Linear Algebra and its Applications 439 (2013) 3115–3119

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Nil-clean matrix rings

S. Breaz [a,1], G. Călugăreanu [a,*], P. Danchev [b], T. Micu [a]

[a] Department of Mathematics and Computer Science, Babeş-Bolyai University, 1 Kogălniceanu street, 400084 Cluj-Napoca, Romania
[b] Department of Mathematics, University of Plovdiv, 24 Tzar Assen Street, 4000 Plovdiv, Bulgaria

ARTICLE INFO

Article history:
Received 31 May 2013
Accepted 23 August 2013
Available online 6 September 2013
Submitted by P. Semrl

MSC:
15B33
16E50
20K30

Keywords:
Nil-clean ring
Frobenius normal form
Companion matrix
Abelian 2-group

ABSTRACT

We characterize the nil-clean matrix rings over fields. As a by product, we obtain a complete characterization of the finite rank Abelian groups with nil-clean endomorphism ring and the Abelian groups with strongly nil-clean endomorphism ring, respectively.

1. Introduction

An element in a ring R is said to be (*strongly*) *nil-clean* if it is the sum of an idempotent and a nilpotent (and these commute). A ring R is called (strongly) nil-clean if all its elements are (strongly) nil-clean. As customarily (for modules), an Abelian group will be called (*strongly*) *nil-clean* if it has (strongly) nil-clean endomorphism ring. All the groups we consider are Abelian (therefore, in the sequel "group" means Abelian group). It is easy to see that every strongly nil-clean element is strongly clean and that every nil-clean ring is clean.

* Corresponding author.
E-mail addresses: bodo@math.ubbcluj.ro (S. Breaz), calu@math.ubbcluj.ro (G. Călugăreanu), pvdanchev@yahoo.com (P. Danchev), tudormicu1@gmail.com (T. Micu).

[1] S. Breaz is supported by the CNCS-UEFISCDI grant PN-II-RU-TE-2011-3-0065.

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.08.027

For a comprehensive study of (strongly) nil-clean rings, we refer to [2] or, more recently, to [3]. For a study of strongly clean matrices over (commutative) projective-free rings, we also mention [1].

The main result which we establish in the present paper is the complete characterization of nil-clean matrix rings over fields (Theorem 3). Incidentally (note that this study began far before the apparition of [3], in an attempt to characterize Abelian groups which have (strongly) nil-clean endomorphism ring), this achievement provides a partial affirmative answer to a question asked in [3, Question 3]: “Let R be a nil-clean ring, and let n be a positive integer. Is $\mathcal{M}_n(R)$ nil-clean?”. As an application, we characterize the finite rank Abelian groups which have nil-clean endomorphism rings and the Abelian groups which have strongly nil-clean endomorphism rings.

For a background material concerning matrix theory that will be used in the sequel without a concrete citation, we refer to [6].

2. Nil-clean matrix rings

First we recall from [3, Proposition 3.14] the following simple but important result.

Lemma 1. *Let R be a nil-clean ring. Then the element 2 is a (central) nilpotent and, as such, is always contained in $J(R)$.*

Further, from the same source, we shall use [3, Corollary 3.10]

Lemma 2. *A unit is strongly nil-clean if and only if it is unipotent.*

Here an element in a ring is called *unipotent* if it has the form $1+n$, for a suitable nilpotent n.

Our main result is the following:

Theorem 3. *Suppose that K is a field. The following are equivalent*:

1. $K \cong \mathbb{F}_2$;
2. *For every positive integer n the matrix ring $\mathcal{M}_n(K)$ is nil-clean*;
3. *There exists a positive integer n such that the matrix ring $\mathcal{M}_n(K)$ is nil-clean.*

Proof. To prove the implication (1) ⇒ (2), since any matrix can be put into Frobenius normal form (a direct sum of companion matrices), and a matrix similar to a nilpotent (or idempotent, or nil-clean) matrix is nilpotent (or idempotent, or nil-clean respectively), it suffices to prove the implication for a single companion matrix. Consider the $n \times n$ companion matrix

$$C_p = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{bmatrix}$$

associated to the polynomial $p(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n$. Notice that in our case $-c_i \in \{0, 1\}$. We distinguish three cases.

Case 1: $-c_{n-1} = 1$. Just take

$$N = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 0 & 0 & \dots & 0 & -c_1 \\ 0 & 0 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \mathbf{1} \end{bmatrix}.$$

Case 2: For $-c_{n-1} = 0$ and $c_{n-2} = 1$, we take

$$N = \begin{bmatrix} \mathbf{0} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & \mathbf{0} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \mathbf{1} & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \mathbf{1} \end{bmatrix}, \qquad E = \begin{bmatrix} \mathbf{0} & \cdots & 0 & 0 & c_0 \\ 0 & \cdots & 0 & 0 & c_1 \\ 0 & \cdots & 0 & 0 & c_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \mathbf{0} & 0 & c_{n-3} \\ 0 & \cdots & 0 & \mathbf{1} & 0 \\ 0 & \cdots & 0 & 0 & \mathbf{1} \end{bmatrix}.$$

Case 3: For $-c_{n-1} = 0$ and $c_{n-2} = 0$, we take

$$N = \begin{bmatrix} \mathbf{0} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & \mathbf{0} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \mathbf{0} & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & \mathbf{1} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} \mathbf{0} & \cdots & 0 & 0 & c_0 \\ 0 & \cdots & 0 & 0 & c_1 \\ 0 & \ddots & 0 & 0 & c_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \mathbf{0} & 1 & c_{n-3}+1 \\ 0 & \cdots & 0 & \mathbf{1} & 0 \\ 0 & \cdots & 0 & 0 & \mathbf{1} \end{bmatrix}.$$

The idempotency is easily checked directly, and the nilpotency reduces to Cayley–Hamilton theorem since all matrices N have X^n as characteristic polynomial.

The implication (2) $\Rightarrow$ (3) is obvious and so its verification will be omitted.

As for the implication (3) $\Rightarrow$ (1) notice that, as a consequence of Lemma 1, the field K is of characteristic 2.

Furthermore, let $a \neq 0$ be an element from K. Then $aI_n = E + N$ with E idempotent and N nilpotent matrices. Since aI_n is central, using Lemma 2 it follows that $(a-1)I_n$ is a nilpotent matrix. This is possible only if $a = 1$, hence $K \cong \mathbb{F}_2$. □

Before stating some consequences and generalizations, recall

Lemma 4. *(See [3, Proposition 3.15].) Let I be a nil ideal of the ring R. Then R is nil-clean if and only if the quotient ring R/I is nil-clean.*

Coming back to Diesl question (see Introduction), notice that, as a consequence of Theorem 3 and Lemma 4, $\mathcal{M}_n(\mathbf{Z}(2^k))$ is another example of nil-clean matrix ring, for a nil-clean ring which is not a field. More can be proved

Corollary 5. *If R is any nil-clean commutative local ring then $\mathcal{M}_n(R)$ is nil-clean.*

Proof. According to [3, Proposition 3.24], a ring R with only trivial idempotents is nil-clean if and only if R is a local ring with $J(R)$ nil and $R/J(R) \cong \mathbb{F}_2$. If R is any commutative local ring, then $J(\mathcal{M}_n(R)) = \mathcal{M}_n(J(R))$ is nil, and the quotient of the matrix ring by its Jacobson radical is isomorphic to $\mathcal{M}_n(\mathbb{F}_2)$. This is nil-clean by our theorem and so $\mathcal{M}_n(R)$ is nil-clean by Lemma 4. □

Corollary 6. *If R is any Boolean ring then $\mathcal{M}_n(R)$ is nil-clean.*

Proof. For a Boolean ring R, let $A \in \mathcal{M}_n(R)$. If S is the subring of R generated by the entries of A, since R is commutative and all elements are idempotents, it is not hard to see that S is (Boolean and) finite. Hence S is isomorphic to a finite direct products of copies of $\mathbb{F}_2$. Therefore $\mathcal{M}_n(S)$ is nil-clean since it is isomorphic to a finite direct product of copies of $\mathcal{M}_n(\mathbb{F}_2)$. Hence, the matrix A is nil-clean in $\mathcal{M}_n(S)$ and so nil-clean in $\mathcal{M}_n(R)$. □

Corollary 7. *If R is any commutative nil-clean ring then $\mathcal{M}_n(R)$ is nil-clean.*

Proof. If R is a commutative nil-clean ring, then $J(R)$ is nil and $R/J(R)$ is Boolean [3, Corollary 3.20]. Hence $J(\mathcal{M}_n(R))$ is nil and since $\mathcal{M}_n(R/J(R))$ is nil-clean, so is $\mathcal{M}_n(R)$ (just use $\mathcal{M}_n(R/J(R)) \cong \mathcal{M}_n(R)/J(\mathcal{M}_n(R))$ and Lemma 4). □

3. Applications to Abelian groups

We will apply here the main theorem from the preceding section to characterize nil-clean Abelian groups of finite rank, i.e., those finite rank Abelian groups whose endomorphism ring is nil-clean. It is worthwhile noticing that a comprehensive investigation of clean p-torsion Abelian groups was given in [5].

The next statement is pivotal:

Lemma 8. *Let G be a finite Abelian 2-group of rank n. Then there is an isomorphism* $\mathrm{End}(G)/2\,\mathrm{End}(G) \cong \mathcal{M}_n(\mathbb{F}_2)$.

Proof. There is a direct decomposition $G = C_1 \oplus \cdots \oplus C_n$, where all C_i are cyclic subgroups of G. Thus $\mathrm{End}(G)$ is isomorphic to the matrix ring $(\mathrm{Hom}(C_i, C_j))_{1 \leqslant i,j \leqslant n}$ (see, e.g., [4]). Moreover, all groups $\mathrm{Hom}(C_i, C_j)$ are cyclic 2-groups. Utilizing this information, it is easy to see $\mathrm{End}(G)/2\,\mathrm{End}(G) \cong \mathcal{M}_n(\mathbb{F}_2)$, as stated. □

Proposition 9. *A finite rank Abelian group G is nil-clean if and only if G is a finite 2-group.*

Proof. If G is nil-clean, then Lemma 1 applies to deduce that the endomorphism $2 \in \mathrm{End}(G)$ is nilpotent, hence G is a (bounded) 2-group. Therefore, G is a finite 2-group.

Conversely, if G is a finite 2-group, then $2\,\mathrm{End}(G)$ is a nilpotent ideal in $\mathrm{End}(G)$. The final conclusion now follows by a combination of Lemmas 4 and 8 with Theorem 3. □

Using similar arguments, one can deduce:

Proposition 10. *An Abelian group G is strongly nil-clean if and only if G is a cyclic 2-group.*

Proof. Suppose G is not cyclic and the endomorphism ring of G is strongly nil-clean. As before, the endomorphism 2 is nilpotent, hence G is a bounded 2-group. As in Lemma 8, we infer that $\mathrm{End}(G)/2\,\mathrm{End}(G)$ is isomorphic to the ring of row-finite matrices over $\mathbb{F}_2$ (see [4]). Therefore, in order to obtain a contradiction, it is enough to prove that if V is an $\mathbb{F}_2$-vector space of dimension $\geqslant 2$ then its endomorphism ring is not strongly nil-clean.

To that aim, let V be such a vector space and let $\varphi : V \to V$ be an injective endomorphism of V. If $\varphi = e + n$ with idempotent e and nilpotent n such that $en = ne$ is a strongly nil-clean decomposition, using Lemma 2, all injective endomorphisms of V can be written as $1 + n$ with n nilpotent, and this holds only if V is 1-dimensional. □

In closing, although we suspect this to be true, we were not able to prove that *the endomorphism ring of a vector space of countable dimension over $\mathbb{F}_2$ is not nil-clean*. Would it be true, we could remove in Proposition 9 the hypothesis "finite rank", thus obtaining the complete algebraic structure of Abelian groups with nil-clean endomorphism ring.

More, we were not able to prove (3) ⇒ (1) in Theorem 3 for division rings (i.e., not necessarily commutative). Would this be done, our main result (Theorem 3) could be extended to division rings. This, in turn, would have an important consequence, namely, *an Artinian ring R is nil-clean if and only if the top ring $R/J(R)$ is a direct product of matrix rings over $\mathbb{F}_2$.*

Acknowledgements

Thanks are due to the referee, for his/her valuable suggestions (especially the Corollaries to Theorem 3) which improved our study.

References

[1] H. Chen, On strongly nil clean matrices, Comm. Algebra 41 (2013) 1074–1086.
[2] A.J. Diesl, Classes of strongly clean rings, PhD Thesis, University of California, Berkeley, 1995.
[3] A.J. Diesl, Nil clean rings, J. Algebra 383 (2013) 197–211.
[4] L. Fuchs, Infinite Abelian Groups, vol. I, Acad. Press, New York, London, 1970;
L. Fuchs, Infinite Abelian Groups, vol. II, Acad. Press, New York, London, 1973.
[5] B. Goldsmith, P. Vamos, A note on clean abelian groups, Rend. Semin. Mat. Univ. Padova 117 (2007) 181–191.
[6] D. Serre, Matrices Theory and Applications, Grad. Texts in Math., Springer-Verlag, New York, 2002.