Contributions to Module Theory, 1–13

Endomorphisms and automorphisms of squares of abelian groups

Grigore Călugăreanu and Phill Schultz

Abstract. A major theme in Tony Corner's work is the interaction between abelian groups and their rings of endomorphisms and groups of automorphisms. Here we study the properties of an abelian group A which are reflected in Aut$(A \oplus A)$.

Key words. Automorphism groups, decompositions, involutions.

AMS classification. Primary 20K30, Secondary 20K15, 20K20.

1 Introduction and Notation

Let A be an abelian group. We say that A is determined in a category $\mathcal{A}$ of abelian groups by its endomorphism ring if whenever $B \in \mathcal{A}$ has the property that $\mathcal{E}(A) \cong \mathcal{E}(B)$ as rings, then $A \cong B$ as abelian groups. Similarly, we say that A is determined in $\mathcal{A}$ by its automorphism group if whenever $B \in \mathcal{A}$ has the property that Aut$(A) \cong$ Aut(B) as groups, then $A \cong B$ as abelian groups.

Usually a group is determined neither by its automorphism group nor its endomorphism ring in a non–trivial category of torsion–free abelian groups closed under isomorphisms. However, if A is a rational group of idempotent type, that is, a subring of the rational numbers, then A is determined by its endomorphism ring in the category of rational groups of idempotent type, [MS00]. In fact, since A is determined by the set of primes p for which $pA = A,$ A is even determined by its additive endomorphism group End(A) in this category. On the other hand, A is determined by its automorphism group in the same category only if $A \cong \mathbb{Z}$, whereas $\mathbb{Q}$ is determined by Aut$(\mathbb{Q})$ in the category of torsion–free divisible groups, as well as by End$(\mathbb{Q})$ in the same category. Of course, if A is determined in a given category by Aut(A) or End(A), then it is also determined by $\mathcal{E}(A)$.

A new phenomenon was considered by the first author in [Cal06]. He shows that more information about A is sometimes provided by Aut$(A^2) =$ Aut$(A \oplus A)$. For example, if A and B are rational groups of idempotent type and both are divisible by exactly n primes, where n is a positive integer or infinity, then always Aut$(A) \cong$ Aut(B) [MS00, Proposition 4.3], and the following are equivalent:

(i) Aut$(A^2) \cong$ Aut(B^2)

(ii) End$(A) \cong$ End(B)

(iii) A and B are both are divisible by the same sets of primes.

(iv) $A \cong B$

We say that A is determined in a category $\mathcal{A}$ by its square automorphism group if whenever $H \in \mathcal{A}$ has the property that Aut$(A^2) \cong$ Aut(H) as groups, then $H = B^2$

with $A \cong B$ as abelian groups. Similarly, A is determined in a category $\mathcal{A}$ by its square endomorphism ring, if whenever $H \in \mathcal{A}$ has the property that $\mathcal{E}(A^2) \cong \mathcal{E}(H)$ as rings, then $H = B^2$ with $A \cong B$ as abelian groups. Determination by the square endomorphism group End(A^2) can be similarly defined.

It is the purpose of this paper to investigate this phenomenon. There are well known examples in the literature [F70, Theorem 90.3] of non–isomorphic indecomposable torsion–free groups A and B for which $A^2 \cong B^2$, so in general Aut$(A^2) \cong$ Aut(B^2) does not imply that $A \cong B$.

One could also consider groups determined by the automorphism group of higher powers; for example, Hahn and O'Meara [HO'M89, 3.3.8 and 3.3.11] showed that A is determined by Aut(A^n) for all $n \geq 3$ in the category of abelian groups whose endomorphism ring is a principal ideal domain, and Krylov et al [KMT03] showed that A is determined by Aut(A^n) for all $n \geq 4$ in the category of abelian groups whose endomorphism ring is commutative. However, in this paper we only consider the square case.

Although we have not obtained definitive results on the structure of groups determined by their square automorphism groups, we have found several properties of pairs of groups whose square automorphism groups are isomorphic. The main result is that if A and H are torsion–free abelian groups divisible by 2 such that Aut$(A^2) \cong$ Aut(H), then $H = B^2$ for some group B such that End$(A) \cong$ End(B). Furthermore, we show that in this case, Aut$(A) \times$ Aut$(A) \cong$ Aut$(B) \times$ Aut(B).

Consequently, A is determined by its square automorphism group in the category of torsion–free abelian groups if A is determined by its endomorphism group in that category. We also pose some intriguing questions which will be the subject of further research.

Of course, the fact that End$(A) \cong$ End(B) does not imply that $\mathcal{E}(A) \cong \mathcal{E}(B)$ is well known. As long ago as 1959, Sasiada [S59] found examples of rank 2 torsion–free groups A and B, A being completely decomposable, satisfying the former but not the latter. Nowadays, such examples are easy to construct using the methods of Mader [M00, Section 15.2].

The notation is mostly standard as in [F70]. In particular, $\mathbb{Z}$ denotes the group or ring of integers and $\mathbb{Q}$ the group or ring of rationals. Unless specifically excepted, the word group will denote a torsion–free abelian group. We use the common symbol $\cong$ for group, abelian group or ring isomorphism, the meaning being specified when it is ambiguous. Similarly, $Z(X)$ denotes the center of the group or ring X and $X \oplus Y$ is the direct sum or direct product of groups or rings X and Y.

Let $G = A \oplus B$ be a direct sum of groups A and B. Then $\mathcal{E}(G)$ can be represented by the ring of 2×2 matrices $\begin{bmatrix} \text{End}(A) & \text{Hom}(A,B) \\ \text{Hom}(B,A) & \text{End}(B) \end{bmatrix}$, the action being given by

$$(a,b)\begin{bmatrix} r & s \\ t & u \end{bmatrix} = (ar + bt, as + bu) \text{ for all } (a,b) \in G.$$

In particular, if $G = A^2$, then we can represent $\mathcal{E}(G)$ by $\mathcal{M} = M(2, \mathcal{E}(A))$, the ring of 2×2 matrices with entries from $\mathcal{E}(A)$, where the slight abuse of notation causes no harm. It is straightforward to verify that $\mathrm{GL}(2, \mathcal{E}(A)) = \mathrm{Aut}(A^2)$ corresponds to the group of invertible matrices in $\mathcal{M}$.

In general, we denote abstract automorphisms of G by Greek lower case letters, but their representation by matrices by capital Latin letters, so for example an automorphism α is represented by $A = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$.

In accordance with common practice in group theory, function names are written on the right of their arguments but functor names are written on the left. For example, $Z(\mathrm{Aut}(A))$ denotes the center of $\mathrm{Aut}(A)$ and $Z(\mathcal{E}(A))$ the center of $\mathcal{E}(A)$. In particular, if F is a sub–functor of the identity in the category under consideration and $f \in \mathrm{Hom}(A, B)$ then $F(A)f \subseteq F(B)$.

When X and Y are groups, abelian or not, the notation $X \leq Y$ means that X is a subgroup of Y and $X \trianglelefteq Y$ means that X is a normal subgroup of Y. When $X \leq Y$,

$$\mathcal{C}_Y(X) = \{y \in Y : yx = xy \text{ for all } x \in X\}$$

is the centralizer of X in Y and

$$\mathcal{N}_Y(X) = \{y \in Y : y^{-1}xy \in X \text{ for all } x \in X\}$$

is the normalizer of X in Y.

For any unital ring R, R^+ denotes the additive group of R and $U(R)$ the unit group of R so that $U(M(2, \mathcal{E}(A))) = \mathrm{GL}(2, \mathcal{E}(A))$.

2 Properties of automorphism groups preserved by an isomorphism

In this section, we pose several naturally arising questions concerning these properties. Throughout the section, A is a group divisible by 2, $G = A^2$ and $\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ is an isomorphism. Note that in general, if A and B are groups such that $2A = A$, $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ does not imply that $2B = B$, which prompts our first question:

Question 2.1. For which $G = 2G$ does it follow that θ maps multiplication by 2 onto mutiplication by 2?

Because of the result of the first author cited above, the answer is affirmative for A a rank 1 group of idempotent type. Incidentally, 2–divisibility is an *additive* property of groups, but in our situation, it can be determined multiplicatively.

Proposition 2.2. *2 is an automorphism of G if and only if* $Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}^2$ *for some* $s \in \mathcal{E}(A)$.

Proof. If 2 is an automorphism of G then A is also 2–divisible so there exists $s \in \mathcal{E}(A)$ such that $2s = 1$. Hence $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2s \\ 0 & 1 \end{bmatrix} = Y$.

Conversely, if $\begin{bmatrix}1 & s\\ 0 & 1\end{bmatrix}^2 = \begin{bmatrix}1 & 2s\\ 0 & 1\end{bmatrix} = Y$, then $1 = 2s$ in Aut(A) so A and hence G are 2–divisible. □

There is a similar criterion using Y^T (Y transpose), but these results are not strong enough to settle Question 2.1.

Since orders and centralizers of automorphisms are preserved by isomorphisms, it is important to consider special elements of G for which these characteristics are known. For example, -1 is a central automorphism of G of order 2, so $(-1)\theta$ is a central automorphism of H of order 2. Hence one may ask:

Question 2.3. When is $(-1)\theta = -1$?

This question will be discussed further in Section 3. Other special automorphisms of G whose algebraic properties are reflected in Aut(H) include

(i) $X = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and its additive inverse $-X$ of order 2.

(ii) $Z = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ of order 4. Its additive and multiplicative inverse coincide.

Note that conjugation by X and Z map upper triangular elements and subgroups to lower triangular elements and subgroups, while conjugation by Y preserves upper and lower triangularity.

3 Involutions

In this section, G is a 2–divisible group and H is a group such that Aut$(G) \cong$ Aut(H).

An involution μ of G is an automorphism of order 2. As shown in [F73, Section 113], μ determines a decomposition $G = G^{\mu} \oplus G^{-\mu}$ where $G^{\mu} = \{x \in G : x\mu = x\}$ and $G^{-\mu} = \{x \in G : x\mu = -x\}$. Conversely, every decomposition $G = A \oplus B$ determines an involution μ for which $G^{\mu} = A$ and $G^{-\mu} = B$, namely $\mu = 2\pi_A - 1$ where π_A is the projection of G along B onto A.

If G is any group, then the involution -1 of G is called the *trivial involution* since the corresponding decomposition is $0 \oplus G$. The importance of involutions for our problem is that if $\theta :$ Aut$(G) \to$ Aut(H) is an isomorphism, then θ maps involutions of G onto involutions of H. This in itself does not imply that non–trivial decompositions of G induce non–trivial decompositions of H or *vice versa*, firstly because H may not be 2–divisible and secondly because $(-1)\theta$ may not equal -1.

We now assume in the rest of this section that H is 2–divisible and study the image of the central involution -1 under θ. Let $\beta = (-1)\theta$. We call θ *good* if $\beta = -1$, otherwise *bad*.

Proposition 3.1. *Let G and H be 2–divisible groups and let $\theta :$ Aut$(G) \to$ Aut(H) be an isomorphism. If θ is bad, then there are decompositions $G = A \oplus B$, $H = C \oplus D$ such that A and B are fully invariant in G and C and D are fully invariant in H.*

Proof. Since θ is bad, β is a non–trivial involution and hence H has a decomposition $H = H^{\beta} \oplus H^{-\beta}$ with respect to which β has the matrix form $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let $C = H^{\beta}$ and $D = H^{-\beta}$.

Since β is central in $\mathrm{Aut}(H)$, β commutes with all automorphisms of the form $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ with $s \in \mathrm{Hom}(C, D)$. But this implies that $s = -s$ for all $s \in \mathrm{Hom}(C, D)$ and hence $\mathrm{Hom}(C, D) = 0$ so C is fully invariant in H. Similarly, by considering automorphisms of the form $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ with $t \in \mathrm{Hom}(D, C)$ we conclude that D is fully invariant in H.

Now consider $-1 \in \mathrm{Aut}(H)$ and $\alpha = (-1)\theta^{-1}$. Since $\alpha \neq -1$, α is a non–trivial central involution in $\mathrm{Aut}(G)$, so by the same argument, we conclude that $G = A \oplus B$ with A and B fully invariant in G. □

Now for the good news. Even though θ may not map -1 to -1, it can always be perturbed to do so.

Proposition 3.2. *If* $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$ *then there exists a good isomorphism* $\phi : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$.

Proof. Let $\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ be an isomorphism. Let $(-1)\theta = \beta$. If $\beta \neq -1$ then by Proposition 3.1, $H = C \oplus D$ such that $\beta | C = 1$ and $\beta | D = -1$. Let $\gamma \in \mathrm{Aut}(H)$ act as -1 on C and the identity on D. Then $\phi = \beta \circ \gamma$ is an isomorphism of $\mathrm{Aut}(G)$ onto $\mathrm{Aut}(H)$ which maps -1 to -1. □

Proposition 3.2 has some useful consequences.

Corollary 3.3. *If* $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$ *then there is an isomorphism* $\phi : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ *such that for all* $\alpha \in \mathrm{Aut}(G)$, $(-\alpha)\phi = -(\alpha\phi)$. □

Corollary 3.4. *Let* $\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ *be a good isomorphism, and let* $G = A \oplus B$ *be any non–trivial decomposition of* G*. Let* α *with matrix* $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *be the involution corresponding to this decomposition of* G*. Then there is a non–trivial decomposition* $H = C \oplus D$ *such that* $\alpha\theta$ *has the same matrix* M *with respect to this decomposition.* □

Given a non–trivial decomposition $G = A \oplus B$ of G, we are interested in involutions ν for which $G^{\nu} = A$ and $G^{-\nu} = C$ for some different complementary summand C. A pair $(r, s) \in \mathrm{Hom}(A, B) \times \mathcal{E}(B)$ is called a *decomposition pair* if $(1 + s)r = 0$ and s is an involution in $\mathrm{Aut}(B)$. Decomposition pairs are used to classify the decompositions of G for which A is fixed as the first summand.

Proposition 3.5. *Let* $G = A \oplus B$*. Then any involution fixing* A *corresponds to a matrix of the form* $M = \begin{bmatrix} 1 & 0 \\ r & s \end{bmatrix}$ *for some decomposition pair* (r, s)*. Conversely, if*

$M = \begin{bmatrix} 1 & 0 \\ r & s \end{bmatrix}$ *for some decomposition pair* (r, s) *then* M *is the matrix corresponding to an involution fixing* A.

Proof. Let M be an involution fixing $\{(a, 0) : a \in A\}$, so its first row has the form $(1, 0)$. Hence $M = \begin{bmatrix} 1 & 0 \\ r & s \end{bmatrix}$ for some pair $(r, s) \in \mathrm{Hom}(B, A) \times \mathcal{E}(B)$. Then $M^2 = 1$ implies that $(1 + s)r = 0$ and $s^2 = 1$. $M \neq I$ implies that $s \neq 1$. Hence (r, s) is a decomposition pair.

Conversely, any such M is an involution and $G = M^+ \oplus M^-$ where $M^+ = \{(a, 0) \in G : a \in A\}$ and $M^- = \{(a, b) \in G : br = -2a \text{ and } bs = -b\}$.

□

Remarks 3.6. (i) $s = -1$ for all decomposition pairs if and only if B is directly indecomposable.

(ii) $(r, -1)$ is a decomposition pair for any $r \in \mathrm{Hom}(B, A)$.

(iii) The given decomposition $G = A^2$ corresponds to the decomposition pair $(0, -1)$.

(iv) We refer to the involution defined in Proposition 3.5 as $\mu(r, s)$, the corresponding matrix M as $M(r, s)$, and the complementary summand C as $B(r, s)$.

Corollary 3.7. *For any fixed involution* s *of* B*, the mapping* $r \mapsto B(r, s)$ *is a bijection from* $\mathrm{Hom}(B, A)$ *onto complementary summands of* A *in* G *and the mapping* $r \mapsto M(r, s)$ *is a bijection of* $\mathrm{Hom}(B, A)$ *onto involutions of* G *fixing* A. □

Now let $\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(H)$ be a good isomorphism and let $\beta \in \mathrm{Aut}(H)$ be the image under θ of $\mu(0, -1)$. Then β is a non–trivial involution of H. Let $C = H^\beta$ and $D = H^{-\beta}$ so that $H = C \oplus D$ and, with respect to this decomposition, β is represented by $M' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. By our remarks above, every complementary summand to C in H has the form $D(r', s')$ for some decomposition pair $(r', s') \in \mathrm{Hom}(D, C) \times \mathcal{E}(D)$. Furthermore, every involution in $\mathrm{Aut}(H)$ fixing C has the form $M'(r', s')$ for such a decomposition pair.

Proposition 3.8. *With the notation above,*

(i) for each decomposition pair $(r, s) \in \mathrm{Hom}(B, A) \times \mathcal{E}(B)$, $M(r, s)\theta = M'(r', s')$ *for some decomposition pair* $(r', s') \in \mathrm{Hom}(D, C) \times \mathcal{E}(D)$.

(ii) the mapping $(r, s) \mapsto (r', s')$ *is a bijection.*

(iii) if $s = -1$ *then* $s' = -1$*. In particular, if* B *is indecomposable, then* B' *is indecomposable.*

Proof. (1) Let $G = A \oplus B$ and (r, s) be a decomposition pair, so that $M(r, s)$ is an involution of G which fixes A. Let $M(r, s)\theta = \beta \in \mathrm{Aut}(H)$. We have seen that H has a decomposition $C \oplus D$ for which $\beta = M'(r', s')$ for some decomposition pair (r', s').

(2) Since (r', s') is determined uniquely by (r, s) and *vice versa*, the correspondence is a bijection.

(3) We have seen that the correspondence maps $(r, -1)$ to $(r', -1)$. □

4 Automorphisms of squares of groups

We now consider the case in which $G = A^2$. Throughout this section, whose two theorems are the main results of this paper, G and H are 2-divisible groups and $\theta :$ Aut$(G) \rightarrow$ Aut(H) is a good isomorphism. Furthermore, $G = A \oplus A$ is a fixed decomposition of G, and $\mu \in$ Aut(G) has matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with respect to this decomposition. Let $\nu = \mu\theta$ and let $H = C \oplus D$ be the corresponding decomposition of H so that by Corollary 3.4 the matrix of ν with respect to this decomposition is also M.

Now Aut(G) contains an involution of a different type, namely χ which maps the first copy of A identically onto the second and *vice versa*. The corresponding matrix is $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has the following properties:

(i) For all $(a, b) \in G,\ (a, b)X = (b, a)$.

(ii) For all decomposition pairs $(r, s),\ XM(r, s)X$ is an involution fixing the second copy of A.

(iii) $XM = -MX$.

Theorem 4.1. *Let $G = A \oplus A,\ H = C \oplus D$ and χ be as described in the two paragraphs above and let $\lambda = \chi\theta$. Then the matrix of λ with respect to this decomposition of H is $L = \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}$ for some isomorphism $b :\ C \rightarrow D$.*

Proof. Let λ have matrix $L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with respect to the decomposition $H = C \oplus D$. Since θ is a group isomorphism, by (3) above, $ML = -LM$, so

$$\begin{bmatrix} a & b \\ -c & -d \end{bmatrix} = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix}$$

Since End(C) and End(D) are torsion–free, this implies that $a = 0 = d$. Since L represents an involution, it follows that $c \in$ Hom(D, C) is an isomorphism with inverse $b \in$ Hom(C, D). ◻

Theorem 4.1 shows that if $G = A^2$ and H are 2–divisible groups with Aut$(G) \cong$ Aut(H), then $H = B^2$ for some group B and there is a good isomorphism $\theta :$ Aut$(G) \rightarrow$ Aut(H) which maps the involution corresponding to a fixed decomposition $G = A \oplus A$ to the involution corresponding to a fixed decomposition $H = B \oplus B$. But as yet, we have found no relationship between A and B. We now construct an additive isomorphism from End(A) to End(B).

Theorem 4.2. *Let $G = A^2$ and $H = B^2$ be 2–divisible groups with* Aut$(G) \cong$ Aut(H). *Then there exists an isomorphism $\theta :$* Aut(G) *onto* Aut(H) *inducing an additive isomorphism $\phi :$* End$(A) \rightarrow$ End(B).

Proof. By Proposition 3.8 there is a good isomorphism $\theta : \text{Aut}(G) \to \text{Aut}(H)$ which induces the matrix bijection $\begin{bmatrix} 1 & 0 \\ r & -1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ r' & -1 \end{bmatrix}$, and hence by Corollary 3.4 the matrix bijection $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ r' & 1 \end{bmatrix}$.

Define $\phi : \text{End}(A) \to \text{End}(B)$ by $r \mapsto r'$. Since

$$\begin{bmatrix} 1 & 0 \\ r+s & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

ϕ is an additive homomorphism which is clearly bijective. □

5 The action of θ on subgroups of $\text{Aut}(G)$

In the final two sections of this paper, we no longer need the hypotheses that G and H are torsion–free and 2–divisible so we replace them by the weaker conditions that G and H are abelian groups satisfying $G = A^2, \ H = B^2$ and $\theta : \text{Aut}(G) \to \text{Aut}(H)$ is an isomorphism mapping $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in $\text{Aut}(G)$ to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in $\text{Aut}(H)$. All matrices are with respect to fixed decompositions $G = A \oplus A$ and $H = B \oplus B$, so we can regard θ as mapping matrices to matrices.

By the fundamental homomorphism theorems, θ maps the lattice of subgroups and the lattice of normal subgroups of $\text{Aut}(G)$ to isomorphic lattices of subgroups and normal subgroups in $\text{Aut}(H)$. Call a subgroup X of a subgroup Y of $\text{Aut}(G)$ *functorial* if there is a sub-functor of the identity F on the category of groups such that $X = F(Y)$. Then $X\theta = F(Y\theta)$. Examples include $F(Y) = Z(Y)$, the center of Y, $\mathcal{C}_{\text{Aut}(G)}(Y)$, the centralizer of Y and $\mathcal{N}_{\text{Aut}(G)}(Y)$ the normalizer of Y. More generally, for all $X \leq Y$ in $\text{Aut}(G)$, $(\mathcal{C}_Y(X))\theta = (\mathcal{C}_{Y\theta}X)\theta$ and similarly for normalizers.

Unfortunately the special subgroups familiar in Linear Algebra and Geometry may not be defined in the present context, and need not be preserved by θ when they are. Nevertheless, some are defined and preserved.

We first consider centralizers of involutions. Recall that $\alpha \in \text{Aut}(G)$ is the involution with matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and that θ maps M to a matrix of the same form as M. Hence θ maps $\mathcal{C}_{\text{Aut}(G)}(M) \to \mathcal{C}_{\text{Aut}(H)}(M)$. It is easy to see that $\mathcal{C}_{\text{Aut}(G)}(M) = \begin{bmatrix} \text{Aut}(A) & 0 \\ 0 & \text{Aut}(A) \end{bmatrix} \cong \text{Aut}(A) \times \text{Aut}(A)$ and similarly, $\mathcal{C}_{\text{Aut}(H)}(M) \cong \text{Aut}(B) \times \text{Aut}(B)$, so that $\text{Aut}(A) \times \text{Aut}(A) \cong \text{Aut}(B) \times \text{Aut}(B)$.

Question 5.1. Does this imply $\text{Aut}(A) \cong \text{Aut}(B)$?

Now consider the corresponding result for the involutions $N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $K =$

$\begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix} \in \text{Aut}(H)$. A simple calculation shows

Proposition 5.2. *With the notation above, θ maps*

$$\mathcal{C}_{\text{Aut}(G)}(N) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in \text{Aut}(G) \right\} \text{ to}$$

$$\mathcal{C}_{\text{Aut}(H)}(K) = \left\{ \begin{bmatrix} x & y \\ b^{-1}yb^{-1} & b^{-1}xb \end{bmatrix} \in \text{Aut}(H) \right\}$$

□

We now consider the isomorphism of the centres of the automorphism groups Aut(G) and Aut(H).

Proposition 5.3. $Z(\text{Aut}(G)) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in Z(\mathcal{E}(A)) \cap \text{Aut}(A) \right\}$ *and there exists an isomorphism* $\phi : Z(\mathcal{E}(A)) \cap \text{Aut}(A) \to Z(\mathcal{E}(B)) \cap \text{Aut}(B)$ *such that* $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \theta = \begin{bmatrix} a\phi & 0 \\ 0 & a\phi \end{bmatrix}$

Proof. The description of the center of Aut(G) follows from routine calculations.

Since the center is a functorial subgroup, $Z(\text{Aut}(G)\theta = Z\,\text{Aut}(H)$. Define $\phi : Z(\mathcal{E}(A)) \cap \text{Aut}(A) \to Z(\mathcal{E}(B)) \cap \text{Aut}(B)$ by $a\phi = b$ if $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \theta = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$. Then ϕ is a multiplication preserving bijection. □

Corollary 5.4. *Let A and B be abelian groups with commutative endomorphism rings. Then* Aut(A^2) $\cong$ Aut(B^2) *implies* Aut(A) $\cong$ Aut(B). □

By considering the example of rational groups of idempotent type, one can see that the converse of Corollary 5.4 generally fails.

6 Triangular subgroups

We now consider the properties of triangular subgroups of Aut(G), defined as follows. An element $X \in \text{Aut}(G)$ is called *upper triangular* if it has the form $X = \begin{bmatrix} r & s \\ 0 & t \end{bmatrix}$. Note that if X is upper triangular, then necessarily r and $t \in \text{Aut}(A)$. A subgroup is upper triangular if all its elements are. The set of all such matrices with r and t in Aut(A) and $s \in \mathcal{E}(A)$ is a subgroup of Aut(G) called the *full upper triangular subgroup*. Similarly, we can define lower triangular elements and subgroups. All the following examples have lower triangular counterparts.

It is not true that the triangular property of automorphisms is preserved by θ, but several useful properties of triangular automorphisms are.

Let $\mathcal{T} = \mathcal{T}(G)$ denote the group of upper triangular matrices in Aut(G). We shall describe the lattices of subgroups and normal subgroups of $\mathcal{T}$ in terms of subgroups of Aut$(A) \times$ Aut(A) acting on $\mathcal{E}(A)$. Let $(B, C) \leq$ Aut$(A) \times$ Aut(A). Then (B, C) is a double operator on $\mathcal{E}(A)$ in the sense that for all $(b, c) \in (B, C)$ and for all $s \in \mathcal{E}(A)$, $bsc \in \mathcal{E}(A)$ is defined by $x(bsc)) = ((xb^{-1})s)c \in A$. A (B, C) module J in $\mathcal{E}(A)$ is an additive subgroup of $\mathcal{E}(A)$ closed under the double operator (B, C).

This seemingly obscure structure is simply the multiplicative version of a bi–module; in fact J is a (B, C) module if and only if J is a left $\mathbb{Z}[B]$–right $\mathbb{Z}[C]$ -bimodule over the group rings $\mathbb{Z}[B]$ and $\mathbb{Z}[C]$. For example, every ideal of $\mathcal{E}(A)$ is an $(\text{Aut}(A), \text{Aut}(A))$ module. However, not every (B, C) module is an ideal; for example, let J be the cyclic additive subgroup of $\mathcal{E}(A)$ generated by the identity map. Then J is a $\{1\} \times \{1\}$ module. The importance of this concept is that (B, C) modules can be used to classify all subgroups and normal subgroups of $\mathcal{T}$.

Proposition 6.1. *Let $U \leq \mathcal{T}$. Then there exists a unique subgroup $(B, C) \leq \text{Aut}(A) \times \text{Aut}(A)$ and a unique (B, C) module J of $\mathcal{E}(A)$ such that*

$$U = \left\{ \begin{bmatrix} b & s \\ 0 & c \end{bmatrix} : (b, c) \in (B, C) \text{ and } s \in J \right\}.$$

Conversely, each such U is a subgroup of $\mathcal{T}$.

Furthermore, $U \trianglelefteq \mathcal{T}$ if and only if $B \trianglelefteq \text{Aut}(A)$ and $C \trianglelefteq \text{Aut}(A)$.

Proof. Let $\begin{bmatrix} b & s \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} u & v \\ 0 & w \end{bmatrix} \in U$. Then

$$\begin{bmatrix} b & s \\ 0 & c \end{bmatrix} \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} = \begin{bmatrix} bu & bv + sw \\ 0 & cw \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} r & s \\ 0 & t \end{bmatrix}^{-1} = \begin{bmatrix} r^{-1} & -r^{-1}st^{-1} \\ 0 & t^{-1} \end{bmatrix}$$

Since $U \leq \mathcal{T}$, the set of diagonals of elements of U is closed under multiplication and inverses, so form a subgroup (B, C) of Aut$(A) \times$ Aut(A). Furthermore, the set of entries in the north-east corners of elements of U is closed under addition and the action of (B, C), so form a (B, C) module. Conversely, if U satisfies these conditions, than U is a subgroup of $\mathcal{T}$.

By the description of multiplication above, U is normal if and only if B and C are. ∎

By conjugation with the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we obtain an equivalent characterization of lower triangular subgroups of Aut(G).

The following examples will play a significant rôle in the sequel.

Examples 6.2. (i) If $(B, C) = \mathrm{Aut}(A) \times \mathrm{Aut}(A)$ and $J = \mathcal{E}(A)$ we recover $\mathcal{T}$.

(ii) If $(B, C) = (\{1\}, \{1\})$ and $J = \mathcal{E}(A)$, we have the group $E(G)$ of upper transvections in $\mathrm{Aut}(G)$.

(iii) Let $Z = Z(\mathcal{E}(A)) \cap \mathrm{Aut}(G)$, and let (B, C) be the diagonal of $Z \times Z$ and $J = \mathcal{E}(A)$. Then we have the group

$$F(G) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in Z,\, b \in \mathcal{E}(A) \right\}.$$

Note that for fixed (B, C) the set of (B, C) submodules of $\mathcal{E}(A)$ is a lattice under inclusion, and we have the following description of subgroups and normal subgroups of $\mathcal{T}$. To simplify the notation, denote by $\mathcal{T}((B, C), J)$ the subgroup of $\mathcal{T}$ described in Proposition 6.1.

Proposition 6.3. *Let $B \leq B'$ and $C \leq C'$ be subgroups of $\mathrm{Aut}(A) \times \mathrm{Aut}(A)$ and let J be a (B, C) submodule of the (B', C') module J'. Then $\mathcal{T}((B, C), J) \leq \mathcal{T}((B', C'), J')$.*

Moreover, if $B \trianglelefteq B'$ and $C \trianglelefteq C'$ then $\mathcal{T}((B, C), J) \trianglelefteq \mathcal{T}((B', C'), J')$.

Proof. This is an immediate consequence of the definitions. ∎

Corollary 6.4. *For fixed (B, C), the lattice $\mathcal{J}$ of (B, C) modules determines a lattice of subgroups $\mathcal{T}((B, C), \mathcal{J}) = \{\mathcal{T}((B, C), J) : J \in \mathcal{J}\}$ of subgroups of $\mathrm{Aut}(G)$. If B and C are normal in $\mathrm{Aut}(A)$, then $\mathcal{T}((B, C), \mathcal{J})$ is a lattice of normal subgroups.* ∎

Examples 6.5. (i) Taking $(B, C) = (\{1\}, \{1\})$ and $\mathcal{J}$ the lattice of additive subgroups of $\mathcal{E}(A)$ we obtain a lattice of normal subgroups of $\mathrm{Aut}(G)$ isomorphic to the lattice of additive subgroups of $\mathcal{E}(A)$.

(ii) Taking $(B, C) = \mathrm{Aut}(A) \times \mathrm{Aut}(A)$ and $\mathcal{J}$ the lattice of ideals of $\mathcal{E}(A)$, we obtain a sublattice of (1) isomorphic to the lattice of ideals of $\mathcal{E}(A)$.

In the following proposition, $F(G)$ and $E(G)$ are the subgroups of $\mathrm{Aut}(G)$ defined in Examples 6.2 (2) and (3).

Proposition 6.6. *(i) $F(G)$ is an abelian subgroup of $\mathrm{Aut}(G)$, normal in $\mathcal{T}(G)$.*

(ii) $F(G)$ is the direct product $Z(\mathrm{Aut}(G)) \times E(G)$.

Proof. (1) It is clear that $F(G)$ is commutative. By Proposition 6.1, $F(G) \trianglelefteq \mathcal{T}(G)$.

(2) $Z(\mathrm{Aut}(G)) \cap E(G) = \{I_2\}$ and $\begin{bmatrix} r & s \\ 0 & r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} 1 & r^{-1}s \\ 0 & 1 \end{bmatrix}$ with $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \in Z(\mathrm{Aut}(G))$ and $\begin{bmatrix} 1 & r^{-1}s \\ 0 & 1 \end{bmatrix} \in E(G)$. ∎

Matrices in $F(G)$ can be characterized as follows.

Lemma 6.7. *Let $M \in \mathrm{Aut}(G)$. Then $M \in F(G)$ if and only if $E(G) \leq \mathcal{C}_{\mathrm{Aut}(G)}(M)$, the centralizer of M in $\mathrm{Aut}(G)$.*

Proof. Note that $E(G) \leq C_{\text{Aut}(G)}(M)$ if and only if for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Aut}(G)$ and all $s \in \mathcal{E}(A)$, $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$.

But this is true if and only if $c = 0$ and $a = d \in Z(\mathcal{E}(A))$ or equivalently, if and only if

$$M \in F(G) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a \in Z(\mathcal{E}(A)) \cap \text{Aut}(A), b \in \mathcal{E}(A) \right\}.$$

□

Corollary 6.8. *Let $F(G) < H \leq \text{Aut}(G)$. Then H is not commutative.*

Proof. Indeed, if $M \in H \setminus F(G)$ there is transvection $X \in E(G) \leq F(G) < H$ such that $MX \neq XM$. □

Recall that a subgroup is called *maximal abelian* if it is maximal among all the abelian subgroups and *abelian maximal* if it is abelian and maximal among all subgroups. Of course, every abelian maximal subgroup is maximal abelian but the converse may fail.

Corollary 6.9. *$F(G)$ is a maximal abelian subgroup in* $\text{Aut}(G)$. □

The next result is folklore

Lemma 6.10. *Let $H \leq G$. Then $H = \mathcal{C}_G(H)$ if and only if H is a maximal abelian subgroup of G.* □

Corollary 6.11. $F(G) = \mathcal{C}_{\text{Aut}(G)}(F(G))$. □

At last we have a property of $\text{Aut}(G)$ that θ transfers to $\text{Aut}(H)$.

Corollary 6.12. *The image $F(G)\theta$ is a maximal abelian subgroup in* $\text{Aut}(H)$. □

Of course this does not yet imply that $F(G)\theta = F(H)$, just that they are both maximal abelian subgroups of $\text{Aut}(H)$. However, we may conclude:

Theorem 6.13. *Let $\mathcal{A}$ be the class of all abelian groups A such that* $\text{Aut}(A \oplus A)$ *has a unique maximal abelian subgroup. If A, $B \in \mathcal{A}$, then* $\text{Aut}(A \oplus A) \cong \text{Aut}(B \oplus B)$ *implies* $\text{End}(A) \cong \text{End}(B)$.

Proof. By Corollary 6.12 and the hypothesis, $F(G)\theta = F(H)$. By Proposition 6.6, $F(G) = Z(\text{Aut}(G)) \times E(G)$ implies $F(H) = Z(\text{Aut}(H)) \times E(G)\theta$ and since also $F(H) = Z(\text{Aut}(H)) \times E(H)$, these direct complements are isomorphic. Hence finally $E(G) \cong E(H)$. It remains only to use the additive embedding

$$f_A : \text{End}(A) \longrightarrow E(G),\ s \mapsto \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix},$$

yielding the isomorphism $\text{End}(A) \xrightarrow{f_A} E(G) \longrightarrow E(H) \xrightarrow{(f_B)^{-1}} \text{End}(B)$. □

Remark 6.14. Let $\mathcal{A}'$ be any class of abelian groups such that for every group $G \in \mathcal{A}'$, all maximal abelian subgroups in Aut(G) are isomorphic and we can cancel $Z(\text{Aut}(G))$ from direct products. Then again, by the proof above, Aut$(A \oplus A) \cong$ Aut$(B \oplus B)$ implies End$(A) \cong$ End(B).

In another direction, notice that Lemma 6.7 above gives a little more than Corollary 6.9:

Proposition 6.15. *Among all the abelian subgroups of* Aut(G) *containing* $E(G)$*,* $F(G)$ *is the greatest* $($*i.e.,* $E(G) \leq L \leq \text{Aut}(G),\ L$ *abelian, implies* $L \leq F(G))$.

Proof. By contradiction: if $L \nleq F(G)$, there is a matrix $M \in L$ such that $M \notin F(G)$. By Lemma 6.7, there is a transvection $X \in E(G)$ with $MX \neq XM$ and so H is not commutative. □

References

[Cal06] Călugăreanu, G. *Determination of torsion-free Abelian groups by their direct powers*, `http://www.math.ubbcluj.ro/~calu/` Preprint, 2006.

[F70] Fuchs, L. *Infinite abelian groups*. Vol. I. Pure and Applied Mathematics. Vol. 36-I. Academic Press, New York-London, 1970.

[F73] Fuchs, L. *Infinite abelian groups*. Vol. II. Pure and Applied Mathematics. Vol. 36-II. Academic Press, New York-London, 1973.

[HO'M89] Hahn, A. J.; O'Meara, O. T. *The classical groups and K-theory*. Grundlehren der Mathematischen Wissenschaften, 291. Springer-Verlag, Berlin, 1989.

[KMT03] Krylov, P. A., Mikhalev, A V., Tuganbaev, A. A. *Endomorphism rings of abelian groups*. Algebras and Applications, 2. Kluwer Academic Publishers, Dordrecht, 2003.

[M00] Mader, A. *Almost comletely decomposable groups*. Algebra, Logic and Applications Series, Vol. 13, Gordon and Breach, 2000.

[MS00] Mader, A. and Schultz, P. *Endomorphism rings and automorphism groups of almost completely decomposable groups* Comm. Agebra, 28 (1), (2000), 51–68

[S59] Sasiada, E. *On two problems concerning endomorphism groups* Ann. Univ. Sci. Budapest, 2 (1959) 65–66

Author information

Grigore Călugăreanu, Department of Mathematics and Computer Science
Faculty of Science
Kuwait University, Kuwait 13060.
Email: `calu@sci.kuniv.edu.kw`

Phill Schultz, School of Mathematics and Statistics
The University of Western Australia
Nedlands, Australia, 6009.
Email: `schultz@maths.uwa.edu.au`