Very strongly (nil)clean elements in rings: **a false problem**

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An element a in a ring R was called *nil-clean* if there exist an idempotent e and a nilpotent $t \in N(R)$ such that $a = e + t$, *uniquely nil-clean* if there is a unique idempotent e such that $a - e$ is nilpotent and *strongly nil-clean* if there exist an idempotent e and a nilpotent $t \in N(R)$ such that $a = e + t$ and $et = te$.

While uniquely nil-clean elements have only one nil-clean decomposition, strongly nil-clean elements may have various nil-clean decompositions. However, notice that (see [1], Corollary 3.8) *if an element of a ring is strongly nil clean, then it has precisely one strongly nil clean decomposition.*

As special cases, the only strongly nil-clean decompositions for idempotents and for nilpotents are the trivial ones.

Summarizing, strongly nil-clean elements may have various nil-clean decompositions, but among these, only one is strongly nil-clean (i.e. the idempotent and the nilpotent commute).

Example. Let a be an element in an arbitrary ring R with identity. It is readily checked that $E_a = \begin{bmatrix} a & a \\ 1-a & 1-a \end{bmatrix}$ is an idempotent in $\mathcal{M}_2(R)$. As such, it is trivially strongly nil-clean: $E_a = E_a + 0_2$ and $E_a 0_2 = 0_2 E_a$.

However $E_a = E + N_a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}$ is *a nil-clean decomposition which is not strongly nil-clean*:
$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & a \\ -a & -a \end{bmatrix} = 0_2 \neq \begin{bmatrix} a & a \\ -a & -a \end{bmatrix} = \begin{bmatrix} a & a \\ -a & -a \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ (if $a \neq 0$).

Therefore a natural problem **seemed to be** to determine *which are the strongly nil-clean elements which have only strongly nil-clean decompositions*. According to the previous observations, these are precisely the strongly nil-clean elements which are also uniquely nil-clean, **so this is a false problem** !

References

[1] Diesl A. J. *Nil clean rings*. J. of Algebra **383** (2013), 197–211.