Determination of torsion-free Abelian groups by their direct powers.

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Abstract

The automorphism groups and the endomorphism rings *of the direct powers*, sometimes give better information on torsion-free Abelian groups than the automorphism groups respectively the endomorphism rings of these groups.

1 Introduction

It is reasonable to consider that for any type of mathematical object, some construction on a direct power sometimes provides more information than the same construction on the object itself. In this paper, the constructions we discuss are the automorphism group and endomorphism ring of some torsion-free Abelian groups.

While Abelian p-groups are determined by their automorphism groups, torsion-free Abelian groups are usually not distinguished by these (not even the rank 1 groups).

A similar situation occurs for endomorphism rings of torsion-free Abelian groups: while a celebrated result of Baer states that torsion Abelian groups are determined by their endomorphism rings, for torsion-free Abelian groups this fails (examples can be given even among rank 1 groups).

For an Abelian group G, one can easily check that $\varphi : (\mathrm{End}(G), +) \longrightarrow (\mathrm{Aut}(G \times G), \circ)$ defined by $\varphi(f) = \begin{bmatrix} 1 & 0 \\ f & 1 \end{bmatrix}$, is a group embedding. This embedding justifies the idea that torsion-free Abelian groups could be better distinguished by the automorphism groups of their squares than by their usual automorphism groups.

Moreover, this opens a new direction of research: determining torsion-free Abelian groups by using the endomorphism rings of their squares.

In this paper, we study to what extent the automorphism group respectively the endomorphism ring of a square (or some larger powers) determine (or not) some special classes of torsion-free Abelian groups. Our main results (referred as square or n-auto-uniqueness respectively square endo-uniqueness) are

Theorem 5. $\mathrm{Aut}(G^2) \cong \mathrm{Aut}(H^2)$ *implies* $G \cong H$ *for any 2-divisible rational groups of idempotent type* G, H.

Proposition 8. *For any positive integer* $n \geq 3$, $\mathrm{Aut}(G^n) \cong \mathrm{Aut}(H^n)$ *implies* $G \cong H$ *for Abelian groups having principal ideal domain endomorphism rings.*

and,

Proposition 9. *For any positive integer* $n \geq 4$, $\mathrm{Aut}(G^n) \cong \mathrm{Aut}(H^n)$ *implies* $G \cong H$ *for Abelian groups having commutative endomorphism rings.*

In the last Section we prove some square endo-uniqueness results.

2 Terminology and Preliminary results

For an Abelian group G, $\mathrm{End}(G)$ denotes its endomorphism ring and $\mathrm{Aut}(G)$ its automorphism group, and, for a ring with identity R, we denote by R^+ the additive group of this ring, by $U(R)$ the group of all units in R and by $GL_n(R) = U(\mathcal{M}_n(R))$ the linear group of invertible matrices having entries in R (here $\mathcal{M}_n(R)$ denotes the full $n \times n$ matrix ring).

For a set of primes P, we denote by $\mathbf{Z}[P^{-1}]$ the subring with identity of $\mathbf{Q}$ generated by $\{p^{-1} | p \in P\}$. Since this will play a role somewhere below, notice that $\mathbf{Z}[P^{-1}]$ is generated, qua ring, by its units. Note that for any rational group R, $\mathrm{End}(R) \cong \mathbf{Z}[P_R^{-1}]$ where $P_R = \{p \in \mathbf{P} | pR = R\}$.

Hence $\mathrm{End}(R^2) \cong \mathcal{M}_2(\mathrm{End}(R)) = \mathcal{M}_2(\mathbf{Z}[P_R^{-1}])$ respectively $\mathrm{Aut}(R^2) \cong GL_2(\mathrm{End}(R)) = GL_2(\mathbf{Z}[P_R^{-1}])$.

If $\mathbf{t}(R)$ is the type of a rational group R, notice that $P_{R_1} \neq P_{R_2} \overrightarrow{\not\Leftarrow} \mathbf{t}(R_1) \neq \mathbf{t}(R_2) \Leftrightarrow R_1 \ncong R_2$. However, the nonreversible implication holds if we consider only idempotent types.

For a torsion-free Abelian group G the symbols $G(\mathbf{t})$, $G^*(\mathbf{t})$ and $G^\sharp(\mathbf{t}) = \langle G^*(\mathbf{t}) \rangle_*$ denote the usual type subgroups and $G^0(\mathbf{t}) = G(\mathbf{t})/G^\sharp(\mathbf{t})$. A type $\mathbf{t}$ is *critical* for G if $G^0(\mathbf{t}) \neq 0$. A group G is called *slim* (see [7]) if $\mathrm{rk}G^0(\mathbf{t}) = 1$ for each critical type $\mathbf{t}$, *block-rigid* if its partially ordered set of critical types is an antichain, and *rigid* if it is both slim and block-rigid. We will use the following

Proposition 1 *([7]) Let* G *be a finite rank (almost) completely decomposable group. Then* $\mathrm{End}(G)$ *is commutative if and only if* $\mathrm{Aut}(G)$ *is commutative and if and only if* G *is rigid.*

Notice that the square of a rational group is block-rigid, but not slim: $\mathrm{rk}(R^2)^0(\mathbf{t}) = 2$ for $\mathbf{t}$ the type of R. Hence neither $\mathrm{End}(R^2)$ nor $\mathrm{Aut}(R^2)$ are commutative.

The following statements are readily checked:

Lemma 2 *For arbitrary rings S and T, any group isomorphism $f : GL_n(S) \to GL_n(T)$ induces a group isomorphism $f^* : U(S) \cap Z(S) \to U(T) \cap Z(T)$ (here $Z(S)$ denotes the center of S).*

Special Case: *For Abelian groups G and H, every group isomorphism $\varphi : \mathrm{Aut}(G^2) \to \mathrm{Aut}(H^2)$ induces an isomorphism $\varphi^* : \mathrm{Aut}(G) \cap Z(\mathrm{End}(G)) \longrightarrow \mathrm{Aut}(H) \cap Z(\mathrm{End}(H))$.*

Special Case: *For Abelian groups having commutative endomorphism rings, $\mathrm{Aut}(G^2) \cong \mathrm{Aut}(H^2)$ implies $\mathrm{Aut}(G) \cong \mathrm{Aut}(H)$.*

Lemma 3 *For arbitrary rings S and T, any ring isomorphism $g : \mathcal{M}_n(S) \to \mathcal{M}_n(T)$ induces a ring isomorphism $g^* : Z(S) \to Z(T)$.*

Special Case: For Abelian groups G and H, every ring isomorphism $\varphi : \mathrm{End}(G^2) \to \mathrm{End}(H^2)$ induces an isomorphism $\varphi^* : Z(\mathrm{End}(G)) \longrightarrow Z(\mathrm{End}(H))$.

Special Case: For Abelian groups having commutative endomorphism rings, $\mathrm{End}(G^2) \cong \mathrm{End}(H^2)$ implies $\mathrm{End}(G) \cong \mathrm{End}(H)$.

In a negative direction, we just point out that there are examples (e.g., see [8]) of nonisomorphic (indecomposable torsion-free) Abelian groups A and B, such that A^2 and B^2 are isomorphic.

This shows that our expectation, that squares sometimes bear more information than the groups, has some limits.

Summarizing, we deal with special cases of the following more general problems:

(i) for two rings S and T, when does $GL_2(S) \cong GL_2(T)$ imply $S^+ \cong T^+$?

(ii) for two rings S and T, when does $\mathcal{M}_2(S) \cong \mathcal{M}_2(T)$ imply $S^+ \cong T^+$?

There is a great deal of literature (maybe more than 100 years long) covering such problems.

As for (i) we refer to Hahn-O'Meara [4] where most of important results are presented. While for $n \geq 3$ there is already a fairly satisfactory theory, unfortunately the case $n = 2$ is kind of pathological (only one page in [4]) and only some special results are proved elsewhere.

In our research we use two papers devoted precisely to GL_2: P.M. Cohn [1] (1966) and M. Dull [2] (1974).

As for (ii) we refer to the excellent updated survey [6] written by T.Y. Lam (1994). For unexplained terminology on Abelian groups the reader is referred to Fuchs [3] and Krylov et alt. [5].

3 Auto-uniqueness

For any rational group R notice that $\mathrm{Aut}(R) \cong \langle \pm 1 \rangle \times F_m$ (here F_m denotes the free Abelian group of rank m) with cardinal $m = |P_R|$. Hence

Lemma 4 *For any two rational groups R_1, R_2, $\mathrm{Aut}(R_1) \cong \mathrm{Aut}(R_2)$ if and only if the cardinals $|P_{R_1}| = |P_{R_2}|$.* $\blacksquare$

In what follows, we prove a square auto-uniqueness theorem for mostly all the rational groups.

Theorem 5 $\mathrm{Aut}(G^2) \cong \mathrm{Aut}(H^2)$ *implies $G \cong H$ for any 2-divisible rational groups of idempotent type G, H.*

Before giving the Proof, some preparatives are necessary.

First notice that if R is *locally free* (i.e., $P_R = \varnothing$) then $\mathrm{Aut}(R^2) \cong GL_2(\mathbf{Z})$, which shows that *automorphism groups of squares do not distinguish locally free rational groups*. Therefore, in the sequel, we consider only rational groups with $P_R \neq \varnothing$ (more precisely $P_R = \{2\}$).

Secondly, since the proof uses a result of M. Dull (see [2]), we first explain Dull terminology (see also [4] or [1]).

An integral domain S (not necessary commutative) is *generalized Euclidean* (or a GE_2-ring) if $GL_2(S) = GE_2(S)$, that is, invertible matrices are generated by elementary transvections and diagonal (both these are also called elementary) matrices. For two rings S, T a *U-homomorphism* $f : S \to T$ is a homomorphism $x \mapsto x'$ of S^+ into T^+ such that $1' = 1$ and $(\alpha a \beta)' = \alpha' a' \beta'$ for all $a \in S$ and α, β units in S (i.e., a generalization of ring homomorphism, which is given in Cohn [1]; a more technical definition for this concept is given by Dull [2], but in the cases we are dealing with, these two definitions are equivalent). *U-anti-homomorphisms* are defined analogously.

The following is elementary: *let S be a ring which, qua ring, is generated by its units. Then any U-homomorphism of S into an arbitrary ring is a ring homomorphism.*

For a commutative integral domain $SL_2 = E_2$ if the special linear group is generated only by elementary trasvections.

While Dull's result is formulated for the automorphism situation only, the method of proof applies to the isomorphism problem as well. So we reformulate his result as follows

Let S, T be commutative integral domains of zero characteristic having 2 as unit and such that $SL_2 = E_2$. Then any isomorphism $\Lambda : GL_2(S) \to GL_2(T)$ is of the form $P_\chi \circ \Phi_g \circ \Gamma$ where, Γ is induced by a U-isomorphism $S \to T$.

We are now ready for the

Proof. Since subrings with 1 of $\mathbf{Q}$ are, as rings, generated by their units, U-isomorphisms are, in this special case, ring isomorphisms. Moreover, since subrings with 1 of $\mathbf{Q}$ are Euclidean, these are also generalized Euclidean and so $GL_2 = GE_2$ and $SL_2 = E_2$ hold.

For a rational group of idempotent type R, assume the first entry in the type $\mathbf{t}(R)$ is nonzero and thus infinity. In this case 2 is a unit in the endomorphism ring (a common hypothesis for automorphism groups of Abelian groups).

Then, for two rational groups R_1 and R_2 with $\mathrm{Aut}(R_1^2) \cong \mathrm{Aut}(R_2^2)$, we deduce $GL_2(\mathbf{Z}[P_{R_1}^{-1}]) \cong GL_2(\mathbf{Z}[P_{R_2}^{-1}])$, and so, by Dull's result, $\mathbf{Z}[P_{R_1}^{-1}] \cong \mathbf{Z}[P_{R_2}^{-1}]$. Hence $P_{R_1} = P_{R_2}$ (indeed, if $P_1 \neq P_2$ are nonempty sets of primes, then the underlying additive subgroups of $\mathbf{Z}[P_1^{-1}]$ and $\mathbf{Z}[P_2^{-1}]$ are not isomorphic, as groups, and therefore, nor as rings) and finally $R_1 \cong R_2$, since we have considered only rational groups of idempotent type. ■

Comparing this result with Lemma 4, shows that automorphism groups of squares, better distinguish rational groups (which are not locally free) than the simple ones.

Example. Let $\mathbf{Q}^{(2,3)}$ and $\mathbf{Q}^{(2,5)}$ be rational groups of type $(\infty, \infty, 0, 0, ...)$ and $(\infty, 0, \infty, 0, 0, ...)$, respectively. Then $\mathrm{Aut}(\ \mathbf{Q}^{(2,3)}) \cong \mathrm{Aut}(\mathbf{Q}^{(2,5)})$ but $\mathrm{Aut}\left((\mathbf{Q}^{(2,3)})^2\right) \ncong \mathrm{Aut}\left((\mathbf{Q}^{(2,5)})^2\right)$, i.e., non-isomorphic rational groups, distinguished by the automorphism groups of their squares but not by their automorphism groups.

Remark 6 *Since rings in Cohn and Dull's results should be domains, a similar result (square auto-uniqueness) for the class of the finite rank completely decomposable rigid groups, with 2-divisible summands, cannot be proved in this way (i.e., by extending the previous Theorem)*

Recall that the *pseudo-socle* of a torsion-free group is the pure subgroup generated by all minimal pure fully invariant subgroups. A torsion-free group is *strongly indecomposable* if it does not have nontrivial quasi-decompositions, and *irreducible* if it has no proper minimal pure fully invariant subgroups.

Proposition 7 $\mathrm{Aut}(G^2) \cong \mathrm{Aut}(H^2)$ *implies* $G \cong H$ *for any strongly indecomposable (torsion-free Abelian of finite rank) groups* G,H *which have* pseudo-socle *equal to the group (in particular, irreducible strongly indecomposables).*

Proof. Torsion-free Abelian groups of finite rank having (not necessarily commutative) domains as endomorphism rings are (see [5]) the strongly indecomposable groups G which have *pseudo-socle* $\mathrm{Psoc}(G) = G$. We just have to use the following Cohn's result, which essentially needs the domain restriction and also some others (but not commutativity):

Theorem 12.2 - *Let S be k-ring and T a k'-ring, both with degree-function, where k and k' are any fields of characteristic $\neq 2$ and S is either a GE_2-ring or all projective S-modules on two generators are free. Then every isomorphism between $GL_2(S)$ and $GL_2(T)$ is obtained by taking the isomorphism induced by a U-isomorphism or U-anti-isomorphism, followed by a central homomothety and an inner automorphism.*

Explaining terminology here would take another half-page (as we already did it for Dull's result), so we skip the details. ■

Since there is an essential difference between results concerning the isomorphism problem for linear groups for $n = 2$ respectively $n \geq 3$, in the problem we discuss, it is easier to obtain better results for cubes and larger direct powers of Abelian groups.

Proposition 8 *For any positive integer $n \geq 3$, $\mathrm{Aut}(G^n) \cong \mathrm{Aut}(H^n)$ implies $G \cong H$ for Abelian groups having principal ideal domain endomorphism rings.*

Proof. Use **3.3.8** from [4]. ■

Characterizations of Abelian groups having principal ideal domain endomorphism rings do exist (for torsion-free Abelian groups see [5]). In particular, these are self-small groups.

Proposition 9 *For any positive integer $n \geq 4$, $\mathrm{Aut}(G^n) \cong \mathrm{Aut}(H^n)$ implies $G \cong H$ for Abelian groups having commutative endomorphism rings.*

Proof. Use **3.3.11** from [4]. ■

4 Square endo-uniqueness

Proposition 10 $\mathrm{End}(G^2) \cong \mathrm{End}(H^2)$ *implies $G \cong H$ for any rational groups of idempotent type G, H.*

Proof. For commutative rings S and T, $M_n(S) \cong M_n(T)$ implies (since the centers are also isomorphic) $S \cong T$. Since endomorphism rings of rational groups are commutative, as already noticed in Section two, $\mathrm{End}(G^2) \cong \mathrm{End}(H^2)$ implies $\mathrm{End}(G) \cong \mathrm{End}(H)$. Finally we have to restrict to rational groups of idempotent type in order to recover $G \cong H$. ■

Proposition 11 $\mathrm{End}(G^2) \cong \mathrm{End}(H^2)$ *implies $G \cong H$ for finite rank completely decomposable rigid groups G,H.*

Proof. Since finite rank completely decomposable rigid groups have commutative endomorphism rings (see Proposition 1), using Proposition 10, what remains is the use of the well-known ring isomorphism $\mathrm{End}(\bigoplus_{i=1}^{n} K_i) \cong \prod_{i=1}^{n} \mathrm{End}(K_i)$ for fully invariant direct summands (and in particular for rigid completely decomposable groups). ■

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