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Self-c-Injective Abelian Groups (*).

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ABSTRACT - Using suitable bibliography, torsion self-c-injective Abelian groups can be characterized, and torsion free self-c-injective Abelian groups turn out to coincide with a widely studied class: the quasi-pure-injective groups. In this paper the mixed case is dealt with: we prove that an honest mixed Abelian group is self-c-injective if and only if it is a direct sum of a divisible group, a quasi-pure-injective torsion free group, and a fully invariant subdirect product of a rigid family of reduced quasi-injective primary groups.

1. Introduction and background.

Recently (2000 and 2004), modules which are self-injective relative to closed submodules were considered by C. Santa-Clara and P.F. Smith (see [13] and [14]) and some Abelian group examples given. The aim of this paper is to determine, up to the infinite rank torsion-free case, the self-c-injective Abelian groups. All the groups we consider in the sequel are Abelian groups. If A is such a group, $T(A)$ denotes the torsion part of A, $T_p(A)$ is the p-component of A, $p(A)$ is the set of all primes p such that $T_p(A) \neq 0$, and $d(A)$ is the set of all primes p such that A is p-divisible. If $x \in A$ we denote by $h_p(x)$ the p-height of x. $\mathbb{P}$ denotes the set of all prime numbers, which we usually denote by p or q. A subgroup $C \leq A$ is *a closed subgroup* if it has no proper essential extensions in A. All unexplained notions and notations can be found in [6] and [5].

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If $\mathcal{P}$ is a property the submodules of an R-module N may have (e.g., closed, pure, essential, uniform), a module M is called *N-$\mathcal{P}$-injective* if every module homomorphism $f : K \to M$ from a submodule K of N having property $\mathcal{P}$ can be extended to an endomorphism of M (i.e., there exists a homomorphism $\overline{f} : N \to M$ such that $\overline{f}|_K = f$). If $M = N$ then M is called *quasi- $\mathcal{P}$-injective* (or *self-$\mathcal{P}$-injective*). Therefore

- for $\mathcal{P} \equiv$ closed submodule, *N-c-injective* and *self-c-injective* modules (considered in the above mentioned papers) are defined,
- for $\mathcal{P} \equiv$ pure submodule, *N-pure-injective* and *quasi-pure-injective* modules are defined,
- for $\mathcal{P} \equiv$ any submodule, *N-injective and quasi-injective (or self-injective) modules are defined.*

Similar notions, defined merely for Abelian groups, were intensively studied in the late 70's. In what follows, we recall some of these, since this will be useful in the sequel:

- for $\mathcal{P} \equiv p$-pure subgroup, the *quasi-p-pure-injective* groups are defined,
- for $\mathcal{P} \equiv$ neat subgroup, the *quasi-neat-injective* groups are defined,
- for $\mathcal{P} \equiv p$-neat subgroup, the *quasi-p-neat-injective* groups are defined.

Here, for a given prime p, a subgroup H of a group G is *p-neat* if $pH = H \cap pG$, *p-pure* if $p^nH = H \cap p^nG$ holds for every positive integer n and is *neat* (*pure*) if it is p-neat (respectively p-pure) for all prime numbers p.

As it is well-known, the closed subgroups ($\mathbb{Z}$-submodules) are exactly the neat subgroups ([12] and [11]). Thus, an alternate denomination for self-c-injective Abelian groups (in the spirit of the late 70's) is *quasi-neat-injective*.

We recall some results which are often used without explicit mention.

LEMMA 1.1. *If H is a neat fully invariant subgroup of a self-c-injective group G then H is self-c-injective.*

LEMMA 1.2. [13, Lemma 2.5]. *Let G be a group and let A_i, $i \in I$, be a family of groups. Then $\prod_{i\in I} A_i$ is G -c-injective if and only if each A_i is G-c-injective.*

LEMMA 1.3. [13, Corollary 2.6]. *A direct summand of a self-c-injective group is a self-c-injective group.*

Torsion self-c-injective groups. By standard arguments, in the torsion case we can reduce our problem to p -groups. Indeed, it is readily checked that *if* $G = \bigoplus_p G_p$ *is the primary decomposition of a torsion group, then* G *is self-c-injective if and only if* G_p *are self-c-injective for all primes* p. Note that for the proof of this result, the fact that for every subgroup H of G_q we have $\mathrm{Hom}(H, \bigoplus_{p \neq q} G_p) = 0$ is essential, that is, the fully invariant property of all the primary components alone is not sufficient in order to complete a proof. Actually, as a consequence of Theorem 2.2 we can see that the group $\mathbb{Z}^p \oplus (\mathbb{Z}(p^2) \oplus \mathbb{Z}(p))$ is not self-c-injective but it is a direct sum of two fully invariant subgroups which are self-c-injective (here $\mathbb{Z}^p = \{m/p^k \mid m, k \in \mathbb{Z}\}$).

Obviously, a subgroup of a p-group is p-neat if and only if it is neat. Hence a p-group is quasi-p-neat-injective if and only if it is quasi-neat-injective. As quasi-neat injective $\equiv$ self-c-injective, a p-group is self-c-injective if and only if it is quasi-p-neat-injective.

In [4], the torsion quasi-p-neat injective groups were characterized:

THEOREM 1.4. *A torsion group* G *is quasi-*p*-neat-injective if and only if* G_p *is the direct sum of two quasi-injective groups and* G_q *is quasi-injective for every prime* $q \neq p$.

In order to have the whole picture, also recall (see [10] and [7]):

THEOREM 1.5. *The following conditions are equivalent for a group* G*:*

i) G *is quasi-injective;*

ii) G *is a fully invariant subgroup of a divisible (injective) group;*

iii) G *is either injective or a torsion group whose* p*-components are direct sums of isomorphic cocyclic groups.*

Therefore, the quasi-injective p-groups are the *homogeneous* p*-groups*, i.e. direct sums of isomorphic cocyclic p-groups ($\mathbb{Z}(p^k)$ with $1 \leq k \leq \infty$), and a self-c-injective p-group is a direct sum of at most two homogeneous p-groups. We can state a complete characterization for torsion self-c-injective groups.

THEOREM 1.6. *A torsion group is self-c-injective if and only if every primary* p *-component is a direct sum of at most two homogeneous* p*-groups.*

From Lemma 1.1 and the previous theorem we obtain

COROLLARY 1.7. *If G is a self-c-injective group then $T(G)$ is a self-c-injective group. Consequently, every p-component of G is a direct summand of G.*

Torsion free self-c-injective groups. For torsion-free groups, neatness and purity are equivalent. Hence, *the self-c-injective torsion free groups are exactly the quasi-pure-injective torsion free groups.*

This class was intensively studied in the last 30 years (though a complete characterization does not exist): rank two self-c-injective torsion-free groups were characterized in [1], completely decomposable self-c-injective groups were characterized in [3] and finally, finite rank self-c-injective torsion free groups were characterized in [2]. Generalizations of this notion were studied in [15], [8] , and [9].

2. Mixed self-c-injective Abelian groups.

According to [6, Exercise 31.17] or [12, Section I.6] every subgroup of a group can be embedded in a neat subgroup which is minimal with respect to this property. For reader's convenience we construct such a subgroup in a particular case.

LEMMA 2.1. *Let G be a group and p a prime number. If $x \in G$ is an element of infinite order such that $h_p(x) = 0$ then there exists a neat subgroup $[x]$ of G such that $x \in [x]$, and $[x]$ is a rank 1 torsion free group which is not p-divisible.*

PROOF. Let D be the divisible hull of G. We consider a direct decomposition $D = T(D) \oplus K$ such that $x \in K$. If $U = \{y \in K \mid \exists\, n \in \mathbb{Z}, (n,p) = 1 : nk \in \langle x \rangle\} \cap G$, we prove that U is a neat subgroup of G. If $z \in G$ is such that $pz \in U$, writing $z = t \oplus k$ with $t \in T(D)$ and $k \in K$, we observe that $pt = 0$ and hence $t \in G$. Therefore we can suppose $z \in K \cap G$ and there exist $m, n \in \mathbb{Z}$ such that $(n,p) = 1$ and $npz = mx$. Since $h_p(x) = 0$, p divides m. We obtain in K the equality $npz = m'px$ which yields $nz = m'x \in \langle x \rangle$ and so $z \in U$. Hence U is p-neat in G. If $q \neq p$ is a prime and $qz \in U$ ($z \in G$) a similar proof allows us to suppose $z \in K \cap G$. In this context using the definition of U we obtain $z \in U$. It is not difficult to observe that $[x] = U$ verifies the required conditions. □

Recall that a group is a *honest mixed group* if it is neither torsion nor torsion free, and it is *torsion reduced* if its torsion part is reduced.

THEOREM 2.2. *Let G be a torsion reduced honest mixed group which is self-c-injective. Then*

a) *If p is a prime such that $T_p(G) \neq 0$ then $G/T_p(G)$ is p-divisible.*

b) *For every prime p the p-component $T_p(G)$ is a quasi-injective group.*

c) *There exists a direct decomposition $G = H \oplus K$ where H is a torsion-free group which is p-divisible for all $p \in p(G)$ and K is a subdirect product of $\prod_p T_p(G)$.*

PROOF. a) Since every p-component of G is a bounded direct summand, it suffices to prove that if H is not torsion, H is not p-divisible and $T_p(H) = 0$, then for every integer $n > 0$ the group $G' = \mathbb{Z}(p^n) \oplus H$ is not self-c-injective. Since H is not p -divisible, there exists $h \in H$ an element of infinite order such that $h_p(h) = 0$. If c be a generator for $\mathbb{Z}(p^n)$ then $x = c + p^n h$ is of infinite order. Since $T_p(H) = 0$ we have $h_p(p^n h) = n$. In what follows we use the neat subgroup $[x] \leq G'$.

Let $\pi : G' \to H$ be the canonical projection. We consider an element $u \in [x]$. Then there exist $q, r \in \mathbb{Z}$, $(r, p) = 1$ such that $ru = qx$. Thus $r\pi(u) = q\pi(x) = qp^n h$ and, since r is coprime with p, we obtain $\pi(u) \in p^n H$. Moreover, since $T_p(H) = 0$, for each $u \in U$, there exists only one element $y_u \in H$ such that $p^n(y_u) = \pi(u)$, and the correspondence $u \mapsto y_u$ defines a homomorphism $\alpha : [x] \to H$. Note that $\alpha(x) = h$.

If G' were self-c-injective there would exist $\beta : G' \to H$ which extends α. This leads to a contradiction since $2n \leq h_p(\beta(p^{2n}h)) = h_p(\alpha(p^n x)) = h_p(p^n h) = n$.

b) Assume that for a prime p the p-component $T_p(G) \neq 0$ is not quasi-injective. Then we can find a direct summand of G isomorphic to a group of the form $L = \mathbb{Z}(p^m) \oplus \mathbb{Z}(p^n) \oplus H$ with $0 < m < n < \infty$, and H a non torsion group with $T_p(H) = 0$. This group is self-c-injective as a direct summand of a self-c-injective group, hence a) shows that H is p-divisible. Therefore, every equation $p^k y = h \in H$ has a unique solution in H, denoted $p^{-k}h$. We consider an element $x = a + p^{n-m}b + c$ where a is a generator for $\mathbb{Z}(p^m)$, b is a generator for $\mathbb{Z}(p^n)$, $c \in H$ an infinite order element and the neat subgroup $[x]$. Then $[x]/p^n[x] \cong \mathbb{Z}(p^n)$ is generated by the coset of x and there exists a homomorphism $\alpha : [x] \to L$ such that $\alpha(x) = b$. Since L is self-c-injective there exists $\beta : L \to L$ extending α. If $\beta(p^{m-n}c) = ka + lb + d$, where k, l are integers, and $d \in H$ then $0 \neq p^m b = \alpha(p^m x) = \alpha(p^m c) = \beta(p^m c) = p^n d$, a contradiction since $p^n d$ is of finite order if and only if $d = 0$.

c) For each prime p we consider a direct decomposition $G = T_p(G) \oplus$

$\oplus G(p)$ with the canonical projection $\pi_p : G \to T_p(G)$ (note that the decomposition is unique as a consequence of a)), and we view π_p as endomorphisms of G; actually these are central endomorphisms. The family of all π_p induces a homomorphism $\psi : G \to \prod_p T_p(G)$, $\psi(g) = (\pi_p(g))_p$. Denoting by $H = \ker(\psi) = \bigcap_p G(p)$, we claim that H is a pure subgroup of G. To prove this, we consider $q \in p(G)$, $n > 0$ an integer and $g \in G$ such that $q^n g = h \in H$. We write $g = g_q + g(q)$ with $g_q \in T_q(G)$ and $g(q) \in G(q)$. Using $h \in G(q)$ we obtain $q^n g_q = 0$, hence we can suppose $g \in G(q)$. Taking another prime $p \neq q$ and the decomposition $g = g_p + g(p)$ with $g_p \in T_p(G)$ and $g(p) \in G(p)$, since $h = q^n g = q^n g_p + q^n g(p) \in G(p)$, we obtain $g_p = 0$ and so $g \in G_p$. Therefore $g \in H$ and H is q-pure for all $q \in p(G)$. Similarly one can see that H is q-pure for all $q \notin p(G)$, hence H is a pure subgroup of G.

Then $H + T(G) = H \oplus T(G)$ is a pure subgroup of G (see [6, Exercise 26.5]). Since G is self-c-injective, it follows that there exists an endomorphism$\varphi : G \to G$ which extends the homomorphism $1_H \oplus 0 : H \oplus T(G) \to$ $\to H \oplus T(G)$. Moreover, since $\varphi(H) \subseteq H$, we can consider the endomorphism $\overline{\varphi} : G/H \to G/H, \overline{\varphi}(g + H) = \varphi(g) + H$. Then $\overline{\varphi}$ induces the endomorphism $\varphi^\star$ of $\psi(G)$ by $\varphi^\star((\pi_p(g))_p) = (\pi_p(\varphi(g)))_p = (\varphi(\pi_p(g)))_p = 0$ for all $g \in G$. Therefore $\overline{\varphi} = 0$, hence $\varphi(G) \subseteq H$, and this shows that φ is an idempotent endomorphism of G. Then $H = \varphi(G)$ is a direct summand, and $G \cong H \oplus \psi(G)$. □

Lemma 2.3. *Let $G = A \oplus B$ be a group such that:*

(1) *for every neat subgroup $K \leq G$,*

$$K \cap B \subseteq \mathrm{rad}_K(A) = \bigcap\{\mathrm{Ker}(f) \mid f : K \to A\},$$

and

(2) *A is injective relative to the exact sequence $0 \to \pi_A(K) \to A$ (here $\pi_A : G \to A$ denotes the canonical projection).*

Then A is G-c-injective.

Proof. Let K be a neat subgroup of G, and $f : K \to A$ a homomorphism. We denote by $\rho : K \to K/K \cap B$ the canonical epimorphism. Since $K \cap B \subseteq \mathrm{rad}_K(A)$ we have $K \cap B \leq \mathrm{Ker}(f)$, and so there exists a homomorphism $f^\star : K/K \cap B \to A$ such that $f = f^\star \rho$. Moreover, there exists $\pi^\star : K/K \cap B \to A$ such that $\pi_{A_{|K}} = \pi^\star \rho$. It is not difficult to prove that $\pi^\star$ is a monomorphism, and $\pi^\star(K/K \cap B) = \pi_A(K)$. Then by (2) there exists $f_1 : A \to A$ such that $f^\star = f_1 \pi^\star$. It follows that $f = f^\star \rho = f_1 \pi^\star \rho = f_1 \pi_{A_{|K}}$. We view the canonical projections $\pi_A : G \to A$ and $\pi_B : G \to B$ as endomorphisms of G. If $\overline{f}: G \to A$ is defined by $\overline{f}\pi_A = f_1$, and $\overline{f}\pi_B = 0$ we obtain $\overline{f}_{|K} = \overline{f}(\pi_A + \pi_B)_{|K} = \overline{f}\pi_{A_{|K}} = f_1\pi_{A_{|K}} = f$, hence A is G-c-injective. □

REMARK 2.4. Generally $\pi_A(K)$ is not a neat subgroup. For example, if $A = \mathbb{Z}$ and $B = \mathbb{Z}(p)$ then $K = \langle(p, \widehat{1})\rangle$ is a neat subgroup in $G = A \oplus B$, but $\pi_A(K) = p\mathbb{Z}$ is not a neat subgroup in $A = \mathbb{Z}$.

COROLLARY 2.5. *Let $G = A \oplus B$ be a group such that for every neat subgroup $K \leq G$, $K \cap B \subseteq \mathrm{rad}_K(A)$. If*

a) *A is quasi-injective, or*

b) *A is a self-c-injective torsion free group and $d(A) \cup d(B) = \mathbb{P}$,*

then A is G-c-injective.

PROOF. a) is obvious, so we only prove that b) is a consequence of the previous lemma. Let p be a prime such that B is p-divisible. If K is a neat subgroup of G and $a \in A$ is such that $pa \in \pi_A(K)$ then there exist $k \in K$ and $b \in B$ such that $k = pa + b$. Since B is p-divisible, p divides k, and there exists $k' \in K$ such that $pk' = k$. Then there exists $a' \in \pi_A(K)$ such that $pa = pa'$. Therefore $\pi_A(K)$ is p-pure in A (A is torsion free) for all primes p for which A is not p-divisible. Thus the pure hull of $\pi_A(K)$, as a subgroup of A, is $\pi_A(K)_\star = \mathbb{Z}_S\pi_A(K) = \{(m/n)x \mid m/n \in \mathbb{Z}_S,\ x \in \pi_A(K)\}$, where S is the set of all primes p such that A is p-divisible, and $\mathbb{Z}_S = \{m/n \mid (n,p) = 1$ for all $p \notin S\}$. If $a : \pi_A(K) \to A$ is a homomorphism then $a^\natural : \pi_A(K)_\star \to A$, $a^\natural((m/n)x) = (m/n)a(x)$ is well defined and extends a. Since A is self-c-injective we can extend $a^\natural$ to a homomorphism $a^\star : A \to A$, and the proof is complete. □

LEMMA 2.6. *If A is a reduced quasi-injective p-group, and B is a p-divisible group then A is $A \oplus B$-c-injective.*

PROOF. Let $n \geq 1$ (the case $n = 0$ is obvious) be the exponent of the p-group A (i.e. $A \cong \bigoplus \mathbb{Z}(p^n)$), K a neat subgroup of $G = A \oplus B$ and $a : K \to A$ a homomorphism. If $l \in K \cap B$, then l is divisible by p in G and there exists $l_1 \in K$ such that $pl_1 = l$. If $\pi_A : G \to A$ is the canonical projection, we have $p\pi_A(l_1) = 0$. In the hypothesis $n > 1$, l_1 is divisible by p, hence we can continue this procedure in order to construct $l_2 \in K$ such that $pl_2 = l_1$. Continuing in the same way we obtain a sequence $l_0 = l, l_1, \dots, l_n$ in K such that for all $i \in \{1, \dots, n\}$ we have $pl_i = l_{i-1}$. Then $K \cap B \subseteq p^nK$ and $K \cap B \subseteq \mathrm{rad}_A(K)$ follows and finally we use Corollary 2.5 in order to conclude that A is $A \oplus B$-c-injective □

The next result is a generalization of [14, Theorem 6].

COROLLARY 2.7. *Let S be an infinite set of primes. If for each prime $p \in S$ T_p denotes a non-zero reduced p-group, then the group $\prod_{p\in S} T_p$ is self-c-injective if and only if every T_p is a quasi-injective group.*

PROOF. Using Lemma 1.1 it suffices to prove that each T_q is $\prod_{p\in S} T_p$-c-injective, and this is a consequence of Lemma 2.6. Conversely, the result follows from Theorem 2.2 since S is an infinite set. □

We are now able to characterize the direct summands K which appear in Theorem 2.2, c).

THEOREM 2.8. *Let G be a group such that $G \leq \prod_p T_p(G)$, and each p-component $T_p(G)$ is a reduced quasi-injective group. The following conditions are equivalent:*

i) *G is a self-c-injective group;*
ii) *G is a fully invariant pure subgroup of $\prod_p T_p(G)$;*
iii) *G is a fully invariant subgroup of $\prod_p T_p(G)$.*

PROOF. i) ⇒ ii) Using Theorem 2.2 a), $G/T_p(G)$ is p-divisible for all $p \in p(G)$ and so $G/T(G)$ is p-divisible for all $p \in p(G)$. Hence it is p-pure in $\prod_p T_p(G)/T(G)$, and it is not difficult to prove that G is p-pure in $\prod_p T_p(G)$ for all $p \in p(G)$.

Since $G/T(G)$ is p-divisible for all $p \in p(G)$ we have

$$\mathrm{Hom}(G/T(G), G) \leq \mathrm{Hom}(G/T(G), \prod_p T_p(G)) = 0.$$

Therefore the exact sequence

$$0 = \mathrm{Hom}(G/T(G), G) \to \mathrm{Hom}(G, G) \to \mathrm{Hom}(T(G), G)$$

proves that two endomorphism α, β of G coincide if and only if $\alpha_{|T(G)} = \beta_{|T(G)}$. Note that $\prod_p T_p(G)$ has the same property.

Let $q \notin p(G)$. The homomorphism $\alpha : T(G) \to T(G)$ given by $q\alpha(x) = x$ is well defined. Since G is self-c-injective there exists $\beta : G \to G$ which extends α. Then $q\beta_{|T(G)} = 1_{|T(G)}$ and, using the previous paragraph, this proves that the multiplication by q is an automorphism of G. Therefore G is q -divisible, and so q-pure in $\prod_p T_p(G)$.

In order to prove that G is fully invariant we consider an endomorphism $\alpha : \prod_p T_p(G) \to \prod_p T_p(G)$. Then there exists $\beta : G \to G$ which extends

$\alpha_{|T(G)}$. Moreover by Corollary 2.7, $\prod_p T_p(G)$ is self-c-injective and G is pure in $\prod_p T_p(G)$, hence there exists $\gamma : \prod_p T_p(G) \to \prod_p T_p(G)$ which extends β. Since $\alpha_{|T(G)} = \gamma_{|T(G)}$ we obtain $\alpha = \gamma$, and so $\alpha(G) \subseteq G$.

The implication ii) $\Rightarrow$ iii) is obvious. As for iii) $\Rightarrow$ ii) observe that for every $q \notin p(G)$ the homomorphism $\alpha : \prod_p T_p(G) \to \prod_p T_p(G)$ given by $q\alpha(x) = x$ is well defined. Since $\alpha(G) \subseteq G$ we obtain $qG = G$, so that G is q-divisible. If $q \in p(G)$ we consider $\alpha : \prod_p T_p(G) \to \prod_p T_p(G)$ defined by $\alpha(T_q(G)) = 0$ and $q\alpha(x) = x$ for all $x \in \prod_{p \neq q} T_p(G)$ and the direct decomposition $G = T_q(G) \oplus (G \cap \prod_{p\neq q} T_p(G))$. It follows that $\alpha(G \cap \prod_{p\neq q} T_p(G)) \subseteq G \cap \cap \prod_{p\neq q} T_p(G)$, and this shows that $G \cap \prod_{p\neq q} T_p(G)$ is q-divisible. Then $G/T(G)$ is q-pure in $(\prod_p T_p(G))/T(G)$, hence G is q-pure in $\prod_p T_p(G)$.

ii) $\Rightarrow$ i) is a consequence of Corollary 2.7 and Lemma 1.1. □

An honest mixed group G which verifies the conditions in the previous theorem is called a *proper mixed self-c-injective* group.

COROLLARY 2.9. *If G is a proper mixed self-c-injective group then $G/T(G)$ is divisible and G is p-divisible whenever $T_p(G) = 0$.*

Recall that a group G is a *splitting* group if $T(G)$ is a direct summand of G.

COROLLARY 2.10. *A self-c-injective group G with $r_0(G) \leq \aleph_0$ is a splitting group.*

PROOF. Using Theorem 2.2, if a self-c-injective group is not splitting, it has a direct summand which is a proper mixed self-c-injective group. Therefore, it suffices to prove that a proper mixed self-c-injective group has the torsion free rank at least $2^{\aleph_0}$.

Let G be a proper mixed self-c-injective group. We consider the group $H = \prod_p T_p(G)$ together with the canonical projections $\pi_p : H \to T_p(G)$. For an element $x \in H$ we consider the *characteristic* $\chi(x) = (h_p(\pi_p(x)))_p$. Since each $T_p(G)$ is a homogeneous bounded p-group, there exists $C_p(x)$ a cyclic direct summand of $T_p(G)$ which contains $\pi_p(x)$. Therefore $x \in \prod_p C_p(x)$ which is a direct summand of H. Thus, for two elements $x, y \in H$ there exists an endomorphism of H which maps x into y if and only if $\chi(x) \leq \chi(y)$.

Therefore if $x \in G$ is an infinite order element, $\chi(x)$ has $\aleph_0$ finite components. If $\chi = (\chi_p)_p$ is a characteristic, for each set $S \subseteq \mathbb{P}$ we consider the characteristic $\chi^S = (\chi_p^S)_p$ with $\chi_p^S = \infty$ for all $p \in S$ and $\chi_p^S = \chi_p$ for all $p \notin S$. If $\chi = \chi(x)$, then χ^S is the characteristic of $x^S \in T(\prod_p C_p(x))$ with

$\pi_p(x^S) = 0$ for all $p \in S$, and $\pi_p(x^S) = \pi_p(x)$ for all $p \notin S$. It follows that the cardinality of $G \cap \prod_p C_p(x)$ is $2^{\aleph_0}$, and since $T(\prod_p C_p(x))$ is a countable group, the cardinality of $(G \cap \prod_p C_p(x))/T(G \cap \prod_p C_p(x))$ is $2^{\aleph_0}$, and this is possible if and only if the torsion free rank of G is at least $2^{\aleph_0}$. $\square$

THEOREM 2.11. *Let G be an honest mixed group. Then G is self-c-injective if and only if $G = D \oplus H \oplus R$ such that*

i) *D is a divisible group,*

ii) *R is a self-c-injective reduced torsion free group which is p-divisible for all $p \in p(H)$, and*

iii) *H is a proper mixed self-c-injective group or it is a quasi-injective reduced torsion group.*

PROOF. If G is self-c-injective then G has the mentioned form (consequence of Theorem 2.2). If G has this form then G is a pure fully invariant subgroup of $G' = D \oplus R \oplus (\prod_p T_p(H))$. Therefore, in order to prove the converse it suffices to show that G' is self-c-injective.

Since D is divisible, D is G'-c-injective. Let K be a neat subgroup of G', and $f : K \to R$ a homomorphism. If q is a prime such that $q \notin p(H)$ then $D \oplus H$ is q-divisible. Let $x \in K \cap (D \oplus H)$. Then q divides x in G', and so there exists $y \in K$ such that $qy = x$. Writing $y = r + z$, with $r \in R$ and $z \in D \oplus H$, we obtain $qr = 0$, hence $r = 0$, since R is torsion free. Then $y \in K \cap (D \oplus H)$ and it follows that $f(K \cap (D \oplus H))$ is q-divisible for all $q \notin p(H)$. Therefore $f(K \cap (D \oplus H))$ is contained in $\bigcap_{q \notin p(H)} q^\omega R$. However, since R is p-divisible for all $p \in p(H)$ we obtain $\bigcap_{q \notin p(H)} q^\omega R$ as the divisible part of the reduced group R. Hence, $f(K \cap (D \oplus R)) = 0$, and R is G'-c-injective as a consequence of Theorem 2.2, Theorem 2.8, and Corollary 2.5. Since $T_p(H)$ is G'-c-injective by Lemma 2.6, an application of Lemma 1.1 completes the proof. $\square$

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