On S.N. Bernstein polynomials. The spectrum of the operator *

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S.N. Bernstein introduced the interpolation polynomials

$$B_n[f;x] = \sum_{i=0}^{n} f(\frac{i}{n})C_n^i x^i(1-x)^{n-i}$$

frequently used in approximation theory of continuous functions on a finite closed interval.

One can search for functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$B_n[f;x] = f(x).$$

Since $B_n[f;x]$ is always a polynomial, f must also be a polynomial. More precisely, we look for polynomials P which are fixed elements for B_n, that is

$$B_n[P(x);x] = P(x).$$

In what follows we prove the following

Theorem 1 *The Bernstein polynomial of a polynomial coincides with this one if and only if the degree of the polynomial is at most one.*

We may equivalently state this theorem also as: *for the linear and positive Bernstein operator, the only fixed polynomials are the polynomials of degree at most one.*

Proof. It is well-known that the condition is *sufficient*. This follows easily from

$$B_n[ax+b;x] = \sum_{i=0}^{n} (a\frac{i}{n} + b)C_n^i x^i(1-x)^{n-i}$$

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together with the formulas

$$\sum_{i=0}^{n} C_n^i x^i (1-x)^{n-i} = 1, \quad \sum_{i=0}^{n} i C_n^i x^i (1-x)^{n-i} = nx \qquad (1).$$

Hence

$$B_n[ax+b;x] = \frac{a}{n} nx + b.1 = ax + b.$$

In order to prove the condition is also *necessary*, it suffices to prove that a polynomial

$$P(x) = a_0 + a_1 x + ... + a_m x^m \qquad (m \leq n)$$

is equal to $B_n[P(x);x]$ only if $a_2 = a_3 = ... = 0$.

Notice that the problem would be easily solved if we succeed in finding type (1) formulas, until

$$\sum_{i=0}^{n} i^m C_n^i x^i (1-x)^{n-i}.$$

However, we will prove the condition is necessary, without finding explicitly values for these expressions, as follows: type (1) formulas are found starting with the binomial Newton formula

$$\sum_{i=0}^{n} C_n^i p^i q^{n-i} = (p+q)^n \qquad (2).$$

In equality (2) take the derivative with respect to p and then multiply by p, and repeat this operation m times. This way we obtain

$$\left.\begin{array}{rcl} \sum\limits_{i=0}^{n} i C_n^i p^i q^{n-i} & = & np(p+q)^{n-1} P_1 \\ \sum\limits_{i=0}^{n} i^2 C_n^i p^i q^{n-i} & = & np(p+q)^{n-2} P_2 \\ ... & .. & ... \\ \sum\limits_{i=0}^{n} i^m C_n^i p^i q^{n-i} & = & np(p+q)^{n-m} P_m \end{array}\right\} \qquad (3).$$

Here P_i $(i = 1, 2, ..., m)$ denotes a homogeneous polynomial relative to p and q, of degree $i-1$. For example $P_1 = 1$, $P_2 = np+q$, $P_3 = n^2p^2+(3n-1)pq+q^2$.

■

Lemma 2 *The polynomials P_i have the form $P_i = (np)^{i-1} + qQ_i$, where Q_i are degree $i-2$ homogeneous polynomials in p and q.*

In order to prove this, it suffices to show that, if

$$P_i = A_1^i p^{i-1} + A_2^i p^{i-2} q + ... + A_i^i q^{i-1} \qquad (4)$$

then $A_1^i = n^{i-1}$ for every $i = 1, 2, 3, ...$

Inspecting more carefully the procedure which gives the formulas (3), we find the recurrence formula

$$P_i = (n-i+2)P_{i-1} + qP_{i-1} + p(p+q)P'_{i-1} \qquad (4').$$

Replacing P_i and P_{i-1} from (4) and identifying the coefficients of p^{i-1}, we obtain

$$A_1^i = (n-i+2)A_1^{i-1} + (i-2)A_1^{i-1} = nA_1^{i-1}.$$

Since $P_1 = 1$ we obtain at once $A_1^i = n^{i-1}$, and the proof (of the Lemma) is complete. $\square$

Proof. Continuing the proof of the theorem, notice that taking $p = 1$, $q = -1$ in (4), we obtain

$$S_i = P_i(1, -1) = A_1^i - E_i$$

where $E_i = A_2^i - A_3^i + ... + (-1)^i A_i^i = n^{i-1} - (n-1)(n-2)...(n-i+1) > 0$, because from (4') we have $S_i = (n-i+1)S_{i-1}$ and so $S_i = (n-1)(n-2)...(n-i+1)$, $i-1$ parentheses. Hence $E_i = n^{i-1} - (n-1)(n-2)...(n-i+1) > 0$, $i = 2, 3, ..., m$.

Replacing p by x and q by $1-x$ we get the formulas

$$\left.\begin{array}{lcc} \sum\limits_{i=0}^{n} iC_n^i x^i (1-x)^{n-i} & = & nxP_1 = nx \\ \sum\limits_{i=0}^{n} i^2 C_n^i x^i (1-x)^{n-i} & = & nxP_2 = nx[nx + (1-x)Q_2] \\ ... & .. & ... \\ \sum\limits_{i=0}^{n} i^m C_n^i x^i (1-x)^{n-i} & = & nxP_m = nx[(nx)^{m-1} + (1-x)Q_m] \end{array}\right\} \qquad (3')$$

where the polynomials Q_i are degree $i-2$ homogeneous in x and $1-x$.

Finally consider $B_n[P(x); x] = B_n[a_0 + a_1 x + ... + a_m x^m; x] = \sum\limits_{i=0}^{n} (a_0 + a_1 \frac{i}{n} + ... + a_m \frac{i^m}{n^m}) C_n^i x^i (1-x)^{n-i} =$

$= a_0.1 + \frac{a_1}{n}.nx + ... + a_m.\frac{1}{n^m}[(nx)^m + nx(1-x)Q_m] =$
$= a_0 + a_1 x + ... + a_m x^m + x(1-x)n^{-1}[a_2 Q_2 + a_3 \frac{Q_3}{n} + ... \frac{a_m Q_m}{n^{m-2}}]$.

Asking for $B_n[P(x);x] = P(x)$ and denoting $K_i = \frac{a_i}{n^{i-2}}$, the proof is reduced to (checking) the equality

$$K_2 Q_2 + K_3 Q_3 + ... + K_m Q_m = 0 \qquad (5)$$

where $Q_i = A_2^i x^{i-2} + A_3^i x^{i-3}(1-x) + ... + A_i^i (1-x)^{i-2} = [A_2^i - A_3^i + ... + (-1)^i A_i^i]x^{i-2} + ... = E_i x^{i-2} + ...$ $(i = 2, 3, ..., m)$.

Identifying with zero the coefficients in (5), let us start with x^{m-2}. Since this gives $K_m E_m = 0$ and $E_m > 0$ we obtain $K_m = 0$ and the last term in (5) vanishes. Continuing with x^{m-3}, we obtain similarly $K_{m-1} = 0$, and step by step $K_{m-2} = K_{m-3} = ... = K_2 = 0$. Hence $a_2 = a_3 = ... = a_m = 0$ a the proof of the theorem is complete. ■

By proving this theorem, we found an eigenvalue for the Bernstein operator

$$B_n[f;x] = \lambda f \qquad (6),$$

namely $\lambda = 1$, and the corresponding eigenvectors: the degree one polynomials.

A natural problem is to find the spectrum of the Bernstein operator, that is, all its eigenvalues λ, and the corresponding eigenvectors f, respectively, which satisfy (6).

Again, since $B_n[P(x);x]$ is a polynomial, the eigenvectors f must be also polynomials; so for $P(x) = a_0 + a_1 x + ... + a_m x^m$, we must study the equality

$$B_n[P(x);x] = \lambda P(x).$$

Coming back to the computation above, we have

$$(1-\lambda)P(x) + \frac{a_2 x(1-x)Q_2}{n} + ... + \frac{a_m x(1-x)Q_m}{n^{m-1}} \equiv 0$$

where Q_m is a degree $m-2$ polynomial, $Q_m = E_m x^{m-2} + ...$ and $E_i = n^{i-1} - (n-1)(n-2)...(n-i+1) > 0$, $i = 2, 3, ..., m$.

Therefore a necessary condition for the existence of λ is $n^{m-1}(1-\lambda)a_m - a_m E_m = 0$, or $a_m(n^{m-1}(1-\lambda) - E_m) = 0$.

This way, since $E_m > 0$, if $\lambda \geq 1$ we have $a_m = a_{m-1} = ... = a_2 = 0$, and so there are no eigenvalues $\lambda > 1$.

The number $\lambda = 1 - \frac{E_m}{n^{m-1}} = (1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{m-1}{n})$ is an eigenvalue, the corresponding eigenvectors being degree m polynomials, depending homogeneously on a_m.

Since $\frac{E_i}{n^{i-1}} = 1-(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{i-1}{n})$, it follows that $\frac{E_i}{n^{i-1}} \neq \frac{E_j}{n^{j-1}}$ whenever $i \neq j$ and so $\lambda = 1-\frac{E_i}{n^{i-1}}$ are (different) eigenvalues for the Bernstein operator, the corresponding eigenvectors being degree i polynomials, $i = 2, 3, ..., m$.

If $\lambda \neq 1 - \frac{E_i}{n^{i-1}}$, $\lambda \neq 1$, $i = 2, 3, ..., m$ then obviously $a_m = a_{m-1} = ... = a_2 = a_1 = a_0 = 0$, the trivial case.

Hence we have proved the following

Corollary 3 *The Bernstein operator $B_n[f;x]$ (on n nodes) has exactly n eigenvalues, all in the real interval $(0,1]$, and these are*

$$\lambda_m = (1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{m-1}{n}), \quad m = 1, 2, ..., n.$$

To each eigenvalue λ_m correspond infinitely many eigenvectors, all degree m polynomials.