Uniquely fine 2×2 integral matrices

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Abstract

Fine elements were defined in [3] for unital rings, as sums of units by nilpotents. Such a sum is called uniquely fine if it is the only decomposition $u+t$ with unit u and nilpotent t. The paper reveals the many facets of this notion for 2×2 integral matrices.

1 Introduction

A nonzero element a in a unital ring R is called *fine* (see [3]) if $a = u + t$ with a unit u and a nilpotent t. It is called *uniquely fine* if there is a unique unit u such that $a - u$ is nilpotent and *strongly fine* if $ut = tu$. Accordingly, one defines fine, uniquely fine and strongly fine rings, respectively.

It was proved in [3] that uniquely fine rings coincide with strongly fine rings and these are precisely the division rings. Elementwise, *a nonzero element is strongly fine iff it is a unit.*

There are *uniquely fine elements which are not units* and *units may not be uniquely fine*.

For example, in $\mathcal{M}_2(\mathbf{F}_2)$, a fine ring together with the two element field $\mathbf{F}_2$, all (the 6) nontrivial idempotents are uniquely fine but not units. From the six units, only the two order 3 units are uniquely fine.

Actually, *units are uniquely fine in a unital ring iff the ring is reduced.* Indeed, if R is not reduced and $0 \neq t$ is nilpotent then $1 + t = (1 + t) + 0$ is a unit which has two different fine decompositions. Conversely, if R is reduced, every fine element is obviously uniquely (and strongly) fine and so, is a unit.

Clearly, if R is not reduced and a unit u commutes with a nonzero nilpotent t, then $u+t$ is (strongly fine, so a unit and) not uniquely fine: $u+t = (u+t)+0$ are different fine decompositions.

The aim of this note is to study the uniquely fine 2×2 integral matrices.

In Section 2 the reduction to similarity classes of fine 2×2 integral matrices is described and fine decompositions which use a positive multiple of the matrix

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unit E_{12} (as special nilpotent) are presented. In the last section, large (infinite) classes of uniquely fine matrices are found, all whose characteristic polynomial factors over the integers and examples of units of infinite fine index are given. Since uniquely fine matrices were not addressed in [3], but several examples of fine or not fine matrices were given, these are revisited from this point of view.

For a unital ring R, $U(R)$ denotes the set of units, $N(R)$ the set of nilpotent elements, $\mathcal{M}_n(R)$ denotes the ring of all $n \times n$ matrices with entries in R, $\mathrm{Tr}(A)$ denotes the trace of the matrix A. By E_{ij} we denote the (so called) matrix unit with all entries zero excepting the (i,j) entry, which is 1. The zero matrix and the identity matrix in $\mathcal{M}_2(R)$ are denoted 0_2 and I_2, respectively.

2 Fine 2×2 matrices and similarity

Definition. Two 2×2 matrices A, B over any unital ring R, are *similar* (or *conjugate*) if there is an invertible matrix U such that $B = U^{-1}AU$. Since similarity is obviously an equivalence relation, a partition of $\mathcal{M}_2(R)$ corresponds to it. The subsets in this partition are called *similarity classes*.

If A is nilpotent (or a unit) and B is similar to A then B is also nilpotent (respectively a unit). This similarity invariance clearly extends to fine matrices and it also restricts to uniquely or strongly fine matrices, respectively. Rephrasing, the notions of fine, uniquely fine and strongly fine are similarity invariants. So is the fine index (a matrix has *fine index* n if it has exactly n different fine decompositions).

In the sequel, $R = \mathbf{Z}$, that is, we deal only with 2×2 integral matrices. To determine all fine matrices actually means to find all the similarity classes of fine matrices. In doing so, it is natural to fix each similarity class by a special representative. In this paper, this will be done into two different ways.

The first has already been done by Behn and Van der Merwe in [2] where an algorithm is presented, which, given a 2×2 matrix, finds a canonical representative (called *reduced*) in its similarity class.

The second consists in fixing the similarity classes by a representative which uses a special nilpotent namely, a positive multiple of E_{12}.

As for the first, recall the following

Definition. For a 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ denote by $D = \mathrm{Tr}(A)^2 - 4\det(A)$. If D is a square (e.g. $\det(A) = 0$), that is, the characteristic polynomial of the matrix factors over the integers, say, $f(x) = (x-\alpha)(x-\delta)$, where $\alpha \geq \delta$, then, for $\alpha \neq \delta$ the matrix $\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}$ (i.e. the SW entry must be 0) is *reduced* if $0 \leq \beta < \alpha - \delta$ and, for $\alpha = \delta$ if $\beta \geq 0$.

Example. The matrix unit $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is reduced, E_{21} is *not reduced*. Actually $E_{21} = UE_{12}U$, with $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so these are similar.

Also from [2] (Theorem 5.2), recall the following result

Theorem 1 *Let $M \in \mathcal{M}_2(\mathbf{Z})$ and assume that the characteristic polynomial of M factors over $\mathbf{Z}$. Then M is similar to a reduced matrix. Moreover, this class representative is unique thus no two different reduced matrices are similar.*

We first classify the nilpotents in $\mathcal{M}_2(\mathbf{Z})$ by similarity classes.

Proposition 2 *Any nonzero 2×2 nilpotent integral matrix is similar to a positive multiple of E_{12}.*

Proof. Any nilpotent matrix is of form $T = \begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ for some $s, x, y \in \mathbf{Z}$ such that $s^2 + xy = 0$. If $s = 0$ then at least one from x, y is also zero and we have *strictly triangular* nilpotents (including 0_2). For such nilpotents we just mention

1) $\{0_2\}$ forms a singleton class.

2) Since $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & -a \\ 0 & 0 \end{bmatrix}$ (conjugation with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$) and $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$ (conjugation with $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$), up to similarity it suffices to consider strictly upper triangular nilpotents $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ with positive integers a.

3) Let $a \neq 0 \neq b$ be positive integers. Then $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$ iff $a \in \{\pm b\}$.

Indeed, suppose $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. Then there exists an invertible matrix U such that $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} U = U \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$. If $U = \begin{bmatrix} p & r \\ s & t \end{bmatrix}$, then $ar = 0 = rb$ and $at = pb$. By hypothesis, $r = 0$ and so $\det(U) = pt = \pm 1$. Hence $p, t \in \{\pm 1\}$ and so $a \in \{\pm b\}$.

If $s \neq 0$ then also $x \neq 0 \neq y$ which we call *strictly nonzero* nilpotents.

As in [2], we show that any matrix $T = \begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ for some $s, x, y \in \mathbf{Z} - \{0\}$ such that $s^2 + xy = 0$, is similar to a positive multiple of E_{12}.

If $\gcd(s; y) = d$ then $s = s'd$, $y = y'd$ and there exist $\delta, \gamma \in \mathbf{Z}$ such that $\gamma s' - \delta y' = 1$. In this case we take, $P = \begin{bmatrix} s' & \delta \\ y' & \gamma \end{bmatrix}$, $P^{-1} = \begin{bmatrix} \gamma & -\delta \\ -y' & s' \end{bmatrix}$ and so $P^{-1}TP = \begin{bmatrix} d & \gamma x + \delta s \\ 0 & 0 \end{bmatrix} P = \begin{bmatrix} * & k \\ 0 & 0 \end{bmatrix} = kE_{12}$ (because $* = 0$ since $s'^2 d + xy' = 0$, $d \neq 0$). By computation $ky' = -d$.

Notice that $s'^2 d + xy' = 0$ implies $y' | s'^2 d$ and since $\gcd(s'; y') = 1$, it follows $y' | d$ and so $k = -\dfrac{d}{y'} \in \mathbf{Z}$, that is, $\begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & -\dfrac{d}{y'} \\ 0 & 0 \end{bmatrix}$.

Finally, if necessary, we just conjugate further with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in order to obtain a positive multiple of E_{12}. $\blacksquare$

Examples. 1) $k = 1$ iff $y' = -d$, i.e. $y = -d^2$. If $\sim$ denotes the similarity relation $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -E_{12}$; $\begin{bmatrix} 12 & 9 \\ -16 & -12 \end{bmatrix} \sim E_{12}$ and $\begin{bmatrix} 418 & 361 \\ -484 & -418 \end{bmatrix} \sim E_{12}$. Here, if s and y have the same sign, we get $-E_{12}$ and, if s and y have the opposite sign, we get E_{12}.

2) $\begin{bmatrix} 12 & 18 \\ -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 12 & 8 \\ -18 & -12 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ (for the first $d = 4$ and $k = -\dfrac{d}{y'} = 2$, for the second $d = 6$ and again $k = -\dfrac{d}{y'} = 2$).

3) If we take $T_{ka} = \begin{bmatrix} ka & ka^2 \\ -k & -ka \end{bmatrix}$, for any $k, a \in \mathbf{N}^*$ then $s = ka$, $y = -k$ give $d = k$, $s' = a$, $y' = 1$ and $T_{ka} \sim kE_{12}$. Same remark with respect to the signs of s and y.

It is now easy to prove the following

Proposition 3 *Nilpotents are not uniquely fine in $\mathcal{M}_2(\mathbf{Z})$.*

Proof. By the above discussion, it suffices to check this for a positive multiple of E_{12}. Suppose kE_{12} is fine. Then there exists a nilpotent $T = \begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ (with $s^2 + xy = 0$) such that $\det(kE_{12} - T) = ky = \pm 1$, and so $k = 1$. Hence only E_{12} may be uniquely fine, but it is not: $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. $\blacksquare$

As mentioned above, by Proposition 2, we may *fix the similarity classes* by selecting kE_{12}, the nilpotent in the fine decomposition. To simplify the wording we call this an *E_{12}-reduction*.

Proposition 4 *A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (is fine and) admits an E_{12}-reduction iff there is a $k \geq 0$ such that* $\det(A) + kc = \pm 1$.

Proof. First, for a matrix A, suppose there is a $k \geq 0$ such that $\det(A) + kc = \pm 1$. Then $\det(A - kE_{12}) = \pm 1$ and so $A - kE_{12}$ is a unit. Hence A is fine with an E_{12}-reduction. Conversely, suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ admits kE_{12} as nilpotent in a fine decomposition. Then $A - kE_{12}$ is a unit and the condition follows. $\blacksquare$

Remark. Not every fine matrix admits an E_{12}-reduction but every fine matrix is similar to a matrix which has an E_{12}-reduction.

Thus, for the similarity classes of fine matrices we can choose as representatives, the E_{12}-reduced (fine) matrices.

Example. In [3], the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ was mentioned as fine, but there is no $k \geq 0$ such that $-6 + 3k = \pm 1$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$. We use Proposition 2 for the nilpotent $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$. For $P = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ we obtain $A' = P^{-1}AP = \begin{bmatrix} 12 & -18 \\ 7 & -11 \end{bmatrix} = \begin{bmatrix} 12 & -19 \\ 7 & -11 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Here $\det(A) = \det(A') = -6$ (and $\mathrm{Tr}(A) = \mathrm{Tr}(A') = 1$). For $k = 1$, $A' - E_{12}$ is a unit and $\det(A' - E_{12}) = \det(A') + c' = \det(A) + 7 = -6 + 7 = 1$ and A and A' are similar.

3 Classes of uniquely fine matrices

Finding integral 2×2 fine matrices amounts to solving some special Diophantine equations as follows from

Theorem 5 *For a 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denote $-\det(A) \pm 1$ by l. Then A is fine iff*

(i) the system $cx + by = l$, $s^2 + xy = 0$ with unknowns x, y, s has integer solutions, whenever $a = d$, or

(ii) the (quadratic) Diophantine equation

$$c^2x^2 + (a-d)^2xy + b^2y^2 - 2clx - 2bly + l^2 = 0$$

has integer solutions x, y, whenever $a \neq d$.

Proof. Since nilpotents in $\mathcal{M}_2(\mathbf{Z})$ have the form $\begin{bmatrix} s & x \\ y & -s \end{bmatrix}$ with $s^2 + xy = 0$, A is fine iff $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) = \pm 1$. This condition can be written $s(a-d) = -cx - by + l$. If $a = d$ we get (i) and if $a \neq d$, eliminating s we obtain the quadratic Diophantine equation. Notice that $-s(a-d) = -cx - by + l$ is also suitable since $(-s)^2 + xy = 0$. ■

Any 2×2 integral matrix A whose characteristic polynomial factors over the integers is similar to a *reduced* matrix, which has the form $C_{abd} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $0 \leq b \leq |a - d|$, if $a \neq d$, and $b \geq 0$ if $a = d$. Hence when addressing uniquely fineness of integral 2×2 matrices in this case, (up to similarity) only such matrices must be studied.

Proposition 6 *Matrices of form $B_{ab} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ are not uniquely fine.*

Proof. The matrix $B_{ab} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is fine iff $\det(B_{ab} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) = a^2 + by = \pm 1$ with $s^2 + xy = 0$, that is iff b divides $a^2 \pm 1$. Clearly, for any given y, we can choose $x = -y = \pm s$ for $s^2 + xy = 0$, but many other choices could be available. Hence such matrices are either not fine or else of fine index at least 2. ■

Example. $B_{53} = \begin{bmatrix} 5 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ 8 & 13 \end{bmatrix} + \begin{bmatrix} 8 & 8 \\ -8 & -8 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -8 & 8 \\ -8 & 8 \end{bmatrix}$.

Proposition 7 *A matrix* $C_{abd} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ *with* $a \neq d$, $0 \leq b \leq |a-d|$ *is fine iff at least one of the (quadratic) Diophantine equations*

$$(a-d)^2xy + b^2y^2 - 2bly + l^2 = 0 \qquad (*)$$

is solvable (over integers), where $l := -ad \pm 1$.

Proof. Indeed, C_{abd} is fine iff $\det(C_{abd} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) = ad + s(a-d) + by = \pm 1$, i.e. $by - l = -s(a-d)$. Hence $s^2 + xy = 0$ leads to the Diophantine equations (*). ■

Notice that (*) can be written $(by - l)^2 = -(a-d)^2xy$, so that $xy \leq 0$ and $xy = -s^2$.

Quadratic Diophantine equations $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ are classified by the sign of $\Delta = B^2 - 4AC$, which, for the equation (*) is $\Delta = (a-d)^4 > 0$ since $a \neq d$. Thus, (*) is a Diophantine equation of hyperbolic type, whose reduced form is a Pell equation

$$X^2 - Y^2 = N = (2bl)^2$$

obtained (for $b \neq 0$) using the substitutions $X = -(a-d)^2x + 2bl$, $Y = (a-d)^2x + 2b^2y - 2bl$. Since the coefficient of Y^2 is a square, this is the less interesting case of a Pell equation, because, for a given integer N, it is easily solved using the factorization of N and identifying the factors (if $N = nm$ then from $X - Y = n$, $X + Y = m$ we get $X = \dfrac{n+m}{2}$, $Y = \dfrac{m-n}{2}$).

However, when studying some given classes of matrices, we shall merely use computer aid (e.g. [1]), instead of the above method.

The following result shows that the study of the matrices C_{abd} can be considerably reduced to a minimum number of representatives.

Proposition 8 *When searching for uniquely fine matrices* C_{abd} *it suffices to take* $a > |d|$, *then take the reminder* r *of the division of* b *by* $a - d$, *and* $2 \leq r \leq a - d - 2$. *Moreover, it suffices to check only half of these, say, the small values of* r *beginning with* 2.

Proof. First notice that if we change (a,d) into $(-a,-d)$, the equation (*) remains the same. Hence we can assume $a-d>0$. Next, if we change (a,d) into (d,a), again the equation remains the same. So we can assume $a>|d|$.

Observe that matrices with negative b must not be treated separately because these (by conjugation with $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$) are similar with those with $b>0$.

Further, for a given $a-d$, only the (reduced) representatives C_{abd} with $0 \leq b < a-d$ must be checked. Indeed, if r is the reminder of the division of b by $a-d$, it is readily checked that

$$\begin{bmatrix} a & (a-d)q+r \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & r \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & q \\ 0 & 1 \end{bmatrix} \text{ for any } 0 \leq r < a-d.$$

Finally, since

$$\begin{bmatrix} a & (a-d)-b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix},$$ only 'half' of these are independent representatives. ■

Remarks. 1) (*) admits $x=0$ in some solution iff b divides l (and then $y=\frac{l}{b} \in \mathbf{Z}$). Special case, $b=1$.

2) (*) admits $y=0$ in some solution iff $l=0$, i.e. for 8 units: $I_2, -I_2, \begin{bmatrix} 1 & m \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & m \\ 0 & 1 \end{bmatrix}$ with $m \in \{0,1,2\}$. None of these is uniquely fine.

Proposition 9 *For a given (a,d), matrices are (fine but) not uniquely fine if $b=(a-d)k \pm 1$, and are fine (but not uniquely fine) only if $a-d$ divides l whenever $b=(a-d)k$.*

Proof. If $|a-d|=b+1$, the minus equation (according to the sign in l) has the solution $(-ad-1, ad+1)$ and the plus equation has $(-ad+1, ad-1)$. None of the corresponding matrices are uniquely fine.

Matrices $\begin{bmatrix} a & (a-d)k \\ 0 & d \end{bmatrix}$, $k \in \mathbf{Z}$ are fine (but not uniquely) only if $a-d$ divides $l=-ad \pm 1$.

As seen in the above proof, it suffices to check this for $k=0$, i.e. for $b=0$. Since such matrices are symmetric, excepting the easy case when these are units, if these admit a fine decomposition, also the transpose decomposition is available, which gives fine index at least 2 (notice that 0_2 is the only symmetric nilpotent in $\mathcal{M}_2(\mathbf{Z})$). ■

Examples. 1) If $d=0$ and $a \geq 2$ (in this case $l=\pm 1$), matrices $\begin{bmatrix} a & ka \\ 0 & 0 \end{bmatrix}$, $k \in \mathbf{Z}$, for $|a| \geq 2$ are never fine.

2) The matrices $\begin{bmatrix} 6 & 7k \\ 0 & -1 \end{bmatrix}$ which are similar to $\begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ have index 2.

3) Take $a-d=2$ and so $(a,d) \in \{(2,0),(1,-1)\}$ with $l \in \{\pm 1, 1 \pm 1\}$.

If $a=2$, $d=0$, $l=\pm 1$ then the matrices are not fine for $b\in\{0,2\}$ and have fine index 2 for $b=1$ (so $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and more general $\begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$, are not fine for any $k\notin\{-1,0,1\}$).

If $a=1$, $d=-1$, $l=1\pm 1$ then matrices have fine index ∞ for $b\in\{0,1,2\}$. Indeed, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -t & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1-t \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -t \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$.

It is worth noting that these are *examples of units of fine index* ∞.

More general, any triangular unit has infinite fine index: $\begin{bmatrix} \pm 1 & a \\ 0 & \pm 1 \end{bmatrix} = \begin{bmatrix} \pm 1 & a-t \\ 0 & \pm 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$ for any $t\in\mathbf{Z}$. Hence there are *no uniquely fine* matrices in this case.

4) Suppose $a-d=3$ and so $(a,d)\in\{(3,0),(2,-1)\}$.

If $b=0$ then the equations (*) are $9xy+l^2=0$, with no solutions if $l\in\{\pm 1, 2-1\}$ but with solutions for $l=2+1$, i.e., $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ (transposes).

If $b=1$ or $b=2=3-1$, both have index at least 2 (see Proposition 9). So there are *no uniquely fine* matrices also in this case.

In the sequel, according to previous remarks, we focus on the following hypothesis: $a>0$, $d\leq 0$ and $a\geq |d|$. Here are our results.

Theorem 10 *1) For small even $a-d$, i.e. $a-d\in\{0,2,4,6,8,10\}$ there are no uniquely fine matrices, for any integer b.*

2) For $a-d\in\{1,3,5\}$ there are no uniquely fine matrices, for any integer b.

3) For any small odd $a-d\in\{7,9,11,13\}$ there are uniquely fine matrices, for suitable $b\leq a-d$.

Proof. The case $a-d=0$ (or $a=d$) was clarified in Theorem 6.

There are only two (idempotent) matrices for $a-d=1$ ($0\leq b\leq a-d$ suffices, up to similarity): $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Both are fine; the first of index 2, the second of index 4.

The cases $a-d\in\{2,3\}$ were presented in the previous examples. The cases $a-d\in\{4,5,6,8,9,10\}$ were verified using [1]. Below we list the complete results for $a-d\in\{7,11,13\}$. Since by Proposition 9, there are no uniquely fine matrices for $r=0$ or $r\in\{1,a-d-1\}$ (here r denotes the reminder of the division of b by $a-d$), we restrict ourselves to $r\in\{2,3,...,a-d-2\}$. ■

Proposition 11 *(a) The matrices* $\begin{bmatrix} 4 & 7k+r \\ 0 & -3 \end{bmatrix}$*, $k\in\mathbf{Z}$ and $r\in\{2,3,4,5\}$ are uniquely fine.*

(b) For $(a,d)=(5,-2)$, all matrices have fine index 2 or 3.

(c) The matrices $\begin{bmatrix} 6 & 7k+r \\ 0 & -1 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \in \{2,3,4,5\}$ are uniquely fine.*

(d) For $(a,d)=(7,0)$, matrices are not fine *excepting (see Proposition 9) $b = 7k \pm 1$ which are of index 2.*

Proposition 12 *(a) The matrices* $\begin{bmatrix} 6 & 11k+r \\ 0 & -5 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $2 \le r \le 9$ are uniquely fine.*

(b) The matrices $\begin{bmatrix} 7 & 11k+r \\ 0 & -4 \end{bmatrix}$, *$k \ge 0$ and $r \ne 0$ have fine index 2.*

(c) The matrices $\begin{bmatrix} 8 & 11k+r \\ 0 & -3 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $2 \le r \le 9$ are uniquely fine.*

(d) The matrices $\begin{bmatrix} 9 & 11k+r \\ 0 & -2 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $2 \le r \le 9$ are uniquely fine.*

(e) The matrices $\begin{bmatrix} 10 & 11k+r \\ 0 & -1 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $2 \le r \le 9$ are uniquely fine.*

(f) The matrices $\begin{bmatrix} 11 & 11k+r \\ 0 & 0 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \ne \pm 1$ are not fine.*

Proposition 13 *(a) The matrices* $\begin{bmatrix} 7 & 13k+r \\ 0 & -6 \end{bmatrix}$, *$r \in \{2,3,4,5,6\}$ and $k \in \mathbf{Z}$ are uniquely fine.*

(b) The matrices $\begin{bmatrix} 8 & 13k+r \\ 0 & -5 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \in \{2,3,4,5,6\}$ are uniquely fine.*

(c) The matrices $\begin{bmatrix} 9 & 13k+r \\ 0 & -4 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \in \{2,3,4,5,6\}$ have index 2.*

(d) The matrices $\begin{bmatrix} 10 & 13k+r \\ 0 & -3 \end{bmatrix}$, *$k \in \mathbf{Z}$ are not fine for $r \in \{2,6\}$, are uniquely fine for $r \in \{3,4\}$and have index 2 for $r=5$.*

(e) The matrices $\begin{bmatrix} 11 & 13k+r \\ 0 & -2 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \in \{2,3,4,5,6\}$ have index 2.*

(f) The matrices $\begin{bmatrix} 12 & 13k+r \\ 0 & -1 \end{bmatrix}$, *$k \in \mathbf{Z}$ are uniquely fine for $r \in \{2,6\}$ and are not fine for $r \in \{3,4,5\}$.*

(g) The matrices $\begin{bmatrix} 13 & 13k+r \\ 0 & 0 \end{bmatrix}$, *$k \in \mathbf{Z}$ and $r \in \{2,3,4,5,6\}$ are not fine.*

Therefore, the following remain open:

Prove or disprove. 1) Matrices $C_{abd} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $0 \le b \le |a-d|$, $a \ne d$ and *even* $a-d$ are not uniquely fine.

2) For any *odd* $a-d \ge 15$ there exist b with $0 \le b \le \dfrac{|a-d|}{2}$, such that C_{abd} is uniquely fine.

Finally recall that in [3], the fineness of 2×2 integral matrices of type $M_{bc} = \begin{bmatrix} 1 & b \\ c & 0 \end{bmatrix}$ was discussed with a special concern for the cases $c = b$, $c = b+1$ and $c = b+2$. While formulas show that matrices M_{bb} and $M_{b,b+2}$ are always fine, this fails for matrices $M_{b,b+1}$. Since finding *uniquely fine* matrices was not addressed in [3], next we provide more details in the first two cases.

Proposition 14 *Matrices M_{bb}, $b \geq 0$ and $M_{b-1,b+1}$, $b \geq 1$ are not uniquely fine.*

Proof. To make this self-contained, we recall the formula (5.17) from [3]

$$M_{bb} = \begin{bmatrix} -b^2 & -b^2+b-1 \\ b^2+b+1 & b^2+1 \end{bmatrix} + (b^2+1) \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

It is easy to show that the only symmetric nilpotent in $\mathcal{M}_2(\mathbf{Z})$ is 0_2. Since M_{bb} are symmetric, but the components of the fine decomposition are not, a different fine decomposition is given by the transposes. Hence, such matrices have index at least 2.

As for $M_{b-1,b+1}$, a fine decomposition (namely (5.18)) was given in [3] (with a unit of determinant $= 1$). Here is a different one (with a unit of determinant $= -1$)

$$M_{b-1,b+1} = \begin{bmatrix} b^2-1 & -b^2+b+1 \\ b^2+b-1 & -b^2+2 \end{bmatrix} + (b^2-2) \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

■

Remark. M_{00} is an idempotent of fine index 2, M_{11} is a unit of infinite fine index, M_{02} is an idempotent of fine index 4.

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