Idempotent von Neumann regular rings

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November 23, 2020

Abstract

We call *idempotent (von Neumann) regular* a unital ring R such that for every $a \in R$ there exists an idempotent $e \in R$ such that $a = aea$. A simple proof shows that these are precisely the Boolean rings (i.e. all elements are idempotent).

1 Introduction

Around 1935, John von Neumann discovered a (now well-known) class of rings called (von Neumann) regular rings, in connection with his work on continuous geometry and operator algebras. An element a of a ring R is said to be (*von Neumann*) *regular* if $a \in aRa$, that is, there exists $x \in R$ such that $a = axa$. The element x is (sometimes) called an "inner inverse" for a. If every $a \in R$ is (von Neumann) regular we say R is (*von Neumann*) *regular*.

To simplify the wording (and writing), in the sequel we use the word "regular" only for "von Neumann regular", elements or rings.

A simple idea of obtaining subclasses of regular rings is to require the inner inverse x above, to have some special properties.

Since $0 = 0 \cdot x \cdot 0$ for any $x \in R$, the zero element is no concern. However, the identity $1 \in R$ should be our concern.

Since regular rings were (merely) introduced for unital rings, we need an inner inverse also for $a = 1$, i.e. $1 = 1 \cdot x \cdot 1 = x$, so if we chose a (common) property for the inner inverses, it should be shared (at least) by the identity 1.

As early as 1968, Gertrude Ehrlich introduced (and further studied in 1976) the so called unit-regular elements and rings, *by requiring the inner inverses to be units*. Notice that this is possible since in any (unital) ring, 1 is a unit. That is, $a \in R$ is called *unit-regular* if $a = aua$ for some unit $u \in U(R)$ and a ring R is called *unit-regular* if all its elements are unit-regular.

Besides the units, there are two other important subsets of elements in Ring Theory: the idempotents and the nilpotents.

We can (and this is the subject of this note) define a subclass of regular rings requiring the inner inverses to be idempotents (because 1 is not only a unit, but is also idempotent), but we cannot do this requiring the inner inverses to be nilpotents (because 1 is not nilpotent for any nonzero ring).

Thus we are ready to give the following

Definition. An element a of a ring R is *idempotent-regular* (id-regular, for short) if there is an idempotent $e \in R$ such that $a = aea$, and a ring is *idempotent-regular* if so are all its elements.

Obviously, idempotents (incl. 0) are id-regular and *Boolean rings* (i.e. every element is idempotent) *are id-regular.*

In the sequel we show that *the id-rings are precisely the Boolean rings.*

2 Idempotent regular rings are Boolean

First an easy

Lemma 1 *The only id-regular unit is* 1.

Proof. Suppose $u = ueu$ with $u \in U(R)$ and $e^2 = e \in R$. Then $ue = 1 = eu$ so $e = u^{-1} = 1$ and $u = 1$. ■

Corollary 2 *If R is id-regular then $U(R) = \{1\}$ and* $\mathrm{char}(R) = 2$.

Proof. The first part follows from the lemma. Since in any unital ring, -1 is also a unit, we must have $-1 = 1$, so $2 = 0$ and $\mathrm{char}(R) = 2$. ■

In a footnote of P.M.Cohn' s 1958 paper "Rings of zero-divisors" [p.913] it is noticed: "M. P. Drazin has proved, somewhat more generally, that in any (not necessarily commutative) ring with 1 and no other invertible elements, every regular element is idempotent."

Since we were not able to find a reference, we supply a proof.

Proposition 3 *In a ring R with $U(R) = \{1\}$, every regular element is an idempotent.*

Proof. [Lam] If r is a nilpotent element, then since $1 + r$ is a unit, $r = 0$. Thus R is a reduced ring, and hence an Abelian ring. For any regular element $a = axa$, the idempotents ax, xa are central. This implies that $a = a^2x = xa^2$ and so a is strongly regular; in particular, a is unit regular. Writing $a = aua$ for some unit u (which must be 1) leads to $a = a^2$. ■

Corollary 4 *The id-regular rings are precisely the Boolean rings.*

Proof. Since an id-regular ring is regular, the statement follows from the previous corollary and proposition. ■

References

[1] P. M. Cohn *Rings of zero-divisors*. Proc. Amer. Math. Soc.**9** (1958), 914-919.