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# Categories whose objects are determined by their rings of endomorphisms

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In an additive category $A_0$, objects are said to be determined by their rings of endomorphisms if for each ring-isomorphism $F$ of the rings of endomorphisms of two objects $A, B$ in $A_0$ there is an isomorphism $f : A \to B$ in $A_0$ such that $F(\alpha) = f\alpha f^{-1}$, for every endomorphism $\alpha$ of $A$. Considering this problem in the context of closed categories (in Eilenberg and Kelly's sense), the author proves a general theorem which generalises results of Eidelheit (for real Banach spaces) and of Kasahara (for real locally convex spaces).

## 0. Introduction

Let $A, B$ be two objects in an additive category $A_0$. We consider the following problem: under what conditions on $A_0$ does a ring isomorphism $F : A_0(A, A) \to A_0(B, B)$ induce an isomorphism $f : A \to B$ in $A_0$, such that $F = A_0\left(f^{-1}, f\right)$, or equivalently, $F(\alpha) = f\alpha f^{-1}$, for every $\alpha \in A_0(A, A)$ ?

Examples of categories in which this problem has an affirmative answer are abundant. We concentrate on three of them, namely: the category of vector spaces and linear transformations over a division ring, the category

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of real Banach spaces and linear continuous transformations (see [1]) and the category of real locally convex spaces and linear continuous transformations (see [3]).

Having these in mind, the notion of closed category (in the sense of [2]) is easily seen to be needed. According to this, we rephrase our initial problem in the following terms.

Let $F : (AA) \to (BB)$ be an isomorphism in the closed category $\mathrm{A} = (\mathrm{A}_0, V, \hom \mathrm{A}, I, i, j, L)$, the subjacency of which is a ring homomorphism (and thus a ring-isomorphism). Under what conditions on $\mathrm{A}$ is there an isomorphism $f : A \to B$ in $\mathrm{A}_0$ such that $F = (f^{-1}, f)$ ? We shall use the notations in the paper by Eilenberg and Kelly [2] and shall also denote by $E(A)$ the ring $\mathrm{A}_0(A, A)$ of the endomorphisms of $A$ .

## 1. Preliminaries at the subjacent level

From now on, let $\mathrm{A}_0$ be an additive category with kernels and finite products (coproducts), and let $A, B$ be two objects in $\mathrm{A}_0$ .

LEMMA 1. *If $P(A)$, $P(B)$ denote the sets of direct factors (summands) of $A$ and $B$, respectively, then each ring-isomorphism $F : E(A) \to E(B)$ induces a canonical bijection $F^* : P(A) \to P(B)$ .*

Proof. By natural restriction, $F$ obviously induces a bijection between $EI(A)$ and $EI(B)$, the sets of idempotent endomorphisms of $A$ and $B$, respectively. It is then sufficient to indicate, for each object $A$ in $\mathrm{A}_0$, a bijection $U_A : P(A) \to EI(A)$ . If $A_1$ is a direct factor of $A$, and $p_1$, $u_1$ are the canonical projection and injection, respectively, then defining $U_A(A_1) = u_1 p_1$ and $U_A^{-1}(\theta) = \ker(1-\theta)$ for every $\theta \in EI(A)$, $U_A$ and $U_A^{-1}$ are easily seen to be mutually inverse (see [4], I, 18.5). So $F^* = U_B^{-1} . F/_{EI(A)} . U_A$ . □

REMARK. Considering $P(A)$ preordered by the well-known relation of comparing subobjects, one can easily verify that $F^*$ is actually a preorder isomorphism. Indeed, one has to show that if $(A_1, u_1) \leq (A_2, u_2)$ then $(F^*(A_1), u_1^*) \leq (F^*(A_2), u_2^*)$ ; but this follows immediately from $(1_B - F(u_2 p_2)).u_1^* = 0$ .

COROLLARY. *Under the assumptions of the previous lemma, if* $A = \prod_{i=1}^{n} A_i$ *then* $B = \prod_{i=1}^{n} F^*(A_i)$ .

Proof. Let $(u_i : A_i \to A)_{i=1}^{n}$ and $(p_i : A \to A_i)_{i=1}^{n}$ be the injections and projections of the biproduct $A$ . We then have $p_i u_j = \delta_{ij}$ and $\sum_{i=1}^{n} u_i p_i = 1_A$ . The morphisms $u_i p_i$ being idempotent, $F(u_i p_i)$ have the same property, and we have $F^*(A_i) = \ker(1_B - F(u_i p_i))$ . From $(1_B - F(u_i p_i)).F(u_i p_i) = 0$ follows the unique existence of morphisms $p_i^*$ such that $F(u_i p_i) = u_i^*.p_i^*$ , $u_i^*$ denoting the injection of the direct factor $F^*(A_i)$ in $B$ .

According to ([4], I, 18.1) we only have to show that $\sum_{i=1}^{n} u_i^* p_i^* = 1_B$ and $p_i^*.u_j^* = \delta_{ij}$ . The first equality is obvious because

$$\sum_{i=1}^{n} u_i^*.p_i^* = \sum_{i=1}^{n} F(u_i p_i) = F\left(\sum_{i=1}^{n} u_i p_i\right) = F(1_A) = 1_B ,$$

any ring-isomorphism being unital. As for the second, from $u_i^* = \ker(1_B - F(u_i p_i)) = \operatorname{equ}(1_B, F(u_i p_i))$ we derive

$$u_i^* = F(u_i p_i).u_i^* = u_i^*.p_i^*.u_i^* ,$$

and $u_i^*$ being mono, we have $p_i^*.u_i^* = 1$ . For $i \neq j$ we also have $u_i.p_i.u_j.p_j = 0$ , and then $F(u_i p_i).F(u_j p_j) = u_i^*.p_i^*.u_j^*.p_j^* = 0$ . So $p_i^*.u_j^* = 0$ follows, $p_j^*$ being epi (in fact, a retraction). □

REMARK. Applying this corollary to $F^{-1}$ and $(F^*)^{-1}$ , one verifies that $A'$ is an indecomposable factor of $A$ iff $F^*(A')$ is an indecomposable factor of $B$ .

LEMMA 2. *If* $(A_1; u_1, p_1)$ *is a direct factor of* $A$ *then there is a ring-isomorphism* $E(A_1) \to (u_1p_1)E(A)(u_1p_1)$ .

Proof. If we define $W_1 : E(A_1) \to (u_1p_1)E(A)(u_1p_1)$ and $W_1^{-1} : (u_1p_1)E(A)(u_1p_1) \to E(A_1)$ by $W_1(\theta_1) = u_1p_1\theta_1u_1p_1$ and $W_1^{-1}(u_1p_1\theta u_1p_1) = p_1\theta u_1$ , respectively; these are easily seen to be mutually inverse ring-homomorphisms. □

COROLLARY. *Under the assumptions of the previous lemma we have a ring-isomorphism* $E(A_1) \to E(F^*(A_1))$ .

Proof. We have only to notice that if $E(A)$ and $E(B)$ are ring-isomorphic then $u_1p_1E(A)u_1p_1$ and $u_1^*.p_1^*E(B)u_1^*.p_1^*$ are also ring-isomorphic. □

We are now in a position to prove the main subjacent-level result:

THEOREM 1. *Let* $A_0$ *be an additive category with kernels and finite products (coproducts), let* $A$, $B$ *be objects in* $A_0$ *, and let* $U$ *be a direct factor of* $A$ . *If* $F : E(A) \to E(B)$ *is a ring-isomorphism then there is a semi-linear isomorphism of abelian groups* $F_U : A_0(U, A) \to A_0(F^*(U), B)$ *, that is to say,* $F_U$ *is a group homomorphism and* $F_U(\alpha\theta) = F(\alpha).F_U(\theta)$ *holds for each* $\alpha$ *in* $E(A)$ *and* $\theta$ *in* $A_0(U, A)$ .

Proof. Let $u$ and $p$ , respectively, be the injection and the projection of $U$ in $A$ . Define $F_U(\theta) = F(\theta p).u^*$ , for each $\theta \in A_0(U, A)$ , where $u^*$, $p^*$ denote the injection and the projection of $F^*(U)$ in $B$ , respectively. It is only routine to verify that $F_U$ is a group-homomorphism which is semi-linear (in the sense described above). $F_U^{-1} : A_0(F^*(U), B) \to A_0(U, A)$ , defined by $F_U^{-1}(\theta^*) = F^{-1}(\theta^*p^*).u$ , is easily checked to be a two-sided inverse for $F_U$ . □

## 2. The main theorem

Let $\mathsf{A} = (\mathsf{A}_0, V, \hom \mathsf{A}, I, i, j, L)$ be a closed category. We shall be concerned with the following conditions:

A1: $\mathsf{A}_0$ is an additive category with kernels and finite (bi)-products;

A2: $V$ is a faithful functor;

A3: for each nonzero object $A$ in $\mathsf{A}_0$, $\mathsf{A}_0(I, A)$ contains a coretraction;

A41: the object $I$ is indecomposable (into direct (bi)products);

A42: according to A3, considering $I$ as a direct factor of $A$, for each ring-isomorphism $F : E(A) \to E(B)$ there exists an isomorphism $w_I : I \to F^*(I)$.

**THEOREM 2.** *Let* $\mathsf{A}$ *be a closed category which satisfies the conditions* A1, A2, A3, *and one of the conditions* A41, A42. *If for two objects* $A, B$ *in* $\mathsf{A}_0$ *there is an isomorphism* $\mathsf{F} : (AA) \to (BB)$ *in* $\mathsf{A}_0$, *the subjacency of which is a ring-homomorphism, then there is a canonical isomorphism* $f : A \to B$ ; *that is* $\mathsf{F} = (f, f^{-1})$ .

**Proof.** According to A3 we shall denote by $p_A$ and $u_A$, respectively, the projection and the injection of $I$ in $A$, and by $p_B^*$, $u_B^*$ the projection and the injection of $F^*(I)$ in $B$, respectively.

First, let us show that the morphism $\mathsf{F}_I : (IA) \to (F^*(I), B)$ in $\mathsf{A}_0$ given by $\mathsf{F}_I = (u_B^*, 1_B).\mathsf{F}.(p_A, 1_A)$ is an isomorphism. We note that if $F = V(\mathsf{F})$, then $F_I = V(\mathsf{F}_I)$. We shall prove that

$\mathsf{F}_I^{-1} : (F^*(I), B) \to (IA)$, given by $\mathsf{F}_I^{-1} = (u_A, 1_A).\mathsf{F}^{-1}.(p_B^*, 1_B)$, is a two-sided inverse for $\mathsf{F}_I$. In order to prove that

$$\mathsf{F}_I^{-1}.\mathsf{F}_I = (u_A, 1_A).\mathsf{F}^{-1}.(u_B^* p_B^*, 1_B).\mathsf{F}.(p_A, 1_A) ,$$

$$\mathsf{F}_I.\mathsf{F}_I^{-1} = (u_B^*, 1_B).\mathsf{F}.(u_A p_A, 1_A).\mathsf{F}^{-1}.(p_B^*, 1_B)$$

are both identities, it is sufficient to prove that

$$\left(u_B^* p_B^*, 1_B\right).\mathsf{F} = \mathsf{F}.\left(u_A p_A, 1_A\right) ,$$

because $p_A u_A = 1$ , $p_B^* u_B^* = 1$ . The subjacency functor being faithful, it is sufficient to check this equality at the subjacent level, namely,

$$\mathsf{A}_0\left(u_B^* p_B^*, 1_B\right).F = F.\mathsf{A}_0\left(u_A p_A, 1_A\right) .$$

Applying both members to a $\theta \in E(A)$ , one has $u_B^* p_B^* = F\left(u_A p_A\right)$ , which is true (see the proof of the corollary of the first lemma).

Now, using the remark following the same corollary, from A41, $I$ being indecomposable, $F^*(I)$ is also indecomposable and so, again by A3, there is an isomorphism $w_I : I \to F^*(I)$ . If we choose the condition A42 instead of A41, such an isomorphism $w_I$ also exists, by hypothesis.

We are now in a position to define the canonical isomorphism $f : A \to B$ as follows: $f = i_B^{-1}.\left(w_I, 1_B\right).\mathsf{F}_I.i_A$ , where $i$ is the natural isomorphism given with the closed category structure in $\mathsf{A}$ . It is clear that $f$ , as composite of isomorphisms, is also an isomorphism.

The functor $V$ being faithful, the two ways of requiring the canonicity of $f$ , namely, $V(\mathsf{F})(\alpha) = f.\alpha.f^{-1}$ and $\mathsf{F} = \left(f^{-1}, f\right)$ , are equivalent. We adopt the first one, which is also equivalent to $F(\alpha).f = f.\alpha$ for each $\alpha \in E(A)$ .

One has to verify that

$$F(\alpha).i_B^{-1}.\left(w_I, 1_B\right).\left(u_B^*, 1_B\right).\mathsf{F}.\left(p_A, 1_A\right).i_A =$$
$$= i_B^{-1}.\left(w_I, 1_B\right).\left(u_B^*, 1_B\right).\mathsf{F}.\left(p_A, 1_A\right).i_A.\alpha .$$

First, we note that

$$i_A.\alpha.i_A^{-1} = \left(1_I, \alpha\right)$$

and

$$i_B.F(\alpha).i_B^{-1} = \left(1_I, F(\alpha)\right)$$

are true, because of the naturality of the isomorphism $i$ . Thus it only remains to show that

$$(u_B^*.w_I,\ F(\alpha)).\mathrm{F}.(p_A,\ 1_A) = (u_B^*.w_I,\ 1_B).\mathrm{F}.(p_A,\ \alpha)\ .$$

But this can be readily checked at the subjactnt level: the equality $F(\alpha).F(\alpha.p_A).u_B^*.w_I = F(\alpha.\theta.p_A).u_B^*.w_I$ is true for every $\theta \in A_0(I, A)$ , $F$ being a ring homomorphism. This completes our proof. □

## 3. Several applications and comments

Let us denote by $K$ the category of vector spaces and linear transformations over a division ring $K$ , by $B$ the category of real Banach spaces and linear continuous transformations, and by $L$ the category of real locally convex spaces and linear continuous transformations.

First, it is obvious that A1 is fulfilled in any one of the three categories considered above. $K$ and $B$ have well-known structures of closed categories, and $L$ also admits such a structure, obtained by considering on the vector space of the linear continuous transformations the locally convex topology of $\sigma$-convergence, with $\sigma$ the family of the bounded subsets of the domain.

Next, for these closed categories, the division ring $K$ in $K$ and the real line in $B$ and $L$ are the corresponding objects $I$ . Indecomposables are one-dimensional spaces and so A41 is fulfilled. All these categories being concrete, the Condition A2 is satisfied. Finally, Condition A3 holds in all of these categories, because for each nonzero element in a space in any of these categories, there is a nontrivial functional which takes the value 1 (identity of $K$ or real number, respectively) on this element.

In fact, the largest category of topological vector spaces in which Condition A3 is satisfied contains all the spaces which admit a nontrivial functional, or equivalently (by a theorem of LaSalle), those which contain a proper open and convex subset. Unfortunately this is not a "nice" category for the rest of our conditions.

From another point of view, let $R$ be an associative ring with identity, and let $R$-mod be the closed category of left unitary

$R$-modules. One can now raise the problem: A1, A2 being satisfied in $R$-mod, under what conditions on the ring $R$ are Condition A3 and one of the Conditions A41 and A42 satisfied?

Unfortunately again, the answer is a deceiving one: $R$ must be a division ring. Indeed, Condition A3 implies the following concrete condition: the left $R$-module ${}_R R$ must be a direct summand of every non-zero left $R$-module.

Let us suppose that for a ring $R$ this condition holds. Let ${}_R M$ be a simple left $R$-module (such ones do exist, for example, $R/m$ for a left maximal ideal $m$ in $R$, considered as a left $R$-module). ${}_R R$ being isomorphic with a cyclic submodule of ${}_R M$, it follows that ${}_R R$ is actually isomorphic with ${}_R M$, and so simple as a left $R$-module.

Then $R$ is a not necessarily commutative division ring. For this last comment, I am indebted to Mr Nae Popescu.

## References


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