^2+1)y - n]^2 < D < [(n^2+1)y - n + 1]^2$, which is true, because for $n \geq 3$, $4ny < 2[(n^2+1)y - n] + 1$ holds. □

Similarly for $y < 0$ and for (-3). □

REMARK 5. M_1 has index 6 (any of the idempotents E_{11}, E_{22}, $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ yield a clean decomposition) and M_2 has index 3 (the idempotents E_{11}, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$).

Further, consider the matrices $L_n = \begin{bmatrix} n(n+1) & -n-1 \\ n & -1 \end{bmatrix}$ for $n \geq 1$.

Proposition 10. *For every positive integer $n \geq 3$, the matrix L_n is uniquely clean.*

Proof. The clean decomposition is $L_n = E_{11} + \begin{bmatrix} n^2+n-1 & -n-1 \\ n & -1 \end{bmatrix}$ (for (+3) and (+2)). Again the proof reduces to the following

Claim. *For $n \geq 3$ equations (± 3), each has* only $(0,0)$ *and* $(-1,0)$ *as solutions.*

The reduction: Equations (± 3) are now

$$-(n+1)x^2 - (n^2+n+1)xy - ny^2 - (n+1)x + (-1 \pm 1)y = 0$$

and equations (± 2) are

$$(n^2+n+1)x + ny - (n+1)z + 1 = \pm 1.$$

The pair (+3) with (+2) has obviously the solution $(0,0)$ with $z=0$ which gives the clean decomposition above.
Has not solution $(-1,0)$: $(n+1)z = -(n^2+n+1)$ implies $n+1$ divides n^2+n+1 (impossible from $n \geq 2$: $\gcd(n+1; n^2+n+1) = 1$).
The pair (-3) with (-2) does not have $(0,0)$ as a solution: $(n+1)z = 2$ has no integer solution for $n \geq 3$.
Has not solution $(-1,0)$: $(n+1)z = -(n^2+n-1)$ implies $n+1$ divides n^2+n-1 (impossible from $n \geq 1$: $\gcd(n+1; n^2+n-1) = 1$).

Proof of Claim. Again (as proofs of the previous two propositions), with similar notations, we reduce the claim (for (+3) and $y > 0$) to

$$[(n^2+n-1)y+n+1]^2 < D < [(n^2+n-1)y+n+2]^2$$

because for $n \geq 3$, $2[(n^2+n-1)y+n+1] > 4(n+1)y$.

A similar proof works for $y < 0$ and for (-3). □

Remarks 6. 1) L_1 is an *idempotent* which is clean of index ∞. Indeed,

$$L_1 = \begin{bmatrix} 1-t & t \\ 1-t & t \end{bmatrix} + \begin{bmatrix} 1+t & -2-t \\ t & -1-t \end{bmatrix}$$

and also

$$L_1 = \begin{bmatrix} 1+2t & -2-4t \\ t & -2t \end{bmatrix} + \begin{bmatrix} 1-2t & 4t \\ 1-t & -1+2t \end{bmatrix}.$$

L_2 has index 3 (the idempotents E_{11}, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$).

2) One can show that the matrices $K_n = \begin{bmatrix} n(n+1) & -n \\ n+1 & -1 \end{bmatrix}$ are uniquely clean for $n \geq 3$. Though $K_n = (L'_n)^T$, where

$$L'_n = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} L_n \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} n(n+1) & n+1 \\ -n & -1 \end{bmatrix},$$

this does not follow directly from the proof for L_n because it is known that a matrix might not be similar to its transpose.

It would be nice to find an example of uniquely clean matrix whose transpose is not uniquely clean.

Next, as already done for nontrivial strongly clean matrices, up to similarity, we want to determine the uniquely clean matrices whose idempotent is E_{11}. According to Corollary 5, we are searching for matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(A) - d = \pm 1$, such that $A = E_{11} + U$ is *the only* clean decomposition, i.e. such that for any idempotent $E \neq E_{11}$, $\det(A - E) \neq \pm 1$.

First we gather some necessary conditions.

PROPOSITION 11. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has a uniquely clean* E_{11}*-decomposition then all entries in* A *are nonzero. Moreover,* $a \neq d$, $b \neq a - d$ *and* $c \neq a - d$.

PROOF. As seen above $\det(A) = d \pm 1$ and for any idempotent $E \neq E_{11}$ we must have $\det(A - E) \neq \pm 1$. The conditions follow from Lemma 7. ☐

Finally, in order to give necessary and sufficient conditions for uniquely clean E_{11}-reduced matrices, we have to consider nontrivial idempotents $E = \begin{bmatrix} x+1 & y \\ z & -x \end{bmatrix}$ with $x^2 + x + yz = 0$ but not all three $x = y = z = 0$. This is somewhat analogous to the contrapositive of the statement of Theorem 3.

THEOREM 12. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix with only nonzero entries,* $a \neq d$ *and* $\det(A) - d = \pm 1$. *Then* A *is uniquely clean iff the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (a-d)x + cy + bz = 0 & (2) \end{cases}$$

has only the zero solution. Equivalently, we can take the system (2) together with

$$bx^2 - (a-d)xy - cy^2 + bx = 0. \quad (3)$$

So what remains to be done is to find necessary and sufficient conditions such that this system has only the zero solution.

In order to find (necessary and) sufficient conditions for such matrices to be uniquely clean, we have to solve the following

PROBLEM. Suppose a, b, c are nonzero integers (with or without b, c do not divide a). What conditions on a, b, c assure that the system

$$x^2 + x + yz = 0 \tag{1}$$

$$ax + by + cz = 0 \tag{2}$$

has only the zero solution *as integer solution.*

From a geometrical point of view, we have to intersect a plane with a hyperboloid of one sheet, both passing through the origin.

Since so far we were not able to solve it, the problem remains open.

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