

# Exchange $2 \times 2$ matrices over principal ideal domains: a characterization

Grigore Călugăreanu, Horia F. Pop

September 26, 2019

## Abstract

We characterize  $2 \times 2$  exchange matrices over commutative domains. In addition, we emphasize large classes of not exchange  $2 \times 2$  and  $3 \times 3$  integral matrices.

## 1 Introduction

An element  $a$  in a unital ring  $R$  is called *clean* ([8]) if  $a = e + u$  with idempotent  $e$  and unit  $u$  and *e-clean*, if we intend to emphasize the idempotent. A clean element is called *trivial* if its decomposition uses a trivial idempotent i.e., 0 or 1. These are the 0-clean and 1-clean elements, respectively.

Also in [8], exchange elements (called *suitable*) were defined into four equivalent ways. One of these is: an element  $a$  in a ring  $R$  is (left) *exchange* if there is an idempotent  $e$  such that  $e - a \in R(a - a^2)$ . Every left exchange element is also right exchange and conversely. Clean elements are exchange ([8]).

Clean  $2 \times 2$  integral matrices are characterized (see e.g. [2]) by pairs of one quadratic Diophantine equation (in two variables) and one linear Diophantine equation (in three variables). Since nowadays such equations are instantly solved by computer (see [1], [9] respectively [7]), it would be useful to have a criterion (checkable by computer) to decide on the exchange property of a matrix in  $\mathcal{M}_2(\mathbb{Z})$ .

In this note we characterize the exchange  $2 \times 2$  matrices over *commutative domains*. It turns out that these are still characterized by pairs of conditions: but now one quadratic equation (in three variables) and one linear equation (in three or four variables).

In addition, we prove two special results, which mainly permit to emphasize large classes of not exchange  $2 \times 2$  but also  $3 \times 3$  *integral* matrices.

Whenever it is more convenient, we will use the widely accepted shorthand “iff” for “if and only if” in the text. All rings we consider are unital.

## 2 Exchange $2 \times 2$ integral matrices

First notice that an element  $a$  in a ring  $R$  is exchange iff there is an element  $m \in R$  such that  $c := a + m(a - a^2)$  is an idempotent. Observe that if this idempotent is trivial, then  $a$  is clean and so exchange. More precisely, we have the following

**Lemma 1** *Let  $a$  be an element in a Dedekind finite ring  $R$ . There exists an element  $m \in R$  such that*

- (i)  $a + m(a - a^2) = 0$  iff  $1 - a$  is a unit (i.e. 1-clean);
- (ii)  $a + m(a - a^2) = 1$  iff  $a$  is a unit (i.e. 0-clean).

**Proof.** The equality can be written as (i)  $(1 - ma)(1 - a) = 1$  and (ii)  $[1 + m(1 - a)]a = 1$ , respectively, so since  $R$  is Dedekind finite,  $1 - a$  is a unit, or  $a$  is a unit. Conversely, if  $1 - a$  is a unit, we can chose  $m = (1 - a)^{-1}$ , and if  $a$  is a unit, we can chose  $m = a^{-1}$ . ■

**Remark.** Notice that *tripotents*, i.e. elements  $f \in R$  such that  $f = f^3$  are (strongly) clean (see e.g. [3]) and so exchange.

In this section,  $R$  denotes a commutative domain. Notice that a  $2 \times 2$  matrix  $E$  is idempotent iff  $E \in \{0_2, I_2\}$ , or else  $\text{Tr}(E) = 1$ ,  $\det(E) = 0$ . Also observe that, since left exchange elements are also right exchange and conversely, a matrix is exchange iff so is its transpose.

Let  $A$  be a  $2 \times 2$  matrix over  $R$ . By Cayley-Hamilton Theorem,

$$A - A^2 = (1 - \text{Tr}(A))A + \det(A)I_2.$$

Next, we prove a characterization for exchange matrices  $A$  with  $\text{Tr}(A) =: t \neq 1$  over any commutative domain. To simplify the writing we shall use the following notations:  $\alpha = \det(A) + (1 - t)(a + t)$ ,  $\beta = (1 - t)c$ ,  $\gamma = (1 - t)b$  and  $\delta = \det(A) - (1 - t)a$ .

**Theorem 2** *A trace  $t \neq 1$ ,  $2 \times 2$  matrix  $A = \begin{bmatrix} a + t & b \\ c & -a \end{bmatrix}$  is exchange over a commutative domain  $R$  iff  $A$  is a unit, or  $I_2 - A$  is a unit, or  $A$  is an idempotent, or*

- (i) if  $\beta = 0$ , there exist  $m_{11}, m_{12}, m_{21} \in R$  such that

$$(a + t + \alpha m_{11})(-a + 1 - t - \alpha m_{11}) = (b + \gamma m_{11} + \delta m_{12})\alpha m_{21}$$

and  $m_{22}$  with  $1 - t = \alpha m_{11} + \gamma m_{21} + \delta m_{22}$ .

- (ii) if  $\delta = 0$ , there exist  $m_{11}, m_{21}, m_{22} \in R$  such that

$$(a + 1 - \gamma m_{21})(-a + \gamma m_{21}) = (b + \gamma m_{11})(c + \alpha m_{21} + \beta m_{22})$$

and  $m_{12}$  with  $1 - t = \alpha m_{11} + \beta m_{12} + \gamma m_{21}$ .

(iii) if  $\beta \neq 0 \neq \delta$ , there exist  $m_{11}, m_{12}, m_{21} \in R$  such that

$$\delta(a + t + \alpha m_{11} + \beta m_{12})[-a + 1 - t - (\alpha m_{11} + \beta m_{12})] =$$

$$(b + \gamma m_{11} + \delta m_{12})\{c\delta + \alpha\delta m_{21} + \beta[1 - t - (\alpha m_{11} + \beta m_{12} + \gamma m_{21})]\}$$

and  $m_{22}$  with  $1 - t = \alpha m_{11} + \beta m_{12} + \gamma m_{21} + \delta m_{22}$ .

**Proof.** As mentioned before  $A - A^2 = (1 - t)A + \det(A)I_2$  and we have only to discuss the case  $C$  is a nontrivial idempotent (otherwise, by Lemma 1,  $A$  is 0-clean or 1-clean).

Now  $C =$

$$\begin{bmatrix} a+t & b \\ c & -a \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \det(A) + (1-t)(a+t) & (1-t)b \\ (1-t)c & \det(A) - (1-t)a \end{bmatrix}$$

and

$$\text{Tr}(C) = 1 \text{ iff } (m_{11} + m_{22}) \det(A) + (1-t)[m_{11}(a+t) + m_{12}c + m_{21}b - m_{22}a - 1] = 0, \text{ and}$$

$$\begin{aligned} \det(C) = 0 \text{ iff } [a+t+m_{11} \det(A) + (1-t)(m_{11}(a+t) + m_{12}c)] [-a+m_{22} \det(A) + \\ (1-t)(m_{21}b - m_{22}a)] = \\ = [b + m_{12} \det(A) + (1-t)(m_{11}b - m_{12}a)] [c + m_{21} \det(A) + (1-t)(m_{21}(a+t) + m_{22}c)]. \end{aligned}$$

For the proof, we show that using  $\text{Tr}(C) = 1$  we can always eliminate  $m_{22}$  or  $m_{12}$  from the equality  $\det(C) = 0$ , this way obtaining a quadratic equation, to be solved in  $R$ .

Using the notations introduced above,  $\text{Tr}(C) = 1$  is equivalent to  $\alpha m_{11} + \beta m_{12} + \gamma m_{21} + \delta m_{22} = 1 - t$  and  $\det(C) = 0$  is equivalent to

$$(a+t+\alpha m_{11}+\beta m_{12})(-a+\gamma m_{21}+\delta m_{22}) = (b+\gamma m_{11}+\delta m_{12})(c+\alpha m_{21}+\beta m_{22}) (**)$$

We go into three cases.

(i)  $\beta = 0$ ; since  $t \neq 1$  this is equivalent to  $c = 0$ . Then we replace  $\delta m_{22} = 1 - t - (\alpha m_{11} + \gamma m_{21})$  into

$$(a + t + \alpha m_{11})(-a + \gamma m_{21} + \delta m_{22}) = (b + \gamma m_{11} + \delta m_{12})\alpha m_{21}$$

and for any given  $m_{11}$ , this is a quadratic equation in unknowns  $m_{12}, m_{21}$ .

(ii)  $\delta = 0$ ; this is equivalent to  $\det(A) = (1 - t)a$ . Now we replace  $\beta m_{12} = 1 - t - (\alpha m_{11} + \gamma m_{21})$  into

$$(a + t + \alpha m_{11} + \beta m_{12})(-a + \gamma m_{21}) = (b + \gamma m_{11})(c + \alpha m_{21} + \beta m_{22})$$

and for any given  $m_{11}$ , this is a quadratic equation in unknowns  $m_{21}, m_{22}$ .

(iii)  $\beta \neq 0 \neq \delta$ ; this is equivalent to  $c \neq 0$  and  $\det(A) \neq (1 - t)a$ . In this case we multiply by  $\beta$ ,  $\alpha m_{11} + \beta m_{12} + \gamma m_{21} + \delta m_{22} = 1 - t$  and by  $\delta$  the equality (\*\*). Now we replace  $\delta m_{22} = 1 - t - (\alpha m_{11} + \beta m_{12} + \gamma m_{21})$  and  $\beta \delta m_{22} = \beta[1 - t - (\alpha m_{11} + \beta m_{12} + \gamma m_{21})]$  into

$$\delta(a+t+\alpha m_{11}+\beta m_{12})(-a+\gamma m_{21}+\delta m_{22}) = (b+\gamma m_{11}+\delta m_{12})(c\delta+\alpha\delta m_{21}+\beta\delta m_{22})$$

and for any given  $m_{11}$ , this is a quadratic equation in unknowns  $m_{12}$ ,  $m_{21}$ .

The equations we obtain are displayed in the statement of the theorem. ■

**Examples.** 1)  $A = \begin{bmatrix} 3 & 3 \\ 0 & -1 \end{bmatrix}$ . For  $m_{11} = 0$  we have to solve the quadratic equation  $(3 - 2m_{12})m_{21} = 1$ . Among the solutions, if we chose  $(m_{12}, m_{21}) = (2, -1)$  we get  $M = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$  and  $C = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ , an idempotent. The linear equation becomes  $3m_{21} + 2m_{22} = 1$ , also verified by  $m_{21} = -1, m_{22} = 2$ . So  $A$  is exchange over any ring.

2)  $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ . Here  $\det(A) = (1 - t)a$ .

The linear equation is  $m_{11} + 3m_{21} = 1$  which has no solution for  $m_{11} = 0$ , unless 3 is a unit.

However, for  $m_{11} = 1$  we have  $m_{21} = 0$  and the linear equation is verified for any  $m_{12}, m_{22}$ . The quadratic equation is  $36m_{21}^2 = 0$ , also verified by  $m_{21} = 0$ . Hence we can chose  $M = \begin{bmatrix} 1 & u \\ 0 & v \end{bmatrix}$  (arbitrary elements  $u, v$ ) and  $C = \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix}$  is idempotent. So  $A$  is exchange over any ring.

3)  $A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ . Here  $c \neq 0$  and  $\det(A) \neq (1 - t)a$ .

For  $m_{11} = 0$  we have to solve the quadratic equation  $3(1 - m_{12})(-2 + 6m_{21}) = 0$  with solution  $m_{12} = 1$  and arbitrary  $m_{21}$ , say  $(m_{12}, m_{21}) = (1, y)$ . Then  $m_{22} = -y$  and for  $M = \begin{bmatrix} 0 & 1 \\ y & -y \end{bmatrix}$  we obtain  $C = \begin{bmatrix} 0 & 0 \\ 1 - 2y & 1 \end{bmatrix}$ , an idempotent. The linear equation becomes  $-m_{12} - 3m_{21} - 3m_{22} = 1$ , also verified by  $m_{12} = 1, m_{21} = y, m_{22} = -y$ . So  $A$  is exchange over any ring.

4)  $A_k = \begin{bmatrix} 2k + 1 & 0 \\ 0 & 0 \end{bmatrix}$  for any integer  $k$ . Here (i) amounts to

$$t(1 - t)[1 + (1 - t)m_{11}](1 - tm_{11}) = 0.$$

Since  $m_{11} = \frac{1}{t-1} \in \mathbb{Z}$  only for  $t \in \{0, 2\}$  (which are not odd), and  $m_{11} = \frac{1}{t} \in \mathbb{Z}$  only for  $t \in \{\pm 1\}$ , such matrices are exchange only if  $t \in \{\pm 1\}$ . Indeed,  $E_{11}$  is an idempotent, so exchange and  $-E_{11}$  is exchange as tripotent. Hence all  $A_k$ ,  $k \notin \{-1, 0\}$  are not exchange over  $\mathbb{Z}$ .

**Remarks.** 1) When checking for exchange a  $2 \times 2$  matrix  $A$ , one should first verify whether  $A$  or  $A - I_2$  are invertible. If this fails then we use the previous (general) theorem. For instance, for the matrix  $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$  (which is a unit) (ii) is not (always) applicable (e.g. over  $\mathbb{Z}$ ). Indeed, by computation

$C = \begin{bmatrix} 1 & 3(1 - m_{11}) \\ 0 & 1 - 3m_{21} \end{bmatrix}$  so  $\text{Tr}(C) = 2 - 3m_{21} \neq 1$  unless 3 is a unit. Actually,  $C = C^2$  iff the left column of  $M$  is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  (arbitrary right column) and so  $C = I_2$ .

2) Related to our research, from [6] we recall

**Corollary 3.7** *Let  $R = M_2(S)$  where  $S$  is any ring, and let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$ .*

*If  $b$  or  $c$  is a unit, then  $a$  is exchange in  $R$ .*

It can be shown that if we replace  $b$  by any unit (or symmetrically  $c$ ) in Theorem 2, the equations there, have solutions. In order not to lengthen this paper we only discuss the  $b$  is a unit case for  $t \neq 1$ , that is, using Theorem 2.

The linear equation is  $\alpha m_{11} + \beta m_{12} + (1 - t)bm_{21} + \delta m_{22} = 1 - t$  which obviously admits the solution  $m_{11} = m_{12} = m_{22} = 0$  and  $m_{21} = b^{-1}$ . It is easy to check that this solution also verifies the quadratic equation and so for  $M = b^{-1}E_{21}$ ,  $C = A + b^{-1}E_{21}(A - A^2)$  is an idempotent. Hence  $A$  is (indeed) exchange over any ring.

### 3 Special characterizations

In this section, in order to simplify the exposition, we deal with *integral* matrices. The astute reader will easily realize how the results that follow may be generalized over principal ideal domains.

We show how one can exploit the situation  $A - A^2 = \mathcal{M}_2(n\mathbb{Z})$ , for a positive  $n \geq 2$ .

First we have the following

**Proposition 3** *Let  $n \in \mathbb{Z}$ ,  $n \geq 2$  and let  $A \in \begin{bmatrix} n\mathbb{Z} + 1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z} + 1 \end{bmatrix} \subset \mathcal{M}_2(\mathbb{Z})$ . Then  $A$  is exchange iff  $A$  is a unit iff  $\det(A) \in \{\pm 1\}$ .*

**Proof.** One way is clear. Conversely, suppose  $A$  is exchange. Notice that  $A^2 \in \begin{bmatrix} n\mathbb{Z} + 1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z} + 1 \end{bmatrix}$  and so  $A - A^2 = nB$  for some  $B \in \mathcal{M}_2(\mathbb{Z})$ . Then  $C = A + M(A - A^2) = A + nMB$  and so  $\text{Tr}(C) \in 2 + n\mathbb{Z}$ .

If  $n = 2$ ,  $C$  is idempotent only if it is trivial (i.e. the trace is 0 or 2). For  $A \in \begin{bmatrix} 2\mathbb{Z} + 1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} + 1 \end{bmatrix}$  we have  $I_2 - A = 2S$ ,  $M(I_2 - A) = 2T$ ,  $I_2 + M(I_2 - A) \in \begin{bmatrix} 2\mathbb{Z} + 1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} + 1 \end{bmatrix}$  and so  $C \neq 0_2$  because  $C = [I_2 + M(I_2 - A)]A \in \begin{bmatrix} 2\mathbb{Z} + 1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} + 1 \end{bmatrix}$ . In the remaining case,  $C = [I_2 + M(I_2 - A)]A = I_2$ , so  $A$  is a unit (we use Lemma 1).

If  $n \geq 3$ , again  $C$  is idempotent only if it is trivial. Now  $C \neq 0_2$  follows from  $C \in \begin{bmatrix} n\mathbb{Z} + 1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z} + 1 \end{bmatrix}$ , and so only  $C = I_2$  remains. Using again Lemma 1,  $A$  must be a unit. ■

Next

**Proposition 4** For  $n \in \mathbb{Z}$ ,  $n \geq 2$ , let  $A \in \mathcal{M}_2(n\mathbb{Z})$ . The following conditions are equivalent

- (i)  $A$  is exchange.
- (ii) there is a matrix  $M$  such that  $A + M(A - A^2) = 0_2$ .
- (iii)  $A - I_2$  is a unit.
- (iv)  $A$  is 1-clean.

**Proof.** (i)  $\Rightarrow$  (ii) Suppose  $A$  is exchange. By hypothesis,  $A$ ,  $A^2$ ,  $A - A^2$ ,  $M(A - A^2)$  and  $C = A + M(A - A^2)$ , all belong to  $\mathcal{M}_2(n\mathbb{Z})$ . Hence  $C \neq I_2$  and since  $\text{Tr}(C)$  is multiple of  $n$ , that is  $\neq 1$  (i.e.  $C$  is not nontrivial), the idempotent  $C = 0_2$ . (ii)  $\Rightarrow$  (iii) Use Lemma 1. (iii)  $\Rightarrow$  (iv) and (iv)  $\Rightarrow$  (i) are obvious. ■

**Corollary 5** If the entries of a  $2 \times 2$  integral matrix  $A$  are not (collectively) coprime then  $A$  is 1-clean (i.e.  $A - I_2$  is a unit) or  $A$  is not exchange.

**Corollary 6** Let  $A$  be an arbitrary  $2 \times 2$  integral matrix and  $m \in \mathbb{Z}$ ,  $m \notin \{-1, 0, 1\}$ . Then  $mA$  is 1-clean or not exchange.

**Examples.** 1)  $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$  is  $I_2$ -clean so exchange but  $\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}$ , for  $m \geq 3$  are not exchange (this improves Example 4, p. 6).

2)  $2I_2$  is ( $I_2$ -clean and so) exchange but  $mI_2$ , for  $m \geq 3$  are not exchange.

Finally, we can now settle two special cases: the *diagonal* and the (upper) *triangular*  $2 \times 2$  integral exchange matrices.

**Proposition 7** A diagonal integral matrix  $A = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$  is exchange iff  $A = \pm I_2$  or  $E_{11}$ ,  $I_2 + E_{22}$ , or else  $u, v$  are coprime and of different parity, and if  $A = \begin{bmatrix} 2k+1 & 0 \\ 0 & 2l \end{bmatrix}$  then there exist integers  $m_{11}, m_{12}, m_{21}$  such that  $(1-2km_{11})[1-(2k+1)m_{11}] = -2l(2l-1)m_{12}m_{21}$  and  $k(2k+1)m_{11} + l(2l-1)m_{22} = -(k+l)$ . The case  $A = \begin{bmatrix} 2l & 0 \\ 0 & 2k+1 \end{bmatrix}$  is recovered by conjugation with  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

**Proof.** Computation shows that

$$C = A + M(A - A^2) = \begin{bmatrix} u[1 + (1-u)m_{11}] & v(1-v)m_{12} \\ u(1-u)m_{21} & v[1 + (1-v)m_{22}] \end{bmatrix}$$

and so  $\text{Tr}(C) = u[1 + (1-u)m_{11}] + v[1 + (1-v)m_{22}]$  and  $\det(C) = uv[1 + (1-u)m_{11} + (1-v)m_{22} + (1-u)(1-v)\det(M)]$ .

If both  $u, v$  are even,  $A$  is not exchange by Corollary 5.

If both  $u, v$  are odd,  $A$  is not exchange by Proposition 3, excepting the units  $\pm I_2$ . Since  $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$  is conjugate to  $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$  by  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  (the exchange property is invariant to conjugations), in the remaining case, using again Corollary 5, we may suppose  $A = \begin{bmatrix} 2k+1 & 0 \\ 0 & 2l \end{bmatrix}$  has coprime entries, that is,  $\gcd(2k+1, 2l) = 1$ . In order to get a characterization we need Theorem 2.

Since  $b = c = 0$ , we also have  $\beta = \gamma = 0$  and we can use case (i). With the corresponding notations,  $u := a + t$ ,  $v := -a$ ,  $t = u + v$ ,  $\alpha = u(1 - u)$  and  $\delta = v(1 - v)$ . Then the Diophantine equations become  $1 - u - v = u(1 - u)m_{11} + v(1 - v)m_{22}$  and  $u(1 - u)[1 + (1 - u)m_{11}](1 - um_{11}) = u(1 - u)v(1 - v)m_{12}m_{21}$ .

Since the case we deal with has odd  $u$  and even  $v$ , we can discard  $u = 0$ . For  $u = 1$  (the equality holds for any  $m_{11}, m_{12}, m_{21}$ ) the linear equation amounts to  $1 - v = \pm 1$ , i.e.  $v \in \{0, 2\}$ . These are the matrices  $E_{11}$  and  $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  (e.g.  $M = I_2 + E_{22}$ ).

So finally the problem is reduced to  $[1 + (1 - u)m_{11}](1 - um_{11}) = v(1 - v)m_{12}m_{21}$  and, with the above notations  $(1 - 2km_{11})[1 - (2k + 1)m_{11}] = -2l(2l - 1)m_{12}m_{21}$ , together with  $k(2k + 1)m_{11} + l(2l - 1)m_{22} = -(k + l)$ . ■

**Examples.** 1)  $\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$  is not exchange. Here the linear equation  $21m_{11} + 6m_{22} = -5$  has no solution, because  $\gcd(21; 6) = 3$  is not a divisor of 5.

2)  $\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$  is exchange. Now the linear equation is  $21m_{11} - m_{22} = -4$ , which has solutions. For  $m_{11} = 1$ , we get  $30 = -2m_{12}m_{21}$  or  $m_{12}m_{21} = -15$ . Since  $\delta m_{22} = 1 - t - \alpha m_{11}$  we obtain  $m_{22} = -17$ . For example, if  $M = \begin{bmatrix} 1 & -3 \\ 5 & -17 \end{bmatrix}$  then  $C = \begin{bmatrix} -35 & 6 \\ -210 & 36 \end{bmatrix}$ , an idempotent.

3) The matrices  $A_l = \begin{bmatrix} 4l - 1 & 0 \\ 0 & 2l \end{bmatrix}$ ,  $l \notin \{0, 1\}$  are not exchange.

The linear equation  $k(2k + 1)m_{11} + l(2l - 1)m_{22} = -(k + l)$  becomes  $(2l - 1)(4l - 1)m_{11} + l(2l - 1)m_{22} = -(3l - 1)$ . Since  $\gcd(4l - 1; l) = 1$  it follows that  $\gcd((2l - 1)(4l - 1); (2l - 1)l) = 2l - 1$  and this (linear Diophantine) equation has solutions iff  $2l - 1$  divides  $3l - 1$ . It is easy to see that this happens only for  $l \in \{0, 1\}$ .

For  $l = 0$ ,  $A_0 = -E_{11}$  is a tripotent, so exchange, and for  $l = 1$ ,  $A_1 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ . For  $M = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$  we get  $C = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$ , an idempotent. So  $A_1$  is exchange.

We just mention that among matrices of form  $\begin{bmatrix} 7 & 0 \\ 0 & s \end{bmatrix}$ , these are exchange for  $s \in \{-4, 2, 6, 8, 12, 24\}$  but not exchange for  $s \in \{-2, 0, 4, 10, 16, 18, 20, 22\}$ .

**Proposition 8** An upper triangular integral matrix  $A = \begin{bmatrix} a + t & b \\ 0 & -a \end{bmatrix}$ , with

$b \neq 0, t \neq 1$  is exchange iff  $a, b, a + t$  are (collectively) coprime,  $1 - t = \alpha m_{11} + \gamma m_{21} + \delta m_{22}$  and

$$(a + t + \alpha m_{11})(-a + 1 - t - \alpha m_{11}) = (b + \gamma m_{11} + \delta m_{12})\alpha m_{21}.$$

**Proof.** This is Corollary 5, and Theorem 2, (i). ■

**Examples.** 1)  $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ . The entries are not coprime, so the matrix is not exchange over  $\mathbb{Z}$ .

2)  $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ . The linear equation amounts to  $2m_{11} + 9m_{21} + 2m_{22} = 3$  with solutions  $m_{11} = 6 + 9u - v, m_{21} = -1 - 2u, m_{22} = v$  (arbitrary elements  $u, v$ ).

The quadratic equation is  $(2 - 2m_{11})(-1 + 2m_{11}) = 2(3 - 9m_{11} - 2m_{12})m_{21}$ . For  $u = -1, v = -2$  we get  $m_{11} = -1, m_{21} = 1, m_{22} = -2$  and then  $m_{12} = 3$ . Indeed, for  $M = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$  we get  $C = \begin{bmatrix} 4 & 6 \\ -2 & -3 \end{bmatrix}$ , an idempotent. Hence the matrix is exchange over any ring.

For  $M = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ , we obtain the trivial idempotent  $C = 0_2$  (not covered by the equations above).

## 4 Application

In [6] we can find the following

**Theorem 5.12** Let  $e$  be an idempotent in a ring  $R$  and  $a = b + \varepsilon$  with  $b \in S := eRe, \varepsilon \in \text{Idem}(\bar{e}R\bar{e})$ . Then  $a$  is exchange in  $R$  iff  $b$  is exchange in  $eRe$  (here  $\bar{e} = 1 - e$ , the complementary idempotent).

There are two special cases which are related to our research.

1)  $R = \mathcal{M}_3(\mathbb{Z})$  with  $e = E_{11} + E_{22}$  and  $\bar{e} = E_{33}$ . In this case we identify  $S = eRe$  with  $\mathcal{M}_2(\mathbb{Z})$  and  $\bar{e}R\bar{e}$  with  $\mathbb{Z}$ .

2)  $R = \mathcal{M}_3(\mathbb{Z})$  with  $e = E_{11}$  and then  $\bar{e} = E_{22} + E_{33}$ . In this case we identify  $S = eRe$  with  $\mathbb{Z}$  and  $\bar{e}R\bar{e}$  with  $\mathcal{M}_2(\mathbb{Z})$ .

Using this, we obtain two consequences. We use block representations of  $3 \times 3$  matrices ( $\mathbf{0}$  denotes a 2-row, or a 2-column).

**Corollary 9** Let  $U \in \mathcal{M}_2(\mathbb{Z})$  and  $\varepsilon \in \{0, 1\} \subset \mathbb{Z}$ . Then  $A = \begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$  is exchange in  $\mathcal{M}_3(\mathbb{Z})$  iff  $U$  is exchange in  $\mathcal{M}_2(\mathbb{Z})$ .

**Corollary 10** Let  $b \in \mathbb{Z}$  and  $E = E^2 \in \mathcal{M}_2(\mathbb{Z})$ . Then  $A = \begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$  is exchange in  $\mathcal{M}_3(\mathbb{Z})$  iff  $b$  is exchange in  $\mathbb{Z}$  iff  $b \in \{-1, 0, 1, 2\} \subset \mathbb{Z}$ .

Therefore, using the results in the previous section we can generate plenty of not exchange  $3 \times 3$  matrices.



**Corollary 11** *The following  $3 \times 3$  matrices are not exchange for any  $n \in \mathbb{Z}$ ,  $n \geq 2$ :*

- (a)  $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$  for  $U \in \begin{bmatrix} n\mathbb{Z} + 1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z} + 1 \end{bmatrix}$ ,  $\det(U) \notin \{\pm 1\}$  and  $\varepsilon \in \{0, 1\}$ ,
- (b)  $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$  for  $U \in \mathcal{M}_2(n\mathbb{Z})$ ,  $\det(U - I_2) \notin \{\pm 1\}$  and  $\varepsilon \in \{0, 1\}$ ,
- (c)  $\begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$  with any  $2 \times 2$  idempotent  $E$  and  $b \in \mathbb{Z} - \{-1, 0, 1, 2\}$ .

**Final comment.** Among other things, Horia F. Pop wrote a program which, given a  $2 \times 2$  matrix  $A$ , prints all the matrices  $M$  such that  $B = A + M(A - A^2) = B^2$ . In order to avoid a redundant search, the matrices  $M$  are searched incrementally with the nonnegative integer  $z$  starting at 0 and incremented by 1 for as long as it is deemed necessary. For each distinct value of  $z$ , only the matrices  $M$  with all elements in the closed interval  $[-z, z]$  and having at least one element of absolute value equal to  $z$  are tested. This procedure has the advantage of splitting the set of all matrices  $M$  with integer elements into distinct subsets, covered one at a time. The program was decisive in proving some of our results.

## References

- [1] D. Alpern *Quadratic equation solver*. [www.alpertron.com.ar/QUAD.HTM](http://www.alpertron.com.ar/QUAD.HTM).
- [2] D. Andrica, G. Călugăreanu *Uniquely clean  $2 \times 2$  invertible integral matrices*. *Studia Universitatis Babeş-Bolyai* **62** (3) (2017), 287-293.
- [3] G. Călugăreanu *Tripotents: a class of strongly clean elements in rings*. *An. Şt. Univ. Ovidius Constanţa* **26** (1) (2018), 69–80.
- [4] G. Călugăreanu *Clean integral  $2 \times 2$  matrices*. *Studia Sci. Math. Hungarica* **55** (1) 2018), 41-52.
- [5] G. Călugăreanu *Nil-clean integral  $2 \times 2$  matrices: The elliptic case*. to appear in *Bull. Math. Soc. Sci. Roum.*
- [6] D. Khurana, T. Y. Lam, P. Nielsen *Exchange elements in rings, and the equation  $XA - BX = I$* . *Trans. Amer. Math. Soc.* **369** (2017), 495-516.
- [7] P. Chevanne *Diophantine equation in 3 unknowns*. <http://mathafou.free.fr/exe-en/exedioph3.html>
- [8] W. K. Nicholson *Lifting idempotents and exchange rings*. *Trans. Amer. Math. Soc.*, **229** (1977), 269-278.
- [9] K. Matthews *Solving the general quadratic Diophantine equation  $ax^2 + bxy + cy^2 + dx + ey + f = 0$* . <http://www.numbertheory.org/php/generalquadratic.html>