Idempotent sums of idempotents

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Exercise. *Determine the (unital) rings with only idempotent sums of two idempotents.*

That is, for any $e = e^2$ and $f = f^2$ in a ring R, $e + f$ is also an idempotent, or equivalently, $Id(R) + Id(R) \subseteq Id(R)$ if $Id(R)$ denotes the set of all the idempotents of R.

Solution. First notice that $1 + 1 = (1 + 1)^2$ yields $2 = 0$, that is, the characteristics of R must be 2.

Secondly, $e + f = (e + f)^2$ gives $ef + fe = 0$ and so $ef = fe$ (using $\mathrm{char}(R) = 2$). Therefore, in such a ring, idempotents commute.

In a ring idempotents commute iff the idempotents are central (see e.g. **22.3.A** in "Exercises in Classical Ring Theory" T. Y. Lam (1995)).

Hence, *such rings are Abelian* (central idempotents) *and of characteristics 2.*

The converse is obvious: $e+f = e+ef+ef+f = e+ef+fe+f = (e+f)^2$.

Remark. Separately, none of these two conditions is sufficient.

The characteristics of $\mathbb{M}_2(\mathbb{F}_2)$ is 2 but $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not idempotent ($\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 = I_2$).

Let R be any (unital) commutative ring of charateristics $\neq 2$. Then $1 + 1$ is not idempotent.