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On Unit Fine Rings

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Abstract. A nonzero element a in a unital ring R is called *unit fine* if there is a unit u in R such that ua is fine (i.e. a sum of a unit and a nilpotent). A ring all whose elements are unit fine is called accordingly. This turns out to be a new class of simple rings, including the class of fine rings (and so the class of simple Artinian rings). The paper studies unit fine elements and rings. For rings such that 1 is a sum of two units, it is proved that matrix rings over unit fine rings, are unit fine.

Keywords. Unit-fine, fine, 2-good, matrix ring.

1. Introduction

In his seminal 1977 paper [7], Nicholson defined *clean* elements in a unital ring R, as sums $e+u$ with idempotent e and unit u . A ring was called *clean* if all its elements are clean. Clean rings turned out to be exchange (suitable).

In his 2006 Ph. D. thesis (see also [5]), Diesl defined *nil-clean* elements in a ring R, as sums $e+t$ with idempotent e and nilpotent t. A ring was called *nil-clean* if all its elements are nil-clean. Nil-clean rings turned out to be clean.

Finally, in their 2014 paper [3], Călugăreanu and Lam defined *fine* elements in a unital ring R, as nonzero sums $u+t$ with unit u and nilpotent t. Fine rings, i.e., rings in which all nonzero elements are fine, (properly) include simple Artinian rings and are (properly) included in the class of all the simple rings.

In the above definitions, if the (two) components of the sums commute, the corresponding elements (or rings) are called *strongly* clean (or nil-clean or fine). If a clean (or nil-clean or fine) element has only one such decomposition, it is called *uniquely* clean (or nil-clean or fine). Rings with only such type of elements are called accordingly.

In [9] (2005) and [2] (2013), an element a in a ring R was called *unit clean* if there exists a unit $u \in R$, such that ua is clean. A ring is *unit clean* if so are all its elements and clearly, such rings generalize the clean rings.

Analogously, in [1] (2019), an element a in a ring R is called *unit nil-clean* if there exists a unit $u \in R$, such that ua is nil-clean. A ring is *unit nil-clean* if so are all its elements. Obviously, nil-clean rings are unit nil-clean.

It is natural to consider the third possible combination. Therefore, we give the following

Definition. A nonzero element $a \in R$ is said to be *unit fine* if there exists a unit $u \in R$, such that ua is fine. A ring is *unit fine* if all its nonzero elements have that property. We could actually call such elements, *left* unit fine, owing to the position of the unit u. However, this definition turns out to be left-right symmetric (see Corollary 7). Clearly, *fine rings are unit fine.* Denote by $u{-}\Phi(R)$ the *set of unit fine elements* of the ring R.

In this paper, we study unit fine elements and rings. It is easy to check the following inclusions for classes of rings

$$\{\text{simple Artinian}\} \subseteq \{\text{fine}\} \subseteq \{\text{unit fine}\} \subseteq \{\text{simple}\},$$

the first and third inclusions being (known to be) irreversible.

Despite the fact that both classes of rings are "between" simple Artinian and simple, it turns out that unit fine elements (and rings) somehow have nicer properties than fine elements (and rings). See for example Corollary 8 and the results in Sect. 3.

Properties of unit fine elements and rings are given in Sect. 2. Unit fine rings are simple and 3-good, but the 2-good property remains an open question. Section 3 presents nice characterizations of special types of 2×2 unit fine matrices, one over any (unital) ring and two others over commutative domains. In Sect. 4 we prove that matrix extensions of unit fine rings are unit fine, in two cases: when the base ring is an elementary divisor ring, and, when the base ring S is unit fine with $1 \in U(S) + U(S)$.

The last section provides an example of unit fine ring which could settle the open question, whether the middle inclusion is (also) irreversible.

Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text. We denote by $U(R)$, $N(R)$, $J(R)$ the set of all units, the set of all nilpotents and the Jacobson radical of R, respectively.

2. Unit Fine Rings

Suppose $a \in R$ is a unit fine element. There are units $u, w \in U(R)$ and nilpotent $t \in N(R)$ such that $ua = w + t$.

Since $ua = w + t$ implies $a = u^{-1}w + u^{-1}t$, obviously, *unit fine elements are fine* whenever units commute with nilpotents.

This occurs for instance for uni rings (see [4]: a ring is called *uni* if its units commute with its nilpotents; examples include unit central rings, nilpotent central rings, rings with commuting units and trivially, reduced rings) or for rings in which $N(R)$ is a left (or right) ideal.

As in the case of fine rings we easily show that *strongly or uniquely unit fine rings are (only the) division rings.*

To be more precise, note that a ring R is uniquely unit fine if R is unit fine, and for any nonzero $a \in R$, if u is a unit, such that ua is fine, then ua is uniquely fine.

Proposition 1. *For any nonzero ring R, the following statements hold.*

(1) *An element $a \in R\backslash(0)$ is strongly unit fine iff $a \in U(R)$.*
(2) *R is strongly unit fine iff R is a division ring.*
(3) *R is unit fine and uni (in particular, reduced) iff it is a division ring.*
(4) *R is uniquely unit fine iff R is a division ring.*

Proof. (1) Units in a nonzero ring are always strongly (unit) fine. Conversely, if $a \in R\backslash(0)$ has a strongly unit fine decomposition $ua = w + t$, then $ua = w(1 + w^{-1}t) \in U(R)$ (since $wt = tw$ implies that $w^{-1}t \in N(R)$ and so $1 + w^{-1}t$ is unipotent). This proves (1), which clearly implies (2).

(3) It suffices to prove the "only if" part, so assume that R is unit fine and uni. Clearly, R is then strongly (unit) fine, so it is a division ring by (2).

(4) Again, it suffices to prove the "only if" part, so assume R is uniquely unit fine. If we can show that R is a reduced ring, it will be a division ring. For any $t \in N(R)$, we have $1 + t \neq 0$ (since $R \neq 0$). From the two (unit) fine decompositions $1 + t = (1 + t) + 0$, we conclude that $t = 0$. □

In [3] it is proved that *fine rings are simple*, and examples show that *simple rings need not be fine.*

Indeed, if we take *any simple domain R that is not a division ring* (e.g. [6], (3.17), or [6], (3.19)), then R cannot be fine since $N(R) = 0$. According to (3), above proposition, these examples also show that *simple rings need not be unit fine.*

Next

Theorem 2. (1) *If $a \in u{-}\Phi(R)$, then $RaR = R$.*
(2) *Any unit fine ring R is a simple ring.*
(3) *For any unital ring homomorphism $f : R \to S$ with $S \neq 0$, $f(u{-}\Phi(R)) \subseteq u{-}\Phi(S)$.*
(4) *Let I be a nil ideal of R, and let $a \in R$. Then $a \in u{-}\Phi(R)$ iff $\overline{a} \in u{-}\Phi(R/I)$.*

Proof. (1) Let $ua = w + t$ be a unit fine decomposition. In the factor ring $\overline{R} = R/RaR$, the unit $\overline{u} = -\overline{t}$ is nilpotent. This implies that $\overline{R} = \{0\}$; that is, $RaR = R$.
(2) follows from (1).
(3) Let $a \in u{-}\Phi(R)$. Then $RaR = R$ by (1). Since $f(R) \neq 0$, we have $f(a) \neq 0$. After knowing this, it is clear that $f(a) \in u{-}\Phi(S)$.
(4) The "only if" part follows easily from (3). Conversely, if $\overline{a} \in u{-}\Phi(R/I)$, take a unit fine decomposition $\overline{ua} = \overline{w} + \overline{t}$ in R/I. Since $I \subseteq J(R)$, we have $u, w \in U(R)$. Moreover, $\overline{ua - w} = \overline{t} \in N(R/I)$ implies that $ua - w \in N(R)$ (since I is a nil ideal). Thus, $ua \in w + N(R)$. Clearly, $a \neq 0$, so we have $a \in u{-}\Phi(R)$.

□

Therefore, for classes of rings

$$\{\text{simpleArtinian}\} \subseteq \{\text{fine}\} \subseteq \{\text{unitfine}\} \subseteq \{\text{simple}\},$$

the first and third inclusions being (known to be) irreversible. We come back to the middle inclusion in the last section.

Referring to *commutative unit fine* rings, recall that a ring R is called *2-primal* if $N(R)$ is contained in the prime radical of R (see [6], p. 195).

The following result gives some other examples of rings whose unit fine elements are just units.

Proposition 3. *If $R \neq 0$ and $N(R) \subseteq J(R)$, then $u{-}\Phi(R) = U(R)$. (In particular, this equality holds for all* local *rings and all non-zero* 2-primal *rings, including all nonzero* commutative *rings.)*

Proof. By hypothesis, if $a \in R$ has a unit fine decomposition $ua = w + t$, then $t \in N(R)$ implies $t \in J(R)$. Therefore, $ua \in w + J(R) \subseteq U(R)$ and so $a \in U(R)$. □

Corollary 4. *If $R \neq 0$, $N(R) \subseteq J(R)$ and R is unit fine then R is a division ring. In particular, the commutative unit fine rings are fields.*

In a ring R, an element $a \in R$ is *equivalent* to $b \in R$ if there are units $p, q \in U(R)$ such that $b = paq$.

Theorem 5. *Let R be a ring. The following conditions are (logically) equivalent:*

(1) *R is unit fine.*
(2) *Every nonzero element in R is equivalent to a fine element.*

Proof. (1) $\Rightarrow$ (2) Obvious (since $a = u^{-1}(ua) \cdot 1$, every element $a \in R$ is equivalent to ua, which is fine by hypothesis).

(2) $\Rightarrow$ (1) Let $0 \neq a \in R$ and assume there exist $p, q \in U(R)$, such that $paq = w + t$, where $w \in U(R)$ and $t \in N(R)$. Therefore, we have $qpa = qwq^{-1} + qtq^{-1}$. Clearly, qwq^{-1} is a unit and qtq^{-1} is nilpotent. In addition, $qp \in U(R)$. This implies that a is unit fine. Therefore, R is unit fine.

□

Corollary 6. *The unit fine property is invariant to conjugation.*

Proof. Since conjugate elements are equivalent, the claim follows by transitivity of equivalence. □

The following result shows that the *unit fineness is a left–right symmetric property.*

Corollary 7. *Let R be a ring. The following conditions are equivalent:*

(1) *R is unit fine.*
(2) *The opposite ring R^{op} is unit fine.*
(3) *For any nonzero $a \in R$, there exists $u \in U(R)$, such that au is fine.*

Proof. (1) $\Rightarrow$ (2) Let $0 \neq a^{op} \in R^{op}$. In view of Theorem 5, there exist $p, q \in U(R)$, such that $paq = w + t$, where $w \in U(R)$ and $t \in N(R)$. Hence, $q^{op}a^{op}p^{op} = w^{op} + t^{op}$ is fine in R^{op}. This yields (2), using Theorem 5 again.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) This follows from Theorem 5 (since $a = 1 \cdot (au)u^{-1}$, an arbitrary $a \in R$ is equivalent to au).

$\square$

Now a property which fails for fine elements.

Corollary 8. *If a is unit fine in R, so are va, av for any $v \in U(R)$.*

Proof. Assume $ua = w + t$ for $u, w \in U(R)$ and $t \in N(R)$. Then $w + t = ua = uv^{-1}(va)$ so va is unit fine. As for av, we use the previous theorem. $\square$

We end this section with a discussion of the 2-good property for unit fine rings.

Recall that, for some positive integer n, an element in a ring R is n *-good* if it is the sum of n units of R.

An example of *unit fine element which is not 2-good* in $\mathbb{M}_2(\mathbb{Z}[X])$, is given below.

First recall from [8] that *a proper matrix ring over an elementary divisor ring is 2-good.* Hence every 2×2 integral matrix is 2-good and in order to find unit fine elements that are not 2-good, $\mathbb{M}_2(\mathbb{Z})$ is not suitable.

Next recall the following

Proposition 9. [8] *Let R be a ring, $n \geq 2$ an integer and let $L = Ra_1 + ...Ra_n$ be a left ideal generated by the elements $a_1, \ldots, a_n \in R$. Let A be the $n \times n$ matrix whose entries are all zero except for the first column which is $[a_1, \ldots, a_n]^T$. Suppose that*

(1) *L cannot be generated by fewer than n elements, and*

(2) *zero is the only 2-good element in L.*

Then A is not 2-good.

It is noticed in [8] that if R is any of the rings $\mathbb{Z}[X]$ or $F[x, y]$, where F is a field, then both conditions are satisfied by any proper ideal of R, so (A and) $\mathbb{M}_n(R)$ is *not* 2-good for all $n \geq 2$.

Example. For $R = \mathbb{Z}[X]$ consider $A = \begin{bmatrix} 1 & 0 \\ 2X & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}[X])$. For $n = 2$ and $L = (1, 2X)$, the conditions in Vamos above proposition are fulfilled, so A is not 2-good (one can also easily show this directly). It is also easy to check that A is not fine. However, it is unit fine, as the following equality shows: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2X & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2X & 1 \\ 2X-1 & 1 \end{bmatrix}$.

Of course, the polynomial $2X$ can be replaced by X^2 or many other polynomials.

However, giving an example of unit fine *element* that is not 2-good implies that *unit fine rings are not 2-good*, only if the example is in a unit fine ring. However $\mathbb{M}_2(\mathbb{Z}[X])$ is not unit fine since it is not simple (has plenty of proper ideals together with $\mathbb{Z}[X]$).

Therefore, the following *question* remains *open*: Are unit fine rings, 2-good ?

The following result is easy

Proposition 10. *Unit fine rings are 3-good.*

Proof. If $ua = w + t$ with $u, w \in U(R)$, $t \in N(R)$ then $a = u^{-1}w + u^{-1}(1 + t) - u^{-1}$. $\square$

3. Unit Fine 2 × 2 Matrices

The goal of this section is to show that there are far more unit fine 2×2 matrices than fine matrices, and special classes of these have nicer characterizations, compared to the fine ones. In a way, the class of unit fine rings seems richer in nice properties compared to the class of fine rings.

Notice that a nilpotent 2×2 matrix over a commutative domain is of form $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta\gamma = 0$. Indeed, let Q be the field of fractions of R. Then in $\mathbb{M}_2(Q)$, B is similar to qE_{12} for some $q \in Q$. So $\mathrm{tr}(B) = 0$ and $\det(B) = 0$.

Recall that two ideals A, B in a commutative ring R are *comaximal* if $A + B = R$, or equivalently, $1 \in A + B$. Notice that Ra and Rb are comaximal (principal) ideals of a commutative ring R iff any common divisor $d \in R$ of a and b is a unit (i.e. is associated in divisibility with 1).

Theorem 11. *Let R be any commutative domain. A diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $(a \neq 0 \neq b)$ is unit fine in $\mathbb{M}_2(R)$ iff Ra, Rb are comaximal.*

Proof. First suppose d divides a and b but d is not a unit. Consider a unit $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ with $xt - yz \in U(R)$ and a nilpotent $T = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta\gamma = 0$. Next, we compute

$$\begin{aligned} \det(UA - T) &= (xa - \alpha)(tb + \alpha) - (yb - \beta)(za - \gamma) = \\ &= ab(xt - yz) + a(\alpha x + \beta z) + b(-\alpha t + \gamma y). \end{aligned}$$

Since d divides a and b, $\det(UA - T) = dm$ for some $m \in R$. Since d is not a unit, so is dm, and A is not unit fine.

Conversely, suppose $ua + vb = 1$, for some $u, v \in R$. Then take $U = \begin{bmatrix} b & -a \\ u & v \end{bmatrix}$, $T = (ab - 1)\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. For this choice, $UA - T = \begin{bmatrix} 1 & -1 \\ ua - ab + 1 & vb + ab - 1 \end{bmatrix}$ for which $\det(UA - T) = 1$. $\square$

Corollary 12. *If a, b are not coprime then $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is not fine.*

In the fine case (this is missing in [3]) we can prove

Proposition 13. *Let R be a commutative domain. A diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (not both a, b zero) is fine in $\mathbb{M}_2(R)$ iff at least one equation $(a-b)s + ab \in U(R)$ is solvable in s over R iff there is a unit $u \in U(R)$ such that $a-b$ divides $ab+u$.*

Remark. The coprime condition is necessary for a diagonal matrix to be *fine*, but is not sufficient.

Indeed, for $R = \mathbb{Z}$, take coprime $a = 5$, $b = -2$. Then $7s - 10 = \pm 1$ has no integer solutions, so $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ is not fine. However $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix}$ shows it is unit fine. Hence this yields *an example of unit fine* 2×2*integral matrix which is not fine.*

Another type of matrices for which the characterization of unit fine ones is nicer and more general than the corresponding one for fine matrices are the matrices $\begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$. In [3], the fineness of such matrices, only over $\mathbb{Z}$, was characterized by some Diophantine equations, namely

$$c^2x^2 + (2bc+1)xy + b^2y^2 - 2crx - 2bry + r^2 = 0$$

where $r := bc \pm 1$.

Using this it was noticed that $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ is fine, but $\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$ is *not* fine in $\mathcal{M}_2(\mathbb{Z})$.

Since $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$, it follows that $\begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$ is *unit fine but not fine.*

However, for unit fine matrices, we can prove a general result:

Theorem 14. *The 2×2 matrices $A = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$, for $b, c \in R$, are unit fine over any unital ring R.*

Proof. For any $a \in R$, observe that

$$\begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} a-c & b(a-c)+1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} 0 & bc-1 \\ 0 & 0 \end{bmatrix}$$

where both $U = \begin{bmatrix} a & -1 \\ 1 & 0 \end{bmatrix}$ and $W = \begin{bmatrix} a-c & b(a-c)+1 \\ 1 & b \end{bmatrix}$ are invertible ($U^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$ and $W^{-1} = \begin{bmatrix} -b & b(a-c)+1 \\ 1 & -a+c \end{bmatrix}$). Hence $UA = W + T$ and A is indeed unit fine. $\square$

Finally, we characterize unit fine 2×2 matrices with zero second row.

Theorem 15. *Let R be any commutative domain. A matrix $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ ($a \neq 0 \neq b$) is unit fine iff Ra and Rb are comaximal ideals.*

Proof. One way, suppose d divides a and b but d is not a unit. Consider a unit $U = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ with $xt - yz \in U(R)$ and a nilpotent $T = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta\gamma = 0$. Next, we compute

$$\det(UA - T) = (xa - \alpha)(zb + \alpha) - (xb - \beta)(za - \gamma) = \\ = a(\alpha x + \beta z) + b(-\alpha z + \gamma y).$$

Since d divides a and b, $\det(UA - T) = dm$ for some $m \in R$. Since d is not a unit, so is dm and A is not unit fine.

Conversely, suppose $ua + vb = 1$, for some $u, v \in R$. Then take $U = \begin{bmatrix} -b & -u \\ a & -v \end{bmatrix}$, $T = \begin{bmatrix} -uv & u^2 \\ -v^2 & uv \end{bmatrix}$. For this choice, $UA - T = \begin{bmatrix} -ab + uv & -b^2 - u^2 \\ a^2 + v^2 & ab - uv \end{bmatrix}$ for which $\det(UA - T) = (ua + vb)^2 = 1$. □

Remark. (1) Two special cases are not covered by Theorem 11 and Theorem 15.

Over any commutative ring, if (say) $b = 0$ (but $a \neq 0$) then $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ is unit fine iff $a \in U(R)$, and $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ is unit fine iff $b \in U(R)$.

(2) It is easy to check that, for any commutative domain, if $A = [a_{ij}]$, $1 \leq i, j \leq 2$ and d is a common divisor of $a_{11}, a_{12}, a_{21}, a_{22}$ that is not a unit, then A is not unit fine.

(3) In [3], examples were given to show that "fine" and "clean" properties are independent for elements. These examples also show the independence of the "unit fine" and "clean" properties, for elements: if b is not a unit, the nilpotent $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ is clean but not unit fine and $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ is unit fine but not clean over any commutative domain S if $a \notin U(S) \cup (1 + U(S))$.

4. Matrix Rings Over Unit Fine Rings

A condition which assures that matrix rings over unit fine rings are unit fine is the elementary divisor property. Actually, this property also assures that matrix rings over *unit nil-clean* rings are unit nil-clean. The proof is similar (see [1]).

No need of this condition for unit clean rings: *matrix rings over unit clean rings are unit clean* (see [2,9]).

Recall that a matrix A (not necessarily square) over a ring R admits a *diagonal reduction* if there exist invertible matrices P and Q, such that PAQ is a diagonal matrix (d_{ij}), for which d_{ii} is a divisor of $d_{i+1,i+1}$ for each i. A ring R is called an *elementary divisor* ring provided that every matrix over R admits a diagonal reduction.

Elementary divisor rings form a subclass of Bezout rings (rings in which every finitely generated left or right ideal is principal) and include PIDs, left PIDs which are Bezout (in particular division rings), valuation rings, the ring of entire functions, etc.

Proposition 16. *Let R be an elementary divisor ring, and let n be a positive integer. If R is unit fine, then $\mathbb{M}_n(R)$ is unit fine, i.e., every $n \times n$ matrix over R is equivalent to a fine matrix.*

Proof. Let $A \in \mathbb{M}_n(R)$. Then, there exist $U, V \in GL_n(R)$, such that $UAV = \operatorname{diag}(a_1, ..., a_n)$. As R is unit fine, we can find $u_i \in U(R)$, such that $u_i a_i = w_i + t_i$ for each i, where $w_i \in U(R)$ and $t_i \in N(R)$. Therefore, $\operatorname{diag}(u_1, ..., u_n)$ $UAV = \operatorname{diag}(w_1, ..., w_n) + \operatorname{diag}(t_1, \cdots, t_n)$ is fine. Using Theorem 5, $\mathbb{M}_n(R)$ is unit fine. $\square$

Using the pattern of the proof from [3], of the Main Theorem " *matrix rings over fine rings are fine*", we prove

Theorem 17. *Let S be a ring such that $1 \in U(S) + U(S)$ and let n be a positive integer. If S is unit fine, so is $R := \mathbb{M}_n(S)$, for any n.*

In the sequel, we just emphasize the modifications in the sequence of results which finally give this proof, for unit fine rings, instead of fine rings.

The first important ingredient is Proposition 3.2 ([3]), for which an adaptation is the following:

Proposition 18. *Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in R = \mathbb{M}_n(S)$, where $A \in \mathbb{M}_k(S)$ and $D \in \mathbb{M}_{n-k}(S)$. If $A \in u{-}\Phi(\mathbb{M}_k(S))$ and $D \in u{-}\Phi(\mathbb{M}_{n-k}(S))$, then $M \in u - \Phi(R)$.*

Proof. Let $UA = W + T$ and $U'D = W' + T'$ be unit fine decompositions in $\mathbb{M}_k(S)$ and $\mathbb{M}_{n-k}(S)$ respectively. Then

$$\begin{bmatrix} U & 0 \\ 0 & U' \end{bmatrix} M = \begin{bmatrix} W & 0 \\ U'C & W' \end{bmatrix} + \begin{bmatrix} T & UB \\ 0 & T' \end{bmatrix}. \qquad (*)$$

Here, the matrix on the left of M and the first matrix on the right-hand side are invertible in R. On the other hand, a high power of the second matrix on the right-hand side is strictly upper triangular, so the second matrix is nilpotent in R. Thus, (*) above is a unit fine decomposition for M in R. $\square$

Applying this proposition with an induction on the number of diagonal blocks of a block matrix, we arrive at the Block Theorem ([3]): *If $M = (A_{ij}) \in R = M_n(S)$ is a block matrix where the blocks $\{A_{ij}\}$ are such that each A_{ii} is a unit fine square matrix, then $M \in u{-}\Phi(R)$.*

Next, recall the

Definition. For $n \geq 2$, we say that a matrix $\begin{bmatrix} A & \beta \\ \gamma & d \end{bmatrix} \in R$ is *in good form* if $A \in \mathbb{M}_{n-1}(S)$ is nonzero and $d \in S$ is also nonzero.

Then there are no differences in the proofs of

Lemma 19. *Suppose there exists an equation* $1 = u+v \in S$ *where* $u, v \in U(S)$*. Then any matrix* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *with* $a \neq 0$ *is similar to a matrix in good form.*

Corollary 20. *Suppose there exists an equation* $1 = u + v \in S$ *where* $u, v \in U(S)$*. Then any nonzero matrix* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is similar to a matrix in good form.*

Proposition 21. *Suppose there exists an equation* $u + v = 1 \in S$ *where* $u, v \in U(S)$*. Then for any* $n \geq 2$*, any nonzero matrix* $M = \begin{bmatrix} A & \beta \\ \gamma & d \end{bmatrix} \in R = \mathbb{M}_n(S)$ *(where* $d \in S$ *and* $A \in \mathbb{M}_{n-1}(S)$*) is similar to a matrix in good form.*

This way we are ready for the proof of Theorem 17, the unit fine version of the Main Theorem in [3].

Proof. In the case $|S| > 2$, if we have $1 \in U(S) + U(S)$, the proof works also for unit fine matrices, just using Proposition 18.

In the case $|S| = 2$, i.e. $S \cong \mathbb{F}_2$ the proof continues by induction on n in a more general case: S is a division ring.

To prove that $M = \begin{bmatrix} A & \beta \\ \gamma & d \end{bmatrix} \neq 0$ is in $u{-}\Phi(R)$, we begin by noting that, we may assume (up to similarity) that $A \neq 0$. If $d \neq 0$, we are done. So we may assume that $d = 0$. By the inductive hypothesis, we have a unit fine decomposition $UA = W + T$ in $\mathbb{M}_{n-1}(S)$.

Thus we can write $\begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} M = \begin{bmatrix} W & U\beta + \delta \\ \gamma & 0 \end{bmatrix} + \begin{bmatrix} T & -\delta \\ 0 & 0 \end{bmatrix}$ where δ is a column vector that is to be chosen. Here, the second summand is nilpotent, so we are done if δ can be chosen such that the *first summand on the right-hand side* is invertible. We go into the following two cases.

Case 1. $\gamma \neq 0$. We perform on the first summand of the right-hand side the following block elementary transformation: $\begin{bmatrix} I_{n-1} & 0 \\ -\gamma W^{-1} & 1 \end{bmatrix} \begin{bmatrix} W & U\beta + \delta \\ \gamma & 0 \end{bmatrix} = \begin{bmatrix} W & U\beta + \delta \\ 0 & -\gamma W^{-1}(U\beta + \delta) \end{bmatrix}$.

In view of this equation, we are done if δ can be chosen such that $\gamma W^{-1}(U\beta + \delta) = 1$, for then the first summand on the right-hand side will be invertible. Since $\gamma \neq 0$, we have $\gamma U^{-1} \neq 0$ too, so we can certainly solve the equation $\gamma W^{-1}(U\beta + \delta) = 1$ *over the division ring* S since (by varying δ) the vector $U\beta + \delta$ can be arbitrarily chosen.

Case 2. $\gamma = 0$. If $\beta \neq 0$, we can repeat the above argument to complete the proof. Thus, we may also assume that $\beta = 0$, so now $M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$. Suppose the i-th row γ' of A is nonzero. Conjugating M by the elementary matrix $I_n + E_{ni}$, we arrive at a new matrix $\begin{bmatrix} A & 0 \\ \gamma' & 0 \end{bmatrix}$ where $\gamma' \neq 0$. We are now back to Case 1, so the proof is complete. $\square$

The following remains an open

Question. Are matrix rings over unit fine rings, unit fine ? That is, can we overcome the obstruction $1 \in U(S) + U(S)$?

5. Example of Unit Fine Ring

Let R be the ring of endomorphisms of a countable-dimensional vector space V over a field k, and I the ideal of elements of R with finite-dimensional image in V. The ring R and the factor ring R/I were widely used over years as examples of several sorts (e.g. R is not unit-regular, R/I is simple). Notice that R is (von Neumann) regular. So is any factor ring, including R/I.

Moreover, R has precisely three ideals: $0, I, R$. Consequently, R/I is simple.

Already in 2015, when the paper [3] was elaborated, it was stated the following

Open question: is R/I a fine ring ?

After 5 years, this question is still unanswered.

This way, the next proposition, can be viewed as a positive answer to a somewhat weaker

Question: Is R/I a unit fine ring ?

In this last section, suggested by George Bergman, we shall use the term "biassociate" instead of "equivalent", in order to avoid the possibly confusing term "equivalent-equivalence class". That is $a, b \in R$ are called *biassociate* if there are units p, q with $b = paq$.

George Bergman proved the following

Proposition 22. *If R is the ring of endomorphisms of a countable-dimensional vector space V over a field k, and I the ideal of elements of R with finite-dimensional image in V, then R/I is unit fine.*

Proof. We write elements of R on the left of their arguments, and compose them accordingly.

Because of the length of this proof some details are only sketched. There are six steps.

(1) In the ring R/I described above, one finds that there are just *four* biassociate-*equivalence classes* of nonzero elements:
   (i) the set of units, i.e. elements that are not left nor right zero-divisors; equivalently, that have left and right inverses,
   (ii) the set of elements that are right but not left zero-divisors; equivalently, elements that have left inverses which are not right inverses,
   (iii) the set of elements that are left but not right zero-divisors; equivalently, elements that have right inverses which are not left inverses,
   (iv) the set of elements that are both left and right zero-divisors; equivalently, elements that have no right nor left inverses.

   For example, here are the details for (ii): if $a'a = 1$ then, by left multiplication with a', $ab = 0$ implies $b = 0$, so a is not a left zero-divisor. If $aa' \neq 1$ then $aa' - 1 \neq 0$ and since $(aa' - 1)a = a - a = 0$, a

is a right zero-divisor; the converse holds if the ring is (*von Neumann) regular*; suppose $a = ara$; then $a(1 - ra) = 0$ and since a is not a left zero-divisor, $1 - ra = 0$, i.e. a has a left inverse.

These classes are clearly disjoint.

Roughly speaking, we deduce (1) using the fact that in R, we can construct elements by specifying where they take each element of a basis of V. In this way, we build elements whose images in R/I are units, which we can put on the right and left side of any element in one of these classes to bring it to a fixed member of that class.

(2) Since *an element is unit fine iff it is biassociate with a fine element*, to show that R/I is unit fine, it suffices to exhibit one fine element in each of these four classes.

(3) This is easy for classes (i) and (iv), because units are fine, respectively, we can embed $\mathbb{M}_2(k)$ in R/I, and get an example there.

Here is how to do it for classes (ii) and (iii).

(4) Verify that R is embeddable in R/I. (Indeed, every endomorphism x of V induces an endomorphism of a countable direct sum of copies of V. Choosing an isomorphism between the vector space V and such a countable direct sum of copies of V, we get a homomorphism $h : R \to R$ which sends each nonzero element of R to an element with infinite-dimensional image; hence the composite homomorphism $R \to R \to R/I$ is an embedding.) Hence to show that there are fine elements of R/I in classes (ii) and (iii) above, it suffices to show that there are such elements in R.

(5) To get an element of R in class (ii), take V to have a basis of elements $b_{i,j}$, where i and j are integers, such that $i \geq 0$, and if $i = 0$, then $j \geq 0$. (So these are pairs that are $\geq (0, 0)$ under lexicographic ordering.)

Let $x : V \to V$ send every $b_{i,j}$ to $b_{i,j+1}$. This has a left inverse, which moves each $b_{i,j}$ to $b_{i,j-1}$, except for $b_{0,0}$, which it sends to 0. Since x is not onto, it can't have a right inverse. Now here are a unit u and a nilpotent element t that sum to x:

$u(b_{i,j}) = b_{i,j+1}$ unless $j = -1$
$u(b_{i,-1}) = b_{i-1,0}$ (for all $i > 0$),
$t(b_{i,j}) = 0$ unless $j = -1$
$t(b_{i,-1}) = b_{i,0} - b_{i-1,0}$ for all $i > 0$.

(6) An element y of R in class (iii) is given by a left inverse of the above x. □

Therefore, it will follow that the class of all the fine rings is *properly* included in the class of all the unit fine rings, using the previous example, if one can show that R/I is not fine.

Another possibility would be to show that R/I is not 2-good (see the end of Sect. 2). Since fine rings are 2-good, this would also prove the proper containment in discussion.

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