STUDIA UNIV. BABEŞ—BOLYAI, MATHEMATICA, XXXI, 3, 1986

ABELIAN GROUPS WITH PSEUDOCOMPLEMENTED LATTICE OF SUBGROUPS

GRIGORE G. CĂLUGĂREANU*

Received: October 13, 1986

ABSTRACT. — In this paper we prove that for the lattice of the subgroups of an abelian group pseudocomplementation and distributivity are equivalent. We also characterize abelian groups which have a Stone lattice or a Heyting algebra of subgroups.

Let L be a lattice with zero and $0 \neq a \in L$. If $C_a = \{x \in L / a \wedge x = 0\}$, the greatest element of C_a (if it exists) is called the pseudocomplement of a in L. (Note that the "pseudocomplement" is differently used for an unspecified maximal element of C_a). If every element in L has a pseudocomplement, L is called a pseudocomplemented lattice.

We first recall the following known facts:

(A) Every distributive compactly generated lattice is pseudocomplemented.

(B) If A is an abelian group, the lattice $L(A)$ of all the subgroups of A is compactly generated.

1. LEMMA. *Let P be an inductive poset. The following conditions are equivalent:* (i) *P has a unique maximal element*; (ii) *P has a greatest element.*

Indeed, if m is the unique maximal element of P and $a \in P$ then $P_a = \{x \in P / a \leqslant x\}$ is inductive and has (by Zorn's lemma) maximal elements which are also maximal in P. So $a \leqslant m$. The converse is obvious.

2. COROLLARY. *Let L be an upper continuous lattice. The following conditions are equivalent:* (i) *C_a has a unique maximal element*; (ii) *C_a has a greatest element.*

Indeed, in an upper continuous lattice, C_a is inductive.

The key result for our paper, from [5] is the following:

(C) Let $B \neq 0$ be a subgroup of an abelian group A. There is a unique B-high subgroup if and only if A/B is a torsion group and for each prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.

3. COROLLARY. *Let $B \neq 0$ be a subgroup of an abelian group A. The following conditions are equivalent:* (i) *B has a pseudocomplement in $L(A)$*; (ii) *there is only one B-high subgroup in A*; (iii) *A/B is a torsion group and for every prime p either $B[p] = A[p]$ or $B[p] = 0$ holds.*

4. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (a) *every nontrivial quotient group of A is a torsion group*; (b) *A is either a torsion group or a torsion-free group of rank* 1.

* *University of Cluj-Napoca, Faculty of Mathematics and Physics, 3400 Cluj-Napoca, Romania*

Proof. Clearly no mixed group has the property (a): if $0 \neq T(A) \neq A$ then $A/T(A)$ is torsion-free. Obviously every torsion group has the property (a). Now, if A is torsion-free of rank $r_0(A) \geqslant 2$ and $0 \neq a \in A$ then $r_0(A/\langle a\rangle) \geqslant 1$ so that $A/\langle a\rangle$ is not torsion. Finally, if A is torsion-free of rank 1, A has the property (a), as every rational group has it.

5. PROPOSITION. *For a torsion group A the following conditions are equivalent:* (c) *for each subgroup B of A and each p prime either $B[p] = A[p]$ or $B[p] = 0$ holds;* (d) *A is a direct sum of cocyclic groups corresponding to different primes.*

Proof. We can obviously reduce our problem to p-groups. But $B[p] = 0$ holds if and only if $B = 0$ so that only the case $B[p] = A[p]$ needs care. If A is a p-group such that $B[p] = A[p]$ holds for each subgroup $B \neq 0$ of A then $A[p] = S(A)$ (the socle) is contained in every nonzero subgroup B of A. In this case, having a smallest nonzero subgroup, A is cocyclic. The converse is obvious.

6. CORROLARY. *If A is an abelian group, the lattice $L(A)$ is pseudocomplemented if and only if A is either a direct sum of cocyclic groups corresponding to different primes or a torsion-free group of rank* 1.

Proof. Using 3, 4 and 5 we only need to observe that (c) is trivially true for torsion-free groups.

7. THEOREM. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a distributive lattice;* (ii) *$L(A)$ is a pseudocomplemented lattice;* (iii) *A is a locally cyclic group;* (iv) $r_0(A) + \max\limits_p r_p(A) \leqslant 1$; (v) *$A$ is either a direct sum of cocyclic groups corresponding to different primes or a torsion-free group of rank* 1.

Proof. One can use [3, p 86, ex. 5] and [2, p 301, T. 78.2]. The rest is done by the previous corollary.

A pseudocomplemented distributive lattice is called a Stone lattice if $a^* \vee a^{**} = 1$, where a^* denotes the pseudocomplement of a in L. If B is a subgroup of A such that A/B is a torsion group, let π be the set of all the primes such that $B[p] = 0$ holds and $B[p] = A[p]$ holds for $p \notin \pi$. Using proposition 2 and 3 from [5] we have $B^* = \bigoplus\limits_{p\in\pi} (T(A))_p$ and $B^{**} = \bigoplus\limits_{p\notin\pi} (T(A))_p$ so that $B^* + B^{**} = T(A)$. Hence only the torsion groups from 7 have Stone lattices of subgroups:

8. PROPOSITION. *For an abelian group A the following conditions are equivalent:* (i) *$L(A)$ is a Stone lattice;* (ii) *A is a direct sum of cocyclic groups corresponding to different primes.*

A lattice with zero is called a Heyting algebra (or a relative pseudocomplemented lattice) if for every $a, b \in L$ the subset $\{x \in L / a \wedge x \leqslant b\}$ has a greatest element denoted $a * b$.

We finally mention the following characterization [1]: (D) A bounded lattice L is a Heyting algebra if and only if L is distributive and for each $b \in L$ the sublattice $1/b = \{x \in L / b \leqslant x\}$ is pseudocomplemented.

The pseudocomplementation and the distributivity of the lattice of all the subgroups of an abelian group being equivalent it immediately follows that

$L(A)$ is a Heyting algebra if and only if $L(A)$ is distributive (any sublattice of a distributive lattice is distributive too).

Remark. The characterization of the class of all abelian groups which have the lattice $L(A)$ a Boole algebra is an easy consequence of 8 (cf. 2, p. 302, Cor. 78.5).

REFERENCES

1. Balbes R., Dwinger P., *Distributive Lattices*, Univ. Missouri Press, Columbia, Miss. 1974.
2. Fuchs L., *Abelian Groups*, Publishing House of the Hungarian Academy of Sciences, Budapest, 1958.
3. Fuchs L., *Infinite Abelian Groups*, vol. 1, Academic Press 1970.
4. Grätzer G., *General Lattice Theory*, Akademie Verlag, Berlin, 1978.
5. Krivonos F. V., *On N-high subgroups of abelian groups*, Vestnik Mosk. Univ., no. 1, 1975, p. 58–64.