MATHEMATICA, Tome 39(62), N° 2, 1997, pp. 195–200

ESSENTIAL-PURE SUBGROUPS OF ABELIAN GROUPS

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1. Introduction

In a joint paper [2] Benabdallah, Charles, Mader found in p-groups a class of subgroups which generalizes the class of all the pure subgroups, still has the inductive property, gives an answer to the question above and in addition has neither of the following two "pathologies": there are subsocles that do not support pure subgroups, and, the p-adic closures of pure subgroups are not in general pure.

These subgroups are called *vertical* and are defined as follows: a subgroup A is vertical in a p-group G if

$$(A + p^nG)[p] = A[p] + p^nG[p], \forall n \in \mathbb{N}^*$$

In what follows, we propose (not only in p-groups, but in arbitrary abelian groups) another generalization of purity which also answers the question we mentioned in the abstract.

2. The general construction

Let P be a property of subgroups of abelian groups and, for every G, let $\mathscr{C}(G)$ be the class of the subgroups of G which have the property P. In what follows we suppose the next conditions fulfilled for each G (*i*): $G \in \mathscr{C}(G)$; (*ii*): $A \in \mathscr{C}(G)$, $A \leq B \leq G \Rightarrow A \in \mathscr{C}(B)$(e.g. pure, neat, essential, direct summand, etc.). Now, if P and Q are two such properties and $\mathscr{C}(G)$, $\mathscr{D}(G)$ are the corresponding classes of subgroups, we suppose that (*iii*): for every G, $\mathscr{C}(G) \cap \mathscr{D}(G) = \{G\}$ i.e. the properties are "far" enough.

A subgroup A has the **property P-Q** in G if there is a subgroup C of G such that A has P in C and C has Q in G. Correspondingly, we denote by $\mathscr{C}\mathscr{D}(G)$ the class of the subgroups of G which have the property P-Q, i.e. $A \in \mathscr{C}\mathscr{D}(G) \Leftrightarrow \exists C\colon A \in \mathscr{C}(C)$, $C \in \mathscr{D}(G)$.

Remark 2.1. Condition (i) implies that the new classes $\mathscr{C}\mathscr{D}(G)$ and $\mathscr{D}\mathscr{C}(G)$ contain the initial ones.

A subgroup A of G is called **$\mathscr{C}$-closed** (or **P-closed**) if $A \in \mathscr{C}(B)$, $A \leq B \leq G \Rightarrow \Rightarrow A = B$ (i.e. A has not the property P in another subgroup of G).

Simple verifications lead to the important general

THEOREM 2.1. *A subgroup* $A \in \mathscr{C}(G)$ *iff* $A \in \mathscr{D}\mathscr{C}(G)$ *and* A is $\mathscr{D}$-closed. □

In the sequel, let P be the purity [5] and Q be the essentialness of the elements in a complete lattice L. In this selection the properties (i) – (iii) are well known in an algebraic modular lattice (the only pure essential element of such a lattice being 1).

An element x is called **essential-pure** in the lattice L if x is essential in some sublattice $c/0$ such that c is a pure element of L (and **pure-essential**, symmetrically). A lattice L satisfies the **restricted socle condition** [3] if for every $a < b$ with a essential in $b/0$, the sublattice b/a contains atoms (e.g. the lattice $L(G)$ of all the subgroups of an abelian group).

Remark 2.2. a) Every element is pure-essential in a pseudocomplemented (see [11] or [3]) modular lattice.

Indeed, if b is a pseudocomplement of a in L then $a \vee b$ is essential in L. Being a complement for b in $a \vee b$, a is surely pure in $a \vee b$.

b) *The pure-closed elements are exactly the essential ones*. Indeed, if x is essential in L and x is pure in $c/0$ $(x \leq c)$ then x is also essential in $c/0$ so $x = c$. Conversely, if x is not essential in L there is a non-zero element a in L such that $x \wedge a = 0$. But in this case x is a complement of a in $x \vee a$ and hence pure so that x is not pure-closed.

c) *The essential-closed elements are* (cf. [6] and [3]) *exactly the neat elements, if L satisfies the restricted socle condition.*

d) *Pure* = *neat* + *(essential-pure), in a lattice L satisfying the restricted socle condition.*

Indeed, from the above theorem one has: x is pure iff x is essential-pure and essential-closed, so (d) follows from (c).

3. Essential-pure subgroups

In what follows we study essential-pure subgroups of an arbitrary abelian group.

Obviously, each pure subgroup and each essential subgroup of a group are essential-pure subgroups. A simple example of essential-pure subgroup which generally needs not to be pure nor essential is the socle of a group.

A useful generalization of this example is obtained for each subset X of a group G as follows

$$S(G,X) = \{g \in G \mid \exists n \in \mathbb{N}^* \text{ square-free}, \ ng \in \langle X \rangle\}$$

$$P(G,X) = \{g \in G \mid \exists n \in \mathbb{N}^*, \ ng \in \langle X \rangle\}.$$

These are subgroups of G which contain $\langle X\rangle$, $S(G, X)$ is essential in $P(G, X)$ (indeed, $P(G, X)/S(G, X)$ is torsion and $S(P(G, X)) \leq S(G) \leq S(G, X)$), and $P(G, X)$ is pure in G. Moreover, $S(G, X)/\langle X\rangle = S(G/\langle X\rangle)$, $P(G, X)/\langle X\rangle = T(G/\langle X\rangle)$, $S(G, \emptyset) = S(G,0) = S(G) \leq S(G, X) \leq S(G,G) = G, P(G, \emptyset) = P(G,0) = T(G) \leq P(G, X) \leq$ $\leq P(G,G) = G$.

Hence, for every subset X of a group G, *$S(G, X)$ is essential-pure in G* so that each subset X of a group G is contained in an essential-pure subgroup. Unfortunately $S(G, X)$ is not the smallest essential-pure subgroup that contains X (e.g. for $G = \mathbb{Z}(p^2)$ select $X = S(G) = pG$). There are "enough" essential-pure subgroups: only the simple groups are essential-pure-simple. Indeed, essential-pure-simple groups must be essential-simple and pure-simple. These classes are known: the elementary groups respectively the rank 1 groups. Between the elementary subgroups of $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ we find only the simple groups $\mathbb{Z}(p)$.

Remark 3.1. A subgroup A is essential-pure in a primary group G iff $S(A)$ supports a pure subgroup of G which contains A.

Subsocles U of G that do not support any pure subgroup of G do exist [7], so the subgroups which support such subsocles U are not essential-pure in G.

Not only $S(G, X)$ – subgroup which contains $S(G)$ – is essential-pure.

THEOREM 3.1. *Every subgroup A which contains the socle $S(G)$ is essential-pure in the group G.*

Observe that for a torsion group G this is immediate A being essential. In the general case $S(P(G, A)) \subseteq S(G) \subseteq A$ and $P(G, A)/A = T(G/A)$ so A is essential in $P(G, A)$□.

Remark 3.2. $\langle X\rangle$ is essential in $P(G, X) \Leftrightarrow S(G) \leq \langle X\rangle$, so that a more general result cannot be obtained in this way.

Consequence 3.1. *In torsion-free groups every subgroup is essential-pure.*
Obvious: $S(G)=0$.□

PROPOSITION 3.1. *Let G be a mixed group; if A is essential-pure in G then $T(A)$ is essential-pure in $T(G)$. Conversely, if $T(A) \neq 0$ is essential-pure in $T(G)$ then A is essential-pure in G.*

Indeed, if $T(A)$ is essential in C then A is also essential in C only if $T(A) \neq 0$.□

PROPOSITION 3.2. *Let G be a torsion group. A is essential-pure in G iff A_p is essential-pure in G_p.*

Indeed, one verifies that A is essential (resp. pure) in C iff A_p is essential (resp. pure) in C_p.□

So, it remains to characterize the essential-pure subgroups only for reduced (see also the Proposition 3.5 below) p-groups and for torsion-free subgroups of mixed groups.

In an arbitrary p-group the socle of an essential-pure subgroup is purifiable (indeed, if A is essential in C then $A[p] = C[p]$). Moreover,

PROPOSITION 3.3. *In a p-group G a subgroup A is essential-pure iff the subsocle $A[p]$ is a kernel of purity (has all the $A[p]$-high subgroups pure).*

Indeed, the subsocle $A[p]$ can always be extended to a neat subgroup K which supports $A[p]$ and contains A (one applies Zorn's lemma). This subgroup K is pure iff $A[p]$ is a kernel of purity. □

Hence, for reduced p-groups, we can use corresponding characterizations of the kernels of purity ([8] and [9]).

However, for quasi-complete groups (e.g. direct sums of cyclic groups, torsion-complete groups) essential-pure subgroups coincide with the vertical ones (see theorem 4.1 below) so a corresponding characterization can be done in terms of the overhang functors (see [2]).

In the second case, the torsion-free essential-pure subgroups of a mixed group can be more proven. First we observe that there are no essential subgroups in this case ($S(G) \neq 0$ is not contained in any torsion-free subgroup) and A is pure in G iff $A \oplus T(G)$ is pure in G.

PROPOSITION 3.4. *In a splitting mixed group all the non-zero torsion-free subgroups are essential-pure.*

Indeed, if $0 \neq A$ is a torsion-free subgroup of a splitting mixed group G let F be a $T(G)$-high pure subgroup of G which includes A (for the existence see [1]). Then A is essential in $P(F, A)$ and $P(F, A)$ (the subgroup pure-generated by A in F) is pure in F and hence also pure in G (by transitivity). So A is essential-pure in G. □

An immediate example of this kind is $n\mathbb{Z}$ essential-pure (but not pure) in $\mathbb{Z}_2 \oplus \mathbb{Z}$. More generally, if F is a torsion-free group, T a torsion group and A is a subgroup of F such that F/A is a non-zero torsion group then A is essential-pure but not pure nor essential subgroup of $F \oplus T$.

In the non-splitting case, the characterization of the torsion-free essential-pure subgroups remains an open question.

PROPOSITION 3.5. *In divisible groups, every subgroup is essential-pure.*

Indeed, one has to take C a divisible hull of A in G. □

PROPOSITION 3.6. *Every essential-pure subgroup A of a p-group G, contained in $p^{\omega}G$, is divisible.*

Indeed, one uses the well-known property: if in a p-group G, every element of order p is of infinite height, then G is divisible. An essential-pure subgroup A has the same socle as a pure subgroup C of G which includes A so that heights are computed in the same way in $G[p]$ and $C[p]$. Hence A is divisible. □

Consequence 3.2. Let G be a reduced p-group with $p^{\omega}G \neq 0$. $p^{\omega}G$ is not essential-pure and contains no essential-pure subgroups. □

Remark 3.3. The essential-pure subgroups have the two "pathologies" mentioned above.

The subsocles which support no pure subgroups, support neither essential-pure subgroups. As for the second "pathology", in the conditions of the last theorem, 0 is pure and so essential-pure in G but closure $p^{\omega}G$ is not.

4. Comparison with vertical subgroups

PROPOSITION 4.1. *Every essential-pure subgroup is vertical. The converse is not in general true.*

Indeed, using Prop.1.6 from [2] one proves that every essential subgroup is vertical. Now, if A is essential-pure in G, A is essential and so, vertical in a pure subgroup C of G. Hence A is vertical in G by the property 2.10 (1') from [2]. Finally, if U is a subsocle of G which supports no pure subgroup of G then U is vertical (cf. Prop. 2.2 [2]) but not essential-pure. □

However,

THEOREM 4.1. *In a reduced p-group G every vertical subgroup is essential pure iff G is quasi-complete* (cf. [4]).

Indeed, in [2] it is proved that in a reduced p-group G all the maximal vertical subgroups are pure iff G is quasi-complete. Every vertical subgroup is essential in a maximal vertical subgroup containing it. Conversely, if M is maximal vertical, M essential in C and C pure in G then $M = C$ and so M is pure. □

5. Comments

One can choose $\mathscr{C}(G)$ the class of all the direct summands or the class of all the neat subgroups. Similarly, all the subgroups of a group are neat-essential and direct summand-essential. The direct summand-closed subgroups are also the essential ones but the neat-closed subgroups form a smaller (unstudied) class.

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Received 18 VIII 1996

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