Integral idempotent 2×2 matrices

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Any 2×2 nontrivial idempotent matrix over an integral domain is of form $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$ (i.e., $\det(E) = 0$, $Tr(E) = 1$).

Over integers we can say much more. So in the sequel $a, b, c \in \mathbb{Z}$.

Start with the degree two equation $a^2 - a + bc = 0$. The discriminant is $\Delta = 1 - 4bc$ is ≥ 0 only if b, c have opposite signs and since $1 - 4bc$ is odd, we must have $1 - 4bc = (2k+1)^2$ for some integer k. This amounts to

$$bc = -k(k+1),$$

a product of 2 consecutive integers.

Further, for $bc = -k(k+1)$, $a_{1,2} = \dfrac{1 \pm (2k+1)}{2} \in \{-k, k+1\}$. Hence finally we have two possible idempotents:

$$E_1 = \begin{bmatrix} k+1 & b \\ -\dfrac{k(k+1)}{b} & -k \end{bmatrix}$$

and

$$E_2 = \begin{bmatrix} -k & b \\ -\dfrac{k(k+1)}{b} & k+1 \end{bmatrix},$$

for every integer k and every (integer) divisor b of $k(k+1)$.

That is, for generate all the nontrivial idempotent 2×2 integral matrices, one starts with an integer k, forms the product $k(k+1)$ and writes all the (integer) divisors of $k(k+1)$. This way b, c are chosen such that $bc = -k(k+1)$ and $a \in \{-k, k+1\}$, as above.