RINGS WITH FINE NILPOTENTS

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Abstract. A nonzero sum of a unit and a nilpotent element in a ring is called a fine element. This is a study of rings in which every nonzero nilpotent is fine, which we call NF rings.

1. Introduction

Throughout, rings are nonzero, associative with identity. Recall from [1] that a nonzero element in a ring is called *fine* if it is a sum of a unit and a nilpotent and a ring is a *fine* ring if every nonzero element is fine. Fine rings form a proper class of simple rings and were the topic of the paper [1]. The rings whose nonzero idempotents are fine turned out to be an interesting class of indecomposable rings and were studied in [4]. In this note, we study rings all whose nonzero nilpotents are fine. Such rings will be called *NF* (nonzero **n**ilpotents are **f**ine). NF rings include fine rings and reduced rings. In Section 2, we determine the NF property of several standard constructions of rings such as ideal extensions, direct products and polynomial (and power series) rings. Section 3 is on the NF property of a matrix ring. A direct proof for "the matrix rings over division rings are NF" is given. For $n \geq 2$, if the matrix ring $\mathbb{M}_n(R)$ is NF, then R is simple; for a commutative ring R, $\mathbb{M}_n(R)$ is NF iff R is a field. Section 4 is about fine nilpotents in a ring. For a GCD domain R (i.e. every pair of nonzero elements has a greatest common divisor), the fine nilpotents in $\mathbb{M}_2(R)$ can be identified by a specific form.

For a ring R, we denote by $J(R)$, $U(R)$ and $\text{nil}(R)$ the Jacobson radical, the unit group and the set of nilpotents of R, respectively. We write $\mathbb{M}_n(R)$ for the ring of $n \times n$ matrices over R whose identity is denoted by I_n and $\mathbb{T}_n(R)$ for the ring of upper triangular $n \times n$ matrices over R. By $E_{ij} \in \mathbb{M}_n(R)$ we denote the standard matrix unit, i.e., E_{ij} has a 1 in the (i, j) position and zeros elsewhere. As in [1], we denote by $\Phi(R)$ the set of fine elements of a ring R.

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2. NF rings

In this section, we present examples of NF rings, through some standard constructions of rings. Our first lemma is a crucial one in the entire discussion. A ring is called *uni* (see [3]) if units commute with nilpotents. A ring is *reduced* if it does not contain any nonzero nilpotents.

Lemma 2.1. (1) *Every reduced ring is an NF ring; the converse holds if R is a uni ring.*

(2) *If R is NF and I is a proper ideal of R, then $\mathrm{nil}(R)\cap I = 0$; particularly, $\mathrm{nil}(R)\cap J(R) = 0$.*

Proof. (1) Since a reduced ring R does not have nonzero nilpotents, R is NF.

Suppose that R is a uni ring that is NF. If $0 \neq a \in \mathrm{nil}(R)$, then $a = u + b$ where u is a unit and b is nilpotent. As u, b commute, $u + b$ is a unit, i.e., a is a unit, a contradiction. So R is reduced.

(2) If $0 \neq a \in \mathrm{nil}(R) \cap I$, then $a = u + b$ where u is a unit and b is nilpotent. So, in R/I, $-\bar{u} = \bar{b}$ with $\bar{b}$ a nilpotent and $-\bar{u}$ a unit, a contradiction. □

By Lemma 2.1(2), the following rings are not NF: the trivial extension $R \propto M$ of a ring R by a nontrivial (R, R)-bimodule M; $R[t]/(t^n)$ $(n \geq 2)$; $\mathbb{T}_n(R)$ $(n \geq 2)$; the formal triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ where R, S are rings and M a nontrivial (R, S)-bimodule.

Another consequence of Lemma 2.1 is

Corollary 2.2. *A local ring is NF iff it is reduced.*

We continue discussing some other standard constructions of rings.

Proposition 2.3. *Let $R = \prod_{\alpha\in\Lambda} R_\alpha$ be a direct product of rings with $|\Lambda| \geq 2$. Then R is NF iff for each $\alpha \in \Lambda$, R_α is reduced.*

Proof. (⇐) If each R_α is reduced, then R is reduced, so R is NF.

(⇒) Assume that R_{α_i} is not reduced. Then R_{α_i} contains a nonzero nilpotent x_{α_i}. Let $x = (x_\alpha) \in R$ where $x_\alpha = 0$ if $\alpha \neq \alpha_i$. Then x is a nonzero nilpotent. But, as $|\Lambda| \geq 2$, x is not a fine element in R. This contradiction shows that R_{α_i} is reduced. □

Let R be a ring and let V be an (R,R)-bimodule which is a non-unital ring in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then the *ideal-extension* $\mathbb{I}(R;V)$ of R by V is defined to be the additive abelian group $\mathbb{I}(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$. Note that if S is a ring and $S = R \oplus A$ where R is a subring and $A \lhd S$, then $S \cong \mathbb{I}(R;A)$.

Theorem 2.4. *Let $S = \mathbb{I}(R;V)$ be the ideal-extension of R by V. Then S is NF iff R is NF and V is reduced.*

Proof. ($\Rightarrow$) Let $0 \neq a \in \text{nil}(R)$. Then $(a,0) \in S$ is a nonzero nilpotent, so $(a,0) = (u,y) + (b,z)$ where $(u,y) \in U(S)$ and $(b,z) \in \text{nil}(S)$. Thus, $a = u + b$ with $u \in U(R)$ and $b \in \text{nil}(R)$. So R is NF.

Assume that $0 \neq x \in V$ and $x^2 = 0$. Then $(0,x) \in S$ is a nonzero nilpotent, so $(0,x) = (u,y) + (b,z)$ where $(u,y) \in U(S)$ and $(b,z) \in \text{nil}(S)$. Thus, $0 = u + b$ with $u \in U(R)$ and $b \in \text{nil}(R)$, a contradiction.

($\Leftarrow$) We first show that $(a,x) \in \text{nil}(S)$ implies $x = 0$. That is, the following statement (P_n) holds for all $n \geq 1$.

(P_n): whenever $(a,x)^n = 0$ in S, $x = 0$.

It is clear that (P_1) holds. Assume that $n \geq 1$ and (P_n) holds. We next show that (P_{n+1}) holds. Suppose $(a,x)^{n+1} = 0$. Thus $(a^2, ax + xa + x^2)^n = ((a,x)^2)^n = (a,x)^{2n} = 0$. By (P_n), we have $ax + xa + x^2 = 0$. Moreover, we have $a^{n+1} = 0$. Then $(axa^n)^2 = 0$. So $axa^n = 0$ as V is reduced. Thus, $(axa^{n-1})^2 = 0$, and so $axa^{n-1} = 0$. Continuing in this way, we see that $axa = 0$. So $(ax)^2 = 0$ and $(xa)^2 = 0$. As V is reduced, we deduce $ax = xa = 0$. It follows from $ax + xa + x^2 = 0$ that $x^2 = 0$, and hence $x = 0$. So (P_{n+1}) holds.

To show that S is NF, let (a,x) be a nonzero nilpotent of S. As proved above, $x = 0$. So a is a nonzero nilpotent of R. Hence $a = u + b$ where $u \in U(R)$ and $b \in \text{nil}(R)$. Then $(a,x) = (u,0) + (b,0)$ with $(u,0) \in U(S)$ and $(b,0) \in \text{nil}(S)$. So S is NF. $\square$

We note that V being reduced alone does not imply that $\mathbb{I}(R;V)$ is NF. To see this, consider $\mathbb{I}(R;V)$ with $V = R/J(R)$ such that $R/J(R)$ is reduced, but R is not NF; for example, let $R = \mathbb{Z}_4$.

Following Krempa [7], a ring R with an endomorphism σ is called *σ-rigid* if $a\sigma(a) = 0$, $a \in R$, implies $a = 0$. By Matczuk [9, Theorem A], R is σ-rigid iff $R[t;\sigma]$ is reduced. By Matczuk [9, Theorem A] and Krempa [7, Corollary 3.5], R is σ-rigid iff $R[[t;\sigma]]$ is reduced.

Corollary 2.5. *Let R be a ring with an endomorphism σ. The following are equivalent:*

(1) *$R[t;\sigma]$ is NF.*

(2) *$R[[t;\sigma]]$ is NF.*

(3) *R is σ-rigid.*

Proof. (1) $\Leftrightarrow$ (3) Note that $R[t;\sigma] \cong \mathbb{I}(R;V)$, where $V = R[t;\sigma]t$.

Suppose (1) holds. Then V is reduced by Theorem 2.4. If $a\sigma(a) = 0$ where $a \in R$, then $(at)^2 = a\sigma(a)t^2 = 0$. So $at = 0$, and hence $a = 0$. So (3) holds.

If (3) holds, then $R[t;\sigma]$ is reduced by [9, Theorem A], so $R[t;\sigma]$ is NF.

(2) $\Leftrightarrow$ (3) Note that $R[[t;\sigma]] \cong \mathbb{I}(R;V)$, where $V = R[[t;\sigma]]t$.

Suppose (2) holds. Then V is reduced by Theorem 2.4. If $a\sigma(a) = 0$ where $a \in R$, then $(at)^2 = a\sigma(a)t^2 = 0$. So $at = 0$, and hence $a = 0$. So (3) holds.

If (3) holds, then R is reduced by [9, Theorem A], and so $R[[t;\sigma]]$ is reduced by [7, Corollary 3.5]. Hence $R[[t;\sigma]]$ is NF. □

Corollary 2.6. *For a ring R, $R[[t]]$ is NF iff $R[t]$ is NF, iff R is reduced.*

Examples 2.7. (1) *If R is a domain and σ is a one-to-one endomorphism of R. Then $R[t;\sigma]$ and $R[[t;\sigma]]$ are NF.*

(2) *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, and σ be the endomorphism of R given by $\sigma(r,s) = (s,r)$. Then R is reduced but not σ-rigid, because, for $a = (1,0) \in R$, $a\sigma(a) = 0$. So $R[[t;\sigma]]$ and $R[t;\sigma]$ are not NF.*

(3) *Let $R = \prod R_i$ be a direct product of domains. Let σ_i be a one-to-one endomorphism of R_i for each i, and let σ be the endomorphism of R given by $\sigma\big((r_i)\big) = \big(\sigma_i(r_i)\big)$. Then R is σ-rigid. So $R[[t;\sigma]]$ and $R[t;\sigma]$ are NF.*

3. Matrix rings

The question here is when a matrix ring is NF. This is a natural approach in order to find non-reduced NF rings. Since matrix rings over fine rings are fine (see [1]), it follows at once that

Theorem 3.1. *Matrix rings over fine rings are NF.*

Since division rings are fine rings we also get

Corollary 3.2. *Matrix rings over division rings are NF.*

First recall that every nilpotent matrix over a field is similar to a block diagonal matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes). Actually, this form is a special case of the Jordan canonical form for matrices. A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix. When $n = 1$, $S = 0$.

We were not able to find a reference for the following

Theorem 3.3. *The following are equivalent for a ring R:*

(1) *Every nilpotent matrix over R is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).*

(2) *R is a division ring.*

Proof. (1) $\Rightarrow$ (2). Let $0 \neq a \in R$. By hypothesis, the nilpotent aE_{12} in $\mathbb{M}_2(R)$ is similar to a block diagonal matrix with each block a shift matrix. It follows that aE_{12} must be similar to E_{12}. So there exists an invertible matrix $U = (u_{ij}) \in \mathbb{M}_2(R)$ such that $U(aE_{12}) = E_{12}U$. It follows that $u_{21} = 0$ and $u_{22} = u_{11}a$. Thus, $U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{11}a \end{bmatrix}$. Let $V = (v_{ij})$ be the inverse of U. From $UV = I_2 = VU$, it follows that

$$u_{11}av_{22} = 1,\ \ v_{11}u_{11} = 1,\ \ v_{21}u_{11} = 0 \ \text{ and } \ v_{21}u_{12} + v_{22}u_{11}a = 1.$$

Thus, u_{11} is a unit, which gives $v_{21} = 0$. So v_{22} is also a unit, and hence $a = u_{11}^{-1}v_{22}^{-1}$ is a unit. Therefore, R is a division ring.

(2) ⇒ (1). This can be proved by induction on the size of the nilpotent matrix, using the argument in Case 3 of the proof of [13, Theorem 5]. □

For a direct proof of the previous corollary, since nilpotents and fine elements of rings are invariant under conjugations, it suffices to show

Lemma 3.4. *Let S_k denote the shift matrix of size k over R.*

(1) *If $n \geq 2$, then $S_n \in \mathbb{M}_n(R)$ is fine.*
(2) *If $A = \begin{bmatrix} S_k & 0 \\ 0 & \mathbf{0} \end{bmatrix} \in \mathbb{M}_n(R)$ with $n > k > 1$, then A is fine.*

Proof. (1) In $\mathbb{M}_n(R)$, $S_n = -E_{n1} + (S_n + E_{n1})$ is a sum of a nilpotent and a unit.

(2) We see that $A = \begin{bmatrix} \mathbf{0} & \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -1 & \mathbf{0} \end{bmatrix} \\ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -1 & \mathbf{0} \end{bmatrix} & -S_{n-k} \end{bmatrix} + (S_n + E_{n1})$ is a sum of a nilpotent and a unit. □

Below is a partial answer to the question of when a matrix ring is NF.

Theorem 3.5. *If $\mathbb{M}_n(R)$ is NF ($n \geq 2$), then R is simple.*

Proof. Let $S = \mathbb{M}_n(R)$. If K is a proper ideal of R, then $KE_{1n} \subseteq \mathbb{M}_n(K) \cap \text{nil}(S)$. So $KE_{1n} = 0$ by Lemma 2.1, i.e., $K = 0$. □

We do not know if R simple implies that $\mathbb{M}_n(R)$ is NF.

Corollary 3.6. *Let R be a commutative ring and $n \geq 2$. Then $\mathbb{M}_n(R)$ is NF iff R is a field.*

Remarks 3.7. (1) The NF property does not pass to subrings: for a division ring D, $\mathbb{M}_n(D)$ ($n \geq 2$) is NF, but $\mathbb{T}_n(D)$ is not NF.

(2) The NF property does not pass to quotient rings: the ring $\mathbb{Z}$ is NF, but $\mathbb{Z}_4$ is not NF. However, if R is NF and nilpotents lift modulo the ideal I, then R/I is NF.

(3) The NF property does not pass from R to $R/J(R)$: let $R := \mathbb{Z}_{(3)}$ be the localization of $\mathbb{Z}$ at the prime ideal $3\mathbb{Z}$, and let Q be the ring of quaternions over R. That is, Q

is the algebra over R with canonical R-basis $\{1, i, j, k\}$, in which the multiplication is R-bilinear and is subject to $i^2 = j^2 = k^2 = -1$ and $ijk = -1$. Then $T := Q \oplus Q$ is a reduced ring, so is NF. By [12, Example 2.3], $Q/J(Q) \cong \mathbb{M}_2(\mathbb{Z}_3)$, so $T/J(T) \cong Q/J(Q) \oplus Q/J(Q) \cong \mathbb{M}_2(\mathbb{Z}_3) \oplus \mathbb{M}_2(\mathbb{Z}_3)$, which is not NF by Proposition 2.3.

(4) The NF property does not pass to matrix rings: the ring $\mathbb{Z}$ is NF but $\mathbb{M}_2(\mathbb{Z})$ is not NF.

Corollary 3.8. *Left R be a semiperfect ring with $J(R)$ nil (e.g., R is a one-sided perfect ring). Then R is NF iff R is the matrix ring over a division ring or a direct sum of division rings.*

Proof. ($\Leftarrow$) The implication is clear.

($\Rightarrow$) Since $J(R)$ is nil, $J(R) = 0$ by Lemma 2.1. So $R = R_1 \oplus \cdots \oplus R_n$ where each R_i is the matrix ring over a division ring. If $n \geq 2$, then R must be reduced by Proposition 2.3, and so R is a direct sum of division rings. □

A ring R is said to be of *bounded index* (*of nilpotence*) if there is a positive integer n such that $a^n = 0$ for all nilpotent elements a of R. The least such integer is called the *index* of R. A ring R is called *potent* if idempotents lift modulo $J(R)$ and every one-sided ideal not contained in $J(R)$ contains a nonzero idempotent. If R is potent, then every nonzero one-sided ideal of $R/J(R)$ contains a nonzero idempotent.

Proposition 3.9. *Let R be a potent ring of bounded index. Then R is NF iff R is a reduced ring or a matrix ring over a division ring.*

Proof. ($\Leftarrow$) The implications is clear.

($\Rightarrow$) Let R be of bounded index n. We may assume that R is not reduced. Then $n > 1$, and there exists $a \in R$ such that $a^n = 0$ and $a^{n-1} \neq 0$. By Lemma 2.1, $a^{n-1} \notin J(R)$, so, in $\overline{R} := R/J(R)$, $\bar{a}^{n-1} \neq 0$ and $\bar{a}^n = 0$. Since R is potent, every nonzero one-sided ideal of $\overline{R}$ contains a nonzero idempotent. Therefore, by [8, Theorem 2.1], there exists $\bar{e}^2 = \bar{e} \in \overline{R}\bar{a}\overline{R}$ such that $\bar{e}\overline{R}\bar{e} \cong \mathbb{M}_n(S)$ where S is a non-trivial ring. Since idempotents lift modulo $J(R)$, we may assume that $e^2 = e$. Thus, $eRe/J(eRe) \cong \bar{e}\overline{R}\bar{e} \cong \mathbb{M}_n(S)$. By [11, Corollary 6], idempotents in $eRe/J(eRe)$ can be lifted to idempotents in eRe, so every complete system of matrix units in $eRe/J(eRe)$ can be lifted to a complete system of matrix units in eRe. Thus $eRe = \mathbb{M}_n(T)$

where T is a non-trivial ring. So, $e = 1$ by Proposition 2.3, and hence $R = \mathbb{M}_n(T)$. Since $\mathbb{M}_n(T)$ is NF, T is simple by Theorem 3.5. Since R is potent, T is potent by [10, Corollary 1.7]. Since R is of bounded index n, for any nonzero idempotent t of T, $e = tI_n$ is an idempotent of R such that $eRe = \mathbb{M}_n(tTt)$, so e is central by [6, Lemma 6.10], and hence t is central. Hence T is a division ring. □

4. Fine nilpotent elements

What can be said about fine nilpotent elements in a non-NF ring? Here the fine nilpotent elements in the 2×2 matrix ring over a GCD domain are characterized. An integral domain is a *GCD* domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$. GCD domains include unique factorization domains, Bézout domains and valuation domains. To simplify the writing, equalities below are used modulo association (in divisibility). For example, $a = b$ means $b = ua$ for a unit u.

Lemma 4.1 lists some well-known properties of a GCD domain.

Lemma 4.1. *Let R be a GCD domain with $a, b, c \in R$.*

(1) $\gcd(ab, ac) = a \gcd(b, c)$.
(2) *If* $\gcd(a, b) = 1$ *and* $\gcd(a, c) = 1$*, then* $\gcd(a, bc) = 1$.
(3) *If* $\gcd(a, b) = 1$ *and* $a \mid bc$*, then* $a \mid c$.

Lemma 4.2. *Let R be a GCD domain and $b, c \in R$.*

(1) $\gcd(b, c) = 1$ *implies* $\gcd(b^n, c) = 1$ *for any* $n \geq 1$.
(2) *Let* $\gcd(b, c) = 1$*. If bc is a square, so are (up to uits) both b and c.*

Proof. (1) This follows from Lemma 4.1(2).

(2) Let $a^2 = bc$. Denote $b_1 = \gcd(b, a)$ and $c_1 = \gcd(c, a)$. Then $b = b_1 b_2$, $c = c_1 c_2$ and $a = b_1 x = c_1 y$ for some $b_2, c_2, x, y \in R$ with $\gcd(b_2, x) = 1 = \gcd(c_2, y)$. Since $\gcd(b, c) = 1$, it follows that $\gcd(b_i, c_i) = 1$, $i \in \{1, 2\}$.

From $a^2 = bc$ we get $b_1 c_1 xy = b_1 b_2 c_1 c_2$, whence $xy = b_2 c_2$. Using this as $x \mid b_2 c_2$ together with $\gcd(b_2, x) = 1$, we obtain $x \mid c_2$. Analogously we derive $y \mid b_2$ and conversely $b_2 \mid y$ and $c_2 \mid x$. Hence $x = c_2$, $y = b_2$.

Finally $b_1c_2 = a = b_2c_1$ used as in the previous two lines gives (together with $\gcd(b_i, c_i) = 1$, $i \in \{1, 2\}$) $b_1 = b_2$ and $c_1 = c_2$, as desired. □

For a square matrix A over a commutative ring R, the determinant and trace of A are denoted by $\det(A)$ and $\mathrm{tr}(A)$, respectively. Notice that a nilpotent 2×2 matrix over an integral domain R is of form $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ with $\alpha^2 + \beta\gamma = 0$. Indeed, let Q be the field of fractions of R. Then in $\mathbb{M}_2(Q)$, B is similar to qE_{12} for some $q \in Q$. So $\mathrm{tr}(B) = 0$ and $\det(B) = 0$.

Proposition 4.3. *Every nonzero nilpotent 2×2 matrix over a Bézout domain R is similar to rE_{12}, for some $r \in R$.*

Proof. Take $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and $x^2 + yz = 0$. We will construct an invertible matrix $U = (u_{ij})$ such that $TU = U(rE_{12})$ with a suitable $r \in R$.

Let $d = \gcd(x, y)$ and denote $x = dx_1$, $y = dy_1$ with $\gcd(x_1, y_1) = 1$. Then $d^2x_1^2 = -dy_1z$ and since $\gcd(x_1, y_1) = 1$ implies $\gcd(x_1^2, y_1) = 1$, it follows y_1 divides d. Set $d = y_1y_2$ and so $T = \begin{bmatrix} x_1y_1y_2 & y_1^2y_2 \\ -x_1^2y_2 & -x_1y_1y_2 \end{bmatrix} = y_2 \begin{bmatrix} x_1y_1 & y_1^2 \\ -x_1^2 & -x_1y_1 \end{bmatrix} = y_2T'$.

Since $\gcd(x_1, y_1) = 1$ there exist $s, t \in R$ such that $sx_1 + ty_1 = 1$. Take $U = \begin{bmatrix} y_1 & s \\ -x_1 & t \end{bmatrix}$ which is invertible (indeed, $U^{-1} = \begin{bmatrix} t & -s \\ x_1 & y_1 \end{bmatrix}$). One can check $T'U = \begin{bmatrix} 0 & y_1 \\ 0 & -x_1 \end{bmatrix} = UE_{12}$, so $r = y_2$. □

Fine nilpotent 2×2 matrices over GCD domains have a specific form.

Theorem 4.4. *Let R be a Bézout domain. A nilpotent matrix $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathbb{M}_2(R)$ with $a^2 + bc = 0$ is fine iff $b = \pm p^2$, $c = \mp q^2$ with coprime $p, q \in R$.*

Proof. We discuss $\det(\begin{bmatrix} a & b \\ c & -a \end{bmatrix} - \begin{bmatrix} s & x \\ y & -s \end{bmatrix}) \in U(R)$ for $s^2 + xy = 0$ and $a^2 + bc = 0$. That is $(a - s)^2 + (b - x)(c - y) \in U(R)$. Equivalently, $a^2 - 2as + s^2 + bc + xy - cx - by = -(2as + cx + by) \in U(R)$. This linear equation has solutions iff $2a, b, c$ are (collectively) coprime. Since $a^2 + bc = 0$, this is equivalent to coprime b, c and finally (using Lemma 4.2 (2)) $b = \pm p^2$, $c = \mp q^2$ with coprime p, q (and so $a = \pm pq$).

Conversely, suppose $up + vq = 1$ for some integers u, v. Then $u^2p^2 + 2uvpq + v^2q^2 = 1$ and so $s = uv$, $x = -v^2$, $y = u^2$ is a solution for the linear equation above. More, it satisfies also

$s^2+xy=0$, as desired. That is, one fine decomposition is $\begin{bmatrix} pq & p^2 \\ -q^2 & -pq \end{bmatrix} = \begin{bmatrix} uv & -v^2 \\ u^2 & -uv \end{bmatrix} +$ $\begin{bmatrix} pq-uv & p^2+v^2 \\ -q^2-u^2 & -pq+uv \end{bmatrix}$ (the determinant of the last matrix is $(pu+qv)^2=1$). □

Remark. In the previous statement and proof, instead of $b=\pm p^2$, $c=\mp q^2$, it should be written $b=up^2$, $c=-u^{-1}q^2$ for some unit $u\in U(R)$. To simplify the writting we took $u=1$

Example 4.5. *For $b=4$, $c=-1$, $a=2$, that is $A=\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$, the linear Diophantine is $4s-x+4y=\pm 1$, with obvious solution $s=y=0$, $x=\pm 1$ (which verifies also $s^2+xy=0$). Indeed, $A=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$.*

The solution given by the proof of the previous proposition ($p=2$, $q=1$) is $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}+$ $\begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$. As noticed in [2], nilpotents are not uniquely fine in $\mathbb{M}_2(\mathbb{Z})$.

Examples 4.6. (1) *The product of two fine nilpotents need not be fine: in $\mathbb{M}_2(\mathbb{Z})$, $A:=\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ is a fine nilpotent, and $B:=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is a fine nilpotent. Here $AB=\begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix}$. Assume AB is fine. Then $AB=C+U$ where $C=\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2+bc=0$ and $U=\begin{bmatrix} 4-a & -b \\ -2-c & a \end{bmatrix}$ is a unit. So $\pm 1=\det(U)=(4-a)a-b(2+c)=4a-a^2-2b-bc=2(2a-b)$, showing that 2 divides 1. This is a contradiction.*

(2) *The power of a fine nilpotent (is nilpotent but) need not be fine: in $\mathbb{M}_2(\mathbb{Z}_4)$, $A:=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a fine nilpotent, since $A^4=0_2$. However $A^2=\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\in J(\mathbb{M}_2(\mathbb{Z}_4))$, so A^2 is not fine. Would A^2 be fine, from a fine decomposition $A^2=U+T$, with unit U and nilpotent T, we get $T=-U+A^2\in -U+J(\mathbb{M}_2(\mathbb{Z}_4))\subseteq U(\mathbb{M}_2(\mathbb{Z}_4))$, a contradiction.*

Over integral domains, an attempt to find a 3×3 fine nilpotent of index 3, whose square is not fine is hard work without computer aid.

Example 4.7. *Consider the fine nilpotent matrix*

$$T=\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}=\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix},$$

whose square is $T^2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. *Then* T^2 *is not fine over any integral domain* D *such that* 2 is not a unit.

Indeed, for any matrix $B = (b_{ij})$, $1 \le i, j \le 3$ *with* $\det(B) = 0$ *(in particular, nilpotent),* $\det(B - T^2) = \det(B) + 2\det \begin{bmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{bmatrix} - 2\det \begin{bmatrix} b_{11} & b_{13} \\ b_{21} & b_{23} \end{bmatrix} \in 2D$, *so* $B - T^2$ *cannot be a unit.*

For fine elements, examples were given in [1], which show that in general $eRe \cap \Phi(R) \nsubseteq \Phi(eRe)$ for full idempotents $e \in R$. Namely, taking $R = \mathbb{M}_3(\mathbb{Z})$ and $e = \operatorname{diag}(1, 1, 0) \in R$, $S := eRe$ was identified with $\mathbb{M}_2(\mathbb{Z})$ (which corresponds to the "2×2 northwest corner" of $\mathbb{M}_3(\mathbb{Z})$) and a 2×2 matrix A which is *not* fine was mentioned, such that the block 3×3 matrix $B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$ is fine.

Hence, for fine nilpotent matrices this is not possible. Therefore, Proposition 4.9 below is encouraging in order to search for a positive answer for

Question 4.8. *If* R *is an NF ring and* $e \in R$ *is a full idempotent, is the corner ring* eRe *necessarily an NF ring?*

Proposition 4.9. *An integral* 3×3 *matrix* $A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$ *is fine nilpotent iff the* 2×2 *matrix* B *is fine nilpotent iff* B *satisfies the characterization in Theorem 4.4.*

Proof. First observe that, by block multiplication, $A^2 = \begin{bmatrix} B^2 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$, so $A^2 = 0$ iff $B^2 = 0$, that is, A is nilpotent (of index 2) iff B is nilpotent.

Suppose, B is a fine 2×2 nilpotent. By Theorem 4.4, say $B = \begin{bmatrix} pq & p^2 \\ -q^2 & -pq \end{bmatrix}$ for some (positive) coprime p, q (the other $\pm$ cases are analogous). Hence $up + vq = 1$ for some integers u, v. Clearly, $|u|\, p - |v|\, q = \pm 1$. Then

$$\begin{aligned} A &= \begin{bmatrix} pq & p^2 & 0 \\ -q^2 & -pq & 0 \\ 0 & 0 & 0 \end{bmatrix} =: T + U \\ &= \begin{bmatrix} -|v| & -|v| & -|v| \\ |u| & |u| & |u| \\ |v| & 2|v| - |u| & |v| - |u| \end{bmatrix} + \begin{bmatrix} pq + |v| & p^2 + |v| & |v| \\ -q^2 - |u| & -pq - |u| & -|u| \\ -|v| & |u| - 2|v| & |u| - |v| \end{bmatrix} \end{aligned}$$

is a fine decomposition for A (the LHS matrix is an index 3 nilpotent and the RHS matrix has $\det = 1$). The computations:

$\det(U) =$

$$(|u|-|v|)\begin{vmatrix} pq+|v| & p^2+|v| \\ -q^2-|u| & -pq-|u| \end{vmatrix} - (|u|-2|v|)\begin{vmatrix} pq+|v| & |v| \\ -q^2-|u| & -|u| \end{vmatrix} - |v|\begin{vmatrix} p^2+|v| & |v| \\ -pq-|u| & -|u| \end{vmatrix}$$

$$= (|u|-|v|)(p-q)(|u|p-|v|q) + (|u|-2|v|)q(|u|p-|v|q) + |v|p(|u|p-|v|q)$$

$$= [(|u|-|v|)(p-q) + (|u|-2|v|)q + |v|p](|u|p-|v|q) = (|u|p-|v|q)^2 = 1.$$

As for T, we have $\det(T) = \mathrm{tr}(T) = \mathrm{tr}(T^2) = 0$, so by Cayley-Hamilton's theorem, $T^3 = 0$.

Conversely, suppose $A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$ is a fine nilpotent. Since $A^3 = \begin{bmatrix} B^3 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} = 0$ then $B^3 = 0$ and so (over $\mathbb{Z}$) $B^2 = 0$ (and so $B = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2+bc = 0$). Hence $A^2 = 0$. Now let $A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} C & \alpha \\ \beta & -\mathrm{tr}(C) \end{bmatrix} + \begin{bmatrix} B-C & -\alpha \\ -\beta & \mathrm{tr}(C) \end{bmatrix}$ a (block) fine decomposition (i.e. $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $\beta = \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix}$) with nilpotent $\begin{bmatrix} C & \alpha \\ \beta & -\mathrm{tr}(C) \end{bmatrix}$ and unit $\begin{bmatrix} B-C & -\alpha \\ -\beta & \mathrm{tr}(C) \end{bmatrix}$. Then

$$\det\begin{bmatrix} C & \alpha \\ \beta & -\mathrm{tr}(C) \end{bmatrix} = 0 = \mathrm{tr}(\begin{bmatrix} C & \alpha \\ \beta & -\mathrm{tr}(C) \end{bmatrix}^2) = \mathrm{tr}(C^2) + \mathrm{tr}^2(C) + \beta\alpha, \quad \text{and}$$

$$\det\begin{bmatrix} B-C & -\alpha \\ -\beta & \mathrm{tr}(C) \end{bmatrix} = \pm 1.$$

We will show that the entries of B on the secondary diagonal (i.e., b, c above) are coprime squares of different sign. As already seen in the proof of Theorem 4.4, since $a^2+bc = 0$, it suffices to show that $\gcd(a,b,c) = 1$.

If $C = [c_{ij}], 1 \leq i,j \leq 2$ then

$$\begin{aligned} \det\begin{bmatrix} C & \alpha \\ \beta & -\mathrm{tr}(C) \end{bmatrix} &= -\det(C)\mathrm{tr}(C) + \beta_1\begin{vmatrix} c_{12} & \alpha_1 \\ c_{22} & \alpha_2 \end{vmatrix} - \beta_2\begin{vmatrix} c_{11} & \alpha_1 \\ c_{21} & \alpha_2 \end{vmatrix} \\ &= -\det(C)\mathrm{tr}(C) + \begin{bmatrix} \beta\begin{bmatrix} c_{12} \\ -c_{11} \end{bmatrix} & \beta\begin{bmatrix} -c_{22} \\ c_{21} \end{bmatrix} \end{bmatrix}\alpha \\ &= -\det(C)\mathrm{tr}(C) + \beta C^*\alpha = 0 \quad \text{with } C^* \\ &= \begin{bmatrix} -c_{22} & c_{12} \\ c_{21} & -c_{11} \end{bmatrix}. \end{aligned}$$

Similarly, $\det(B-C)\mathrm{tr}(C) + \beta(B^*-C^*)\alpha = \pm 1$ (here $(B-C)^* = B^* - C^*$ with $B^* = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$). Replacing $\beta C^*\alpha = \det(C)\mathrm{tr}(C)$ in the last equality yields

$$[\det(B-C) - \det(C)]\mathrm{tr}(C) + \beta B^*\alpha = \pm 1$$

which finally can be written

$$[-c_{11}^2 + c_{22}^2 - \alpha_1\beta_2 + \alpha_2\beta_1]a + [c_{21}\mathrm{tr}(C) + \alpha_1\beta_1]b + [c_{12}\mathrm{tr}(C) + \alpha_2\beta_2]c = \pm 1,$$

showing that a, b, c are indeed collectively coprime. □

Remark 4.10. When one makes an attempt to prove that *corners of NF rings are NF*, the start is considering $t \in \text{nil}(eRe)$, for some (full) idempotent $e \in R$. Since $\text{nil}(eRe) \subseteq \text{nil}(R)$, by hypothesis there are $t' \in \text{nil}(R)$ and $u \in U(R)$ such that $t = t' + u$. By multiplication, $t = ete = et'e + eue$. However, $et'e$ may not be nilpotent in eRe (as seen below, it is a unit) and eue may not be a unit in eRe (as seen below, it is nilpotent):

Taking $R = \mathbb{M}_3(\mathbb{Z})$ and $e \in R$ to be the full idempotent $\text{diag}\,(1,1,0)$, $S := eR\,e$ is identified with $\mathbb{M}_2(\mathbb{Z})$. Now $T' = \begin{bmatrix} -\mathbf{1} & -\mathbf{1} & -1 \\ -\mathbf{1} & \mathbf{0} & -1 \\ 1 & 1 & 1 \end{bmatrix}$ is an index 3 nilpotent with unit 2×2 N-W corner and $U = \begin{bmatrix} -\mathbf{1} & -\mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a unit with nilpotent 2×2 N-W corner.

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