BREAKING POINTS IN SUBGROUP LATTICES

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Dedicated to R.R. Khazal

Abstract

The paper classifies those locally finite groups having a proper nontrivial subgroup which is comparable with any other element of the subgroup lattice.

1 Introduction

Let G be a group and let $L(G)$ denote its subgroup lattice. The description of groups G with $L(G)$ a chain is well-known. In a chain, every element is comparable with the others. This raises the natural question of seeing what can be said about groups G having a proper nontrivial subgroup H with the property that for every subgroup X of G one has either $X \leqslant H$ or $H \leqslant X$. Such a subgroup H will be called a *breaking point* for the lattice $L(G)$. For the sake of convenience, we shall call these groups BP-groups.

Of course, BP-groups cannot be decomposed as nontrivial direct products. Moreover, if G is a BP-group with breaking point H, then every subgroup K of G strictly containing H is itself a BP-group with breaking point H. These simple considerations are valuable in what follows and we shall use them without any further reference.

Standard results from abelian group theory dispose of the structure of abelian BP-groups: these are cyclic p-groups in the finite case and Prüfer p-groups $Z(p^\infty)$ in the infinite case. This focuses the discussion on nonabelian BP-groups.

As more exotic examples, the so-called extended Tarski groups, see Ol'shanskii [3], p. 344 are also BP-groups. If G is one of these groups, the largest breaking point is $Z(G)$, which is a finite cyclic p-group (p is a rather large prime) and $G/Z(G)$ is an infinite simple p-group of exponent p (a Tarski group). These (quasi finite) examples show that BP-groups need not be soluble, nor locally finite; the class of BP-groups is thus large enough to warrant a more serious investigation.

We shall restrict ourselves here to the particular case of locally finite BP-groups; the cyclic p-groups of order at least p^2 and the generalized quaternion groups are examples of finite BP-groups. As we have seen, the Prüfer p-groups exhaust the infinite abelian BP-groups - these groups are also locally finite. But there also exist infinite nonabelian locally finite BP-groups, as for example Szele's group S discussed in [2] and [6]: $S = A\langle x\rangle$, where A is a Prüfer 2-group and x has order

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four, x^2 is the unique element of order two of A and x acts on the normal subgroup A by inverting all elements of A.

The main result of this note shows that the examples described above exhaust all locally finite BP-groups:

Theorem 1.1 *A locally finite BP-group is isomorphic to one of the following groups: finite cyclic p-groups of order at least p^2, generalized quaternion groups, Prüfer groups $Z(p^\infty)$ and Szele's group S.*

The notation is standard and the proofs are elementary.

2 The proof of the Theorem

We need first some general information on BP-groups.

Lemma 2.1 *Let G denote a nonabelian BP-group. Then:*
1) G is a p-group for some prime p and if H is a beaking point of $L(G)$, then H is a finite cyclic group contained in $Z(G)$. In particular, G has a unique subgroup of order p.
2) $Z(G)$ is the largest breaking point of $L(G)$.
3) If G is infinite, a proper infinite abelian subgroup A of G has finite index if and only if A is normal in G.
4) If G is infinite and if p is odd, then a proper infinite abelian subgroup A of G has infinite index if and only if $A = N_G(A)$.

Proof 1) Note first that if H is a breaking point for $L(G)$ and if $g \in G \setminus H$, then H is a proper subgroup of $\langle g \rangle$, whence H is cyclic and central.

We prove next that H is finite. If $g \in G \setminus H$, then $H = \langle h \rangle < \langle g \rangle$, so we can write $h = g^a$ for some integer a with $|a| \geq 2$. Apply the same argument to the element $hg = g^{a+1} \in G \setminus H$ to obtain $h = g^{(a+1)b}$ for some integer b with $|b| \geq 2$.

Suppose now that H is infinite. Then both $\langle g \rangle$ and $\langle hg \rangle = \langle g^{(a+1)} \rangle$ are infinite and $g^a = h = g^{(a+1)b}$. One must have $a = (a+1)b$, for otherwise g would have finite order, a contradiction. Elementary divisibility arguments force $a = -2$ and $b = 2$, thus $h = g^{-2}$. The above argument, applied to the element $g^{-1} \in G \setminus H$, gives $h = g^2$, whence the equality $g^2 = h = g^{-2}$, forcing $g^4 = 1$, another contradiction.

We are now ready to show that G is a p-group. To prove this, we show first that G is periodic. If $g \in G \setminus H$ would be of infinite order, then $\langle g \rangle$ would contain a nontrivial finite subgroup, namely $H = \langle h \rangle$, a contradiction. Thus G is periodic indeed. If $g \in G \setminus H$, then the finite cyclic group $\langle g \rangle$ is a BP-group with breaking point H; thus $\langle g \rangle$ is indecomposable, forcing $\langle g \rangle$ to be a p-group for some prime p. This implies at once that G is a p-group. In particular, H is a finite cyclic p-group and G has a unique subgroup of order p. This concludes the proof of 1).
2) Suppose that $Z(G)$ is not a breaking point for $L(G)$. Then there exists some $x \in G \setminus Z(G)$ such that $Z(G) \nsubseteq \langle x \rangle$. The abelian group $K = Z(G)\langle x \rangle$ is not cyclic, nor a Prüfer group, which contradicts the fact that it must have a unique

subgroup of order p. Thus $Z(G)$ is a breaking point for $L(G)$. The maximality of $Z(G)$ follows from 1).
3) Let A be an infinite abelian subgroup of G. Note that A is not normal in G if and only if $core_G(A)$ is a proper subgroup of A if and only if $core_G(A)$ is finite - these follow from A being a Prüfer group. Also note that if A has finite index in G, then $core_G(A)$ has finite index in G. Thus A being not normal in G and having finite index are contradictory.
4) Let A be an infinite proper subgroup of G, so that A is a Prüfer p-group. Then A is a maximal abelian subgroup of G (for otherwise one would find a larger Prüfer subgroup B containing A, which is impossible), so $A = C_G(A)$. Now $N_G(A)/A$ is isomorphic to a subgroup of $Aut(A)$ and by 1) this factor group is periodic. This forces $N_G(A) = A$ since if p is odd $Aut(A)$ has no nontrivial elements of p-power order. This completes the proof of the Lemma. □

We are now in a position to give a proof of the Theorem.

Proof Let G be a locally finite BP-group. If G is finite, then either G is cyclic, or $p = 2$ and G is a generalized quaternion group. This follows from the lemma and from Satz 8.2, p. 310 of Huppert [1]. If G is infinite, then any two nontrivial elements of G generate a finite subgroup T of G which has just one minimal subgroup. For p odd, this subgroup T is cyclic. Thus G is abelian, which implies at once that $G \cong Z(p^\infty)$.

We reached the stage where all locally finite BP-groups were classified, except those which are infinite nonabelian 2-groups. From now on, G will denote a locally finite infinite nonabelian BP-group which is a 2-group.

Since G is nonabelian, there exists a pair of non commuting elements in G which generate a nonabelian subgroup K of G, which is a generalized quaternion 2-group. Thus, if X is any finite subset of G, then X is contained in the subgroup $L = \langle X, K\rangle$, which is also a generalized quaternion group. These groups are discussed in [2], p. 48, where it is proved that the unique such locally generalized quaternion group is actually Szele's group S described in the introduction. This concludes the proof of the Theorem. □

3 Final remarks

1) Using our Theorem and Theorem 2.4.16 of Schmidt [4], it is easy to prove that if an infinite nonabelian BP-group G has modular lattice $L(G)$, then G must be an extended Tarski group.

2) The Theorem implies that for an infinite BP-group G the following are equivalent: a) G is a Černikov group; b) G is locally finite; c) G is locally nilpotent; d) G is locally soluble.

3) By parts 2) and 1) of the Lemma, it follows that if G is a nonabelian BP-group, then there are only finitely many breaking points in $L(G)$. Therefore, if the subgroup lattice of a group has infinitely many breaking points, then the group must be a Prüfer p-group.

4) After a preliminary version of this note was written, the authors received a reprint of Prof. R. Schmidt's recent paper [5]. In Satz 3.1 of [5], it is shown that odd order p-groups are BP-groups precisely when there exist a so-called supermodular subgroup of G (which is a breaking point of $L(G)$). It is also shown there (for such infinite nonabelian p-groups of odd order) that $Z(G)$ is the largest breaking point of $L(G)$. A number of interesting questions remain still open:

a) Are there infinite BP-groups which are not locally finite and with non modular lattice $L(G)$?

b) Are there infinite BP-groups of infinite exponent which are not locally finite? Such groups would necessarily have infinite proper abelian subgroups of infinite index.

Prof A. Yu. Ol'shanskii, in a personal communication, kindly pointed out that the construction of such groups would be possible, but not very simple. This hints that a complete classification of BP-groups is far from being an easy task.

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