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Abelian groups with left morphic endomorphism ring

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The main purpose of this paper is to determine as much as possible, large classes of Abelian groups which have left morphic endomorphism ring. We show that morphic Abelian groups and Abelian groups whose endomorphism ring is left (or right) morphic coincide in the torsion, divisible, torsion-free and splitting mixed cases. In some cases, we obtain more general results concerning image-projective (or image-injective) Abelian groups.

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1. Introduction

An endomorphism α of a module ${}_RM$ is called *morphic* if $M/\mathrm{im}\alpha \cong \ker\alpha$, that is, if the dual of the Noether isomorphism theorem holds for α. The module ${}_RM$ is called *morphic* if every endomorphism is morphic. A ring R is called *left morphic* (see [7]) if ${}_RR$ is a morphic module, that is, for every right multiplication $\alpha = \cdot a :_R R \to_R R$ by any element $a \in R$, the condition $R/Ra \cong l(a)$, where $l(a)$ denotes the left annihilator, holds.

While mostly all morphic Abelian groups were determined in [2], the determination of the Abelian groups whose endomorphism ring is left morphic was postponed to a future paper. This is what we do here.

For almost all tractable classes of Abelian groups, that is, divisible groups, torsion groups, torsion-free groups and splitting mixed groups, respectively, we show that morphic Abelian groups and Abelian groups with left morphic endomorphism ring coincide.

Moreover, a comprehensive information is given in the remaining nonsplitting case and with respect to the divisible part decomposition.

The word group will be used to designate an Abelian group G. $T(G)$ is its torsion subgroup, G_p its p-primary component for each prime p and $D(G)$ denotes the maximal divisible subgroup of G. For a given prime p, a p-group is called *homocyclic* if it is a direct sum of isomorphic cyclic p-groups. For unexplained terminology, we refer to Fuchs [3].

2. Image-Projective

A module ${}_RM$ was called *image-projective* if whenever $\gamma(M) \subseteq \alpha(M)$ for $\alpha, \gamma \in E = \mathrm{End}_R(M)$ then $\gamma \in \alpha E$, that is, if an endomorphism δ exists in the following diagram, when endomorphisms α and γ are given.

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \delta & \downarrow \gamma & & \\ M & \xrightarrow{\alpha} & \alpha(M) & \longrightarrow & 0. \end{array}$$

This property was introduced in [8], in order to compare morphic modules to modules with left morphic endomorphism ring. In the same context, a module M is *kernel-direct* if $\ker(\alpha)$ is a direct summand of M, for every $\alpha \in E$.

From [8], we note the following chart: for any module M

$$\begin{array}{ccccc} E \text{ unit-regular} & \Rightarrow & & E \text{ left morphic} & \Rightarrow E \text{ DF} \\ \Downarrow & & & \Downarrow & \\ E \text{ regular} & \Rightarrow M \text{ ker-direct} & \Rightarrow & M \text{ image-projective,} & \end{array}$$

where DF stands for Dedekind finite.

Next, recall that a module M is *quasi-projective* if for every morphism γ there exists an endomorphism δ making commutative the diagram

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \delta & \downarrow \gamma & & \\ M & \xrightarrow{p_N} & M/N & \longrightarrow & 0, \end{array}$$

where p_N denotes the canonical projection. Clearly, quasi-projective modules are image-projective.

Quasi-projective Abelian groups were determined by Fuchs and Rangaswamy in [4]: these are *either free or else torsion groups with homocyclic p-components*. Actually, for Abelian groups, projective $\equiv$ free $\Rightarrow$ quasi-projective $\Rightarrow$ image-projective. $\mathbf{Z}(p^\infty)$ is not image-projective and any group whose *endomorphism ring is a field* is image-projective. Such groups were characterized by Szele: $\mathbf{Q}$ or $\mathbf{Z}(p)$ for some prime p.

The characterization of quasi-projective Abelian groups mentioned above, uses six lemmas which are stated for modules. Similar results can be proved for image-projective modules. We state three of these without proofs below and add a fourth which is [8, Lemma 2.16].

Lemma 1. (1) *Every direct summand of an image-projective module is image-projective.*

(2) *If* M_i $(i \in I)$ *are image-projective and* $\mathrm{Hom}(M_i, M_j) = 0$ *for* $i \neq j$, *then* $M = \bigoplus M_i$ *is also image-projective.*
(3) *For an image-projective module* M, *let* $\alpha \in \mathrm{End}_R(M)$ *be such that* $\alpha(M)$ *is isomorphic to a direct summand of* M. *Then* $\ker(\alpha)$ *is a summand of* M.
(4) *If* $M \oplus N$ *is image-projective and* $\alpha : M \to N$ *is an epimorphism, then* $\ker(\alpha)$ *is a direct summand of* M.

Proposition 2. *If* G *is an image-projective group then*

(1) *The divisible part of* G *is torsion-free*;
(2) *The torsion part of* G *is quasi-projective*;
(3) *If* p *is a prime such that the* p*-component* G_p *of* G *is nonzero then* $G/T(G)$ *is* p*-divisible.*

Proof. (1) If p is a prime, then the multiplication by p induces an epimorphism of $\mathbf{Z}(p^\infty)$. Therefore, its kernel has to be a direct summand, and this is not true. It follows that for every prime p the group $\mathbf{Z}(p^\infty)$ is not image-projective. Therefore, the divisible part of every image-projective group is torsion-free.

(2) Let p be a prime, and suppose that G_p is not quasi-projective. Therefore, G has a direct summand isomorphic to $\mathbf{Z}(p^n) \oplus \mathbf{Z}(p^m)$ for some positive integers m, n with $n < m$. This direct summand cannot be image-projective since there exists an epimorphism $\mathbf{Z}(p^m) \to \mathbf{Z}(p^n)$ whose kernel is not a direct summand.

(3) Let p be a prime such that $G_p \neq 0$. By (2), the p-component G_p is bounded, hence, it is a direct summand of G, and so we have a decomposition $G = G_p \oplus H$. Suppose that H is not p-divisible. It follows that for every positive integer n the group H/p^nH has a direct summand of order p^n. Therefore, if $\langle g \rangle$ is a cyclic direct summand of G_p, then it is a direct summand of G and there exists an epimorphism $H \to \langle g \rangle$. The kernel of this epimorphism has to be a direct summand of H, which is not possible since the p-component of H is zero. □

Thus, we obtain the following characterizations for some classes of image-projective groups.

Corollary 3. (1) *A divisible group is image-projective iff it is torsion-free, i.e. a direct sum of* $\mathbf{Q}$.
(2) *A torsion group is image-projective iff it is quasi-projective.*

Proof. (1) Suppose G is torsion-free divisible. Since images of endomorphisms α of G are also torsion-free, these have pure kernels, so these are also divisible, and so direct summands. Hence, G is ker-direct and so, image-projective.

(2) Obvious. □

3. Left Morphic Endomorphism Ring

Recall that an element $a \in R$ *is left morphic* iff $Ra = l(b)$ and $l(a) = Rb$ for some $b \in R$.

It is easy to determine the divisible and torsion-free groups which have left (or right) endomorphism ring, independently of the results obtained on image-projective groups in the previous section.

First, it is easy to check the following.

Proposition 4. *Suppose $a \in R$ is left morphic. Then the following conditions are equivalent*

(i) *a is left cancellable (or $l(a) = 0$),*
(ii) *a is right cancellable (or $r(a) = 0$),*
(iii) *a has a left inverse (or $Ra = R$),*
(iv) *a is a unit.*

This elementary result (and its left-right symmetry) already produce useful consequences.

Corollary 5. *The only left (or right) morphic domains are the fields.*

Corollary 6. *Let ${}_RM$ be a module, $E = \mathrm{End}_R(M)$ and let $\mu \in E$. If μ is left (or right) morphic, then μ is surjective iff μ is injective.*

Proof. We write *composition* of endomorphisms *in the arrows order.*

If μ is surjective, then μ is left cancellable. By the proposition, it is a unit and so injective.

Conversely, if μ is injective, then μ is right cancellable and again by the proposition it is a unit and so surjective.

A right-left symmetry applies for right morphic endomorphisms. □

Corollary 7. *Let ${}_RM$ be a module, and $E = \mathrm{End}_R(M)$. If E is right (or left) morphic, then M is Hopfian and co-Hopfian.*

For a survey on Hopfian, co-Hopfian and Dedekind modules (with special attention on Abelian groups) see also [1].

Corollary 8. *Let G be a group with right (or left) morphic $E = \mathrm{End}(G)$. Then G is divisible iff G is torsion-free.*

Proof. For any positive integer n, denote $\mu_n \in E = \mathrm{End}(G)$ the multiplication by n. As it is well known, G is divisible (or torsion-free) iff μ_n is surjective (respectively injective). □

Next, using Proposition 2, we determine all the classes of groups (i.e. torsion, torsion-free, divisible and splitting mixed) mentioned in the Introduction, which have left morphic endomorphism ring as consequences of the following.

Theorem 9. *Let G be an Abelian group such that $E = \mathrm{End}(G)$ is left morphic. Then:*

(1) *The divisible part $D(G)$ of G is torsion-free;*

(2) *The torsion part of G is quasi-projective and every p-component of G is finite*;
(3) *$G/T(G)$ is divisible*;
(4) *If $D(G) \neq 0$, then $G/D(G)$ is torsion and quasi-projective.*

Proof. (1) This a consequence of Proposition 2.

(2) Using Proposition 2, we obtain that every p-component of G is quasi-projective, hence, we have an isomorphism $G = G_p \oplus H$ such that $H_p = 0$. It follows that G_p has to be co-Hopfian, and this is possible only if G_p is finite.

(3) We use Proposition 2 to obtain that $G/T(G)$ is divisible for every prime p such that G_p is nonzero. If p is a prime with $G_p = 0$, then the multiplication by p induces a monomorphism $\mu_p : G \to G$. Using Corollary 7, it follows that μ_p is an isomorphism, hence G is p-divisible.

(4) Write $G = R \oplus D(G)$ with R reduced and $D(G)$ divisible. Then $\mathrm{End}(R \oplus D(G))$ can be viewed as matrix ring

$$\begin{bmatrix} \mathrm{End}(R) & \mathrm{Hom}(R, D(G)) \\ \mathrm{Hom}(D(G), R) & \mathrm{End}(D(G)) \end{bmatrix}$$

(again, we view the endomorphisms acting on the left, so the composition is written in the arrows order). Since $\mathrm{Hom}(\mathrm{D(G)}, R) = 0$, it follows from [7, Proposition 18] that $\mathrm{Hom}(R, \mathbf{Q}) = 0$. Hence, R is torsion and so $R = T(G)$. The conclusion follows from (2). □

Corollary 10. (1) *The endomorphism ring of a torsion group G is left morphic if and only if every p-component of G is a finite direct sum of copies of a cyclic p-group $\mathbf{Z}(p^k)$.*
(2) *The endomorphism ring of nonreduced group G is left morphic if and only if $G = D(R) \oplus R$ such that the divisible part $D(G)$ of G is torsion-free of finite rank, and R is a torsion group whose endomorphism ring is left morphic.*

Corollary 11. *Morphic groups and groups with left morphic endomorphism ring coincide in the following cases*: *divisible groups*, *torsion groups*, *torsion-free groups and splitting mixed groups.*

In the remaining of this section, we give some examples for negative situations.

There are large classes of mixed groups whose endomorphism ring is not left morphic.

Since the only left morphic domains are the division rings, the endomorphism ring of $\mathbf{Z}$ is not left (nor right) morphic. Therefore, the endomorphism ring of *mixed groups which have an infinite cyclic* (*or free*) *direct summand* is **not** left (nor right) morphic. By the above theorem, *so are mixed groups which have a quasicyclic* (i.e. $\mathbf{Z}(p^\infty)$ for some prime p) *direct summand*, or, more general, for groups whose divisible part has nonzero torsion.

In some important cases, the groups whose endomorphism ring is left morphic are easy to determine because *unit regular rings are left and right morphic*, and *every left or right morphic ring is Dedekind finite*. Finally, recall that a direct product $\prod R_i$ of rings is left (or right) morphic iff each R_i is left (respectively right) morphic.

Recall (see [2]) that the usual expected reconstructions, "if both $D(G)$ and reduced R are *morphic*, then $G = D(G) \oplus R$ is *morphic*", and "if both $T(G)$ and $G/T(G)$ are *morphic*, then G is *morphic*", both fail.

In the sequel, we show that these reconstructions also fail for groups with left morphic endomorphism ring.

In one direction, since the property of having a left morphic endomorphism ring passes to direct summands, if a (mixed) group $G = D(G) \oplus R$ is not reduced and has left morphic endomorphism ring, then both $\mathrm{End}(D(G))$ and $\mathrm{End}(R)$ are left morphic (i.e. $D(G)$ is a finite direct sum of $\mathbf{Q}$). Notice that from the proof of (4), Theorem 9, we know that *if* $\mathrm{End}(G \oplus \mathbf{Q})$ *is left morphic, then either* $\mathbf{Q}$ *is a subgroup of* G *or else* G *is torsion and reduced.*

Here are the

Examples. (1) Consider (as in [2]) $G = \mathbf{Q} \oplus (\prod_p \mathbf{Z}(p))$. $\mathrm{End}(\mathbf{Q})$ is left morphic as a field, and $\mathrm{End}(\prod_p \mathbf{Z}(p))$ is unit-regular (as product of fields) and so left morphic. It is known that $\mathrm{End}(G)$ is 2-regular (for every x there is a y such that $x^2yx^2 = x^2$) but not regular (and so, nor unit-regular). $\mathrm{End}(G)$ is neither left morphic by the previous paragraph.

(2) If $G = \prod_p \mathbf{Z}(p)$, then $\mathrm{End}(G/T(G))$ is not left morphic but $\mathrm{End}(G)$ is left morphic.

Indeed, as direct product $\prod_p \mathbf{Z}_p$ of fields, the endomorphism ring of G is commutative, so G is DF, but $G/T(G)$ is not DF and so $\mathrm{End}(G/T(G))$ is not left morphic (it is infinite rank torsion-free divisible). Moreover, since every field is unit-regular, and a direct product of rings is unit-regular if and only if each component is unit-regular, $\mathrm{End}(G)$ is left morphic because $\mathrm{End}(G)$ is unit regular.

(3) Let $H = \mathbf{Q} \oplus P$ with the subgroup $P = P(G, a) = \{g \in G : ng \in \langle a \rangle$ for some positive integer $n\}$ of all elements in G that depend on $\{a\}$ (here once again $G = \prod_p \mathbf{Z}(p)$ and a is the infinite order element $(\overline{1}, \overline{1}, \ldots,))$. Then $T(H)$ and $H/T(H)$ have both left morphic endomorphism rings, but $\mathrm{End}(H)$ is not left morphic by Theorem 9.

4. Right Morphic Endomorphism Ring

We conclude this paper with a glimpse on groups whose endomorphism ring is right morphic.

A ring was called *right morphic* if R_R is morphic. An element $a \in R$ is right morphic (as the endomorphism $x \mapsto ax$) iff $aR = r(b)$ and $r(a) = bR$ for some $b \in R$ (here $r(a)$ denotes the right annihilator). A ring R is called *left P-injective* if, for each principal left ideal Ra of R, each R-morphism $Ra \to R$ extends to R; equivalently (see [5, Theorem 1]) if $lr(a) = aR$.

We call a module $_RM$ *image-injective* if R-linear maps $\beta(M) \to M$ extend to M for each $\beta \in E = \mathrm{End}(_RM)$. Note that $_RR$ is image-injective if and only if R is left P-injective. If E is right morphic, then $_RM$ is image-injective.

Also recall that *quasi-injective* groups (i.e. homomorphisms from subgroups can be extended to endomorphisms) were determined by Kil'p [6] and these are *either divisible or torsion groups with homocyclic p-components*. Clearly, quasi-injective groups are image-injective.

As already noticed for left morphic rings, unit-regular rings are right morphic, right morphic rings are DF, right morphic rings are closed under corners and the only right morphic domains are the division rings. This way, we recapture at once the analogue of Corollary 11, for divisible groups and torsion-free groups.

As customarily, the determination of the torsion groups with right morphic endomorphism ring reduces to reduced p-groups and relies on a result which is dual to **2**, Proposition 2.

Proposition 12. *If G is an image-injective group, then the torsion part of G is quasi-injective.*

Proof. The proof is similar to the one given for image-projective groups and relies on the dual of (4) Lemma 1: If $M \oplus N$ is image-injective and $\alpha : M \to N$ is an monomorphism then α splits. □

Corollary 13. *A p-group has right morphic endomorphism ring iff it is homocyclic and finite.*

Corollary 14. *Morphic groups and groups with right (or left) morphic endomorphism ring coincide in the following cases: divisible groups, torsion groups, torsion-free groups and splitting mixed groups.*

In [7], an example of left morphic ring which is not right morphic is given. It would be nice to have an example of Abelian group with left morphic but not right morphic endomorphism ring.

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