IDEMPOTENT MATRICES SIMILAR TO TRIANGULAR MATRICES ARE ALSO SIMILAR TO DIAGONAL ONES.

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We first quote from [2] introduction:

"In 1945, Foster examined the following questions: for a commutative ring R, when can we find an invertible matrix P over R such that $PAP^{-1} = \mathrm{diag}(e_1, ...e_n)$ for a given idempotent matrix A over R? The problem concerns not only matrix theory but also module theory and algebraic K-theory. He proved the following theorem (cf. [1], Theorem 10).

Foster's theorem. *The following are equivalent for a commutative ring R with identity:*

(a) Each idempotent matrix over R is diagonalizable under a similarity transformation.

(b) Each idempotent matrix over R has a characteristic vector.

In 1966, Steger in [3] utilized Foster's theorem to prove the following theorem.

Steger's theorem. *Let R be a commutative ring with identity and A be an $n \times n$ idempotent matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix, then there is an invertible matrix U over R such that UAU^{-1} is a diagonal matrix*".

In 1999, Song, Guo [2], proved that Foster's theorem and Steger's theorem can be generalized to an arbitrary (possibly not commutative) ring with identity.

Moreover, *in an arbitrary (unital) ring R* they proved that *equivalent idempotents are (also) conjugate.*

Using Steger's terminology, a ring R was called an *ID* ring if every idempotent matrix over R is similar to a diagonal matrix. Thus, by the result of Song and Guo (Corollary 5), if every idempotent matrix over R is equivalent to a diagonal matrix, then R is an ID ring. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

Starting with this summary on similarity of idempotent matrices to diagonal matrices, the following is a natural

Question. Can we describe the rings over which every idempotent matrix over R is similar to a triangular matrix ?

Since diagonal matrices are (upper and lower) triangular, one should expect to find a class of rings, larger than the class of ID rings, mentioned above. In what follows, as an easy consequence of a corollary from [2], we prove the following

Theorem 1. *Over any (unital) ring R, an idempotent matrix E is similar to a triangular matrix if and only if it is similar to a diagonal matrix.*

Therefore, **the class of rings over which every idempotent matrix is similar to a triangular matrix is not larger: it is precisely the same class of ID rings**.

For the proof we recall from [2]

Corollary 8. *Let E be an idempotent matrix over a ring R, and let A be similar to a block written matrix* $\begin{bmatrix} B_{11} & \mathbf{0} \\ B_{21} & B_{22} \end{bmatrix}$. *Then A is also similar to* $\begin{bmatrix} B_{11} & \mathbf{0} \\ \mathbf{0} & B_{22} \end{bmatrix}$.

Proof. [of the theorem]. Let E be an $n \times n$ idempotent matrix. The statement is trivial for $n = 1$ and obvious for $n = 2$.

By induction, suppose it holds for $(n-1) \times (n-1)$ idempotent matrices. Decompose E into its left-upper $(n-1) \times (n-1)$ block B_{11} and the 1×1 block e_{nn}, i.e., $E = \begin{bmatrix} B_{11} & \mathbf{0} \\ \beta & e_{nn} \end{bmatrix}$ (here β is an $1 \times (n-1)$ row and e_{nn} is an idempotent in R). Since $B_{11}^2 = B_{11}$, by the previous corollary, we can replace the row β be the zero row. Finally, by induction hypothesis, we can replace B_{11} by a diagonal matrix, and the proof is complete. □

References

[1] A.L. Foster *Maximal idempotent sets in a ring with units*. Duke J. Math. **13** (1946), 247-258.

[2] G. Song, X. Guo *Diagonability of idempotent matrices over noncommutative rings*. Linear Alg. Appl. **297** (1999), 1-7.

[3] A. Steger *Diagonability of idempotent matrices*. Pacific J. of Math. **19** (3) (1966), 535-542.