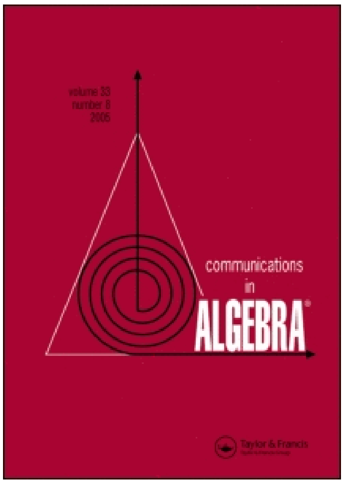

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RINGS WITH LATTICES OF IDEMPOTENTS

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The central idempotents of any ring with identity form a Boolean algebra. This result is largely extended for rings with generalized commuting idempotents.

Key Words: Boolean algebra; Generalized commuting idempotents; Principal ideals generated by idempotents.

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1. INTRODUCTION

In associating lattices to rings with identity, attention was mainly paid to right (or left) principal ideals, that is, the set $\{aR : a \in R\}$ naturally ordered by inclusion. A celebrated class of rings, the *regular* rings, was defined by John von Neumann. For such rings, every right principal ideal is (also) generated by an idempotent, and $\{aR : a \in R\} = \{eR : e \in \mathrm{Id}(R)\}$ forms a complemented distributive lattice (i.e., a Boolean algebra and/or a Boolean ring) with respect to addition and intersection of such ideals. Therefore, this is a sublattice of the (complete) lattice $I_r(R)$ of all the right ideals of R.

Since right annihilators in any ring always form a (complete) lattice, which is not generally a sublattice of $I_r(R)$, conditions were imposed in order to connect this lattice to the principal ideals: a ring is *right Rickart* if every right annihilator of any element is (a principal right ideal) generated by an idempotent, and *right Baer*, if all the right annihilators are principal right ideals generated by idempotents.

Even more special conditions can be found in the literature: a ring is an *ISS-ring* (see [3]) if for any idempotents $e, f \in R$ there exists an idempotent $g \in R$ with $eR + fR = gR$. It is proven that R has ISS and the ACC on idempotent generated right ideals if and only if the idempotent generated right ideals form a ACC lattice under addition and intersection. Or, a ring R has *AC1* (see [1]) if the set of all the right annihilators forms a sublattice of $I_r(R)$, and has *AC3* if this is a complete sublattice of $I_r(R)$. Of course, left versions of all these are available.

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However, when it comes to $(\mathrm{Id}(R), \leq)$ (the poset of all the idempotents), there seems to be no progress from another celebrated result: $B(R) = Z(R) \cap \mathrm{Id}(R)$, that is, the set of all the central idempotents, forms a Boolean algebra (and so a Boolean ring and/or a distributive complemented lattice) with respect to $\inf\{e, f\} = ef$ and $\sup\{e, f\} = e + f - ef$ (together with ring multiplication and the special addition $e \boxplus f = e + f - 2ef = (e - f)^2$).

The aim of this note is to extend this result.

In the sequel, $\mathrm{Id}(R)$ denotes the set of all the idempotent elements in a ring with identity R, and, for $a, b \in R$ we use the notation $a \leq b$ if $a = ab = ba$. It is readily checked that this (binary) relation is *transitive* and *antisymmetric* on R. However, it is *reflexive* (and so, a partial order) only when restricted to the set $\mathrm{Id}(R)$. Two idempotents e, f are *orthogonal* if $ef = fe = 0$, and *isomorphic* if $e = ab$ and $f = ba$ for suitable elements $a, b \in R$. Zero and 1 are the *trivial* (orthogonal) idempotents of any ring. For an idempotent e, $e' = 1 - e$ is the *complementary* idempotent. For two idempotents e, f, we denote by $\langle e, f\rangle_s$ the subsemigroup of the multiplicative monoid of the ring, generated by e and f.

The main results we obtain are the following ones.

Theorem. *In the poset* $(\mathrm{Id}(R), \leq)$, *two idempotents e, f have a greatest lower bound and a least upper bound with* $\inf\{e, f\} \in \langle e, f\rangle_s$ *if and only if e and f are generalized commuting idempotents.*

Proposition. *The poset* $(\mathrm{Id}(R), \leq)$ *has only generalized commuting idempotents if and only if all idempotents are central* (*i.e.*, $\mathrm{Id}(R) = B(R)$).

2. GENERALIZED COMMUTING IDEMPOTENTS

For the first part of this section, suppose R is a (multiplicatively written) semigroup. Two idempotents e, f are called *generalized commuting idempotents* if there exists a positive integer n such that $(ef)^n = (fe)^n$ or, $(ef)^n e = (fe)^n f$. It is easy to prove that there is a strict hierarchy for these conditions, namely,

$$ef = fe \Longrightarrow efe = fef \Longrightarrow \cdots \Longrightarrow (ef)^n = (fe)^n \overset{(*)}{\Longrightarrow} (ef)^n e = (fe)^n f \Longrightarrow \cdots .$$

For instance, (*) is proved as follows: by left multiplication with e, we first obtain $(ef)^n = e(fe)^n$; then by left multiplication with f, we obtain $f(ef)^n = (fe)^n$. Hence also $(ef)^n e = e(fe)^n = f(ef)^n = (fe)^n f$.

These implications cannot be reversed. As an example, for the first implication: in the 2×2 matrix ring over $\mathbf{Z}$, take $e = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$. Then $efe = fef = 0_2$, the zero matrix (and so e, f are generalized commuting idempotents), but $ef = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = fe$.

If $f' = I_2 - f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ is the complementary idempotent, then $e = ef' = ef'e = (ef')^2 = \cdots$ and $f' = f'e = f'ef' = (f'e)^2 = \cdots$. Hence e and f' are not generalized commuting idempotents.

Remarks. 1) More can be proved: if for instance $(ef)^n = (fe)^n$, then not only (as we already saw) $(ef)^n e = (fe)^n f$, but $(ef)^n = (ef)^n e = (fe)^n f = (fe)^n$, and all elements "to the right" are equal.

2) Therefore, using the above sequence, a *commuting index* can be introduced for any two idempotents: the least positive integer n, if any, such that $(ef)^n = (fe)^n$ or, $(ef)^n e = (fe)^n f$.

3) Clearly, all elements above are contained in the subsemigroup of R generated by e and f, denoted $\langle e, f\rangle_s$. Actually, since we only consider idempotents, elements in $\langle e, f\rangle_s$ are of four possible forms: $(ef)^n$, $(ef)^n e$, $(fe)^n$, and $(fe)^n f$, for some positive integer n.

Proposition 1. *Two idempotents e, f are generalized commuting if and only if $\langle ef\rangle_s \cap \langle fe\rangle_s \neq \varnothing$.*

Proof. The condition is clearly necessary. To check that it is also sufficient, suppose $(ef)^n = (fe)^m$ for positive integers n and m. If $k = \max(n, m)$, we show that $(ef)^k = (fe)^k$, and so these are generalized commuting idempotents. Indeed, if $(ef)^n = (fe)^m$, then checking $(ef)^n = (ef)^{n+1} = (ef)^{n+2} = \cdots$ and $(fe)^m = (fe)^{m+1} = (fe)^{m+2} = \cdots$ will do. Here is a sample: $(ef)^{n+1} = (ef)^n ef = (fe)^m ef = (fe)^m f = (ef)^n f = (ef)^n$.

Once again, more can be shown about the intermediate products: $(ef)^n e = (fe)^m e = (fe)^m = (ef)^n$. □

Further, idempotents in a ring R (with identity) are called *generalized commuting* whenever they are generalized commuting in the multiplicative monoid of the ring.

We first prove some technical equalities gathered in the following lemma.

Lemma 2. *For any two idempotents e, f in a ring R, the following hold*:

a) *If $s = e + f - ef - fe + efe + fef - \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (ef)^n$ then $es = s$ and $sf = f$; but*
b) *$se = e + (fe)^n - (ef)^n e$ and $fs = f + (fe)^n - f(ef)^n$*;
c) (*de Morgan*) *$s = 1 - [(1-e)(1-f)]^n$, and similarly*;
d) *$e + f - ef - fe + efe + fef - \cdots - (ef)^n - (fe)^n + (ef)^n e = 1 - [(1-e)(1-f)]^n (1-e)$.*

Proof. We will just mention the induction step ($n \to n+1$) in the proof of an equality which is equivalent to (c), that is,

$$a = [(1-e)(1-f)]^n = 1 - e - f + ef + fe - \cdots + (ef)^n$$

(clearly true for $n = 1$ or 2).

$$\begin{aligned}[(1-e)(1-f)]^{n+1} &= (1-e)(\underline{1} - \underline{f})[(1-e)(1-f)]^n \\ &= a - (1-e)fa = a - (1-e)f(1-s)\end{aligned}$$

$$\overset{(b)}{=} a - (1-e)(-(fe)^n + f(ef)^n)$$
$$= a + (fe)^n - f(ef)^n - e(fe)^n + (ef)^{n+1},$$

as desired. □

Theorem 3. *In* $(\mathrm{Id}(R), \leq)$, *generalized commuting pairs of idempotents have a greatest lower bound and a least upper bound.*

Proof. *Case* 1. We show that if $(ef)^n = (fe)^n$, then for the idempotents e, f, the greatest lower bound exists and

$$\inf\{e, f\} = (ef)^n = (fe)^n,$$

respectively, the least upper bound exists and

$$\sup\{e, f\} = e + f - ef - fe + efe + fef - \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (ef)^n.$$

First note that $(ef)^n \leq e$ holds since $e(ef)^n = (ef)^n$ is true for any idempotents and $(ef)^n e = e(fe)^n = e(ef)^n = (ef)^n$, using the hypothesis. Similarly, $(fe)^n \leq f$ and so, $(ef)^n = (fe)^n$ is a lower bound for e, f. If $a \in \mathrm{Id}(R)$ and $a \leq e$ and $a \leq f$, any product of a with any product of e and f equals a. Hence also $a = a(ef)^n = (ef)^n a$ and so $a \leq (ef)^n$, which solves the inf part.

Notice that $(ef)^n = (fe)^n$ is an idempotent: indeed,

$$(ef)^n(ef)^n = e(fe)^n f(ef)^{n-1} = e(ef)^n f(ef)^{n-1} = (ef)^n(ef)^{n-1} = \cdots = (ef)^n.$$

As for the sup, one first verifies $e \leq e + f - ef - fe + \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (ef)^n$ (half is (a) in the previous lemma, the other half needs—see (b)—the hypothesis $(ef)^n = (fe)^n$), respectively, $f \leq e + f - ef - fe + \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (fe)^n$ (similar). Therefore, (by hypothesis) the common element is an upper bound for e, f. Finally, if $a \in \mathrm{Id}(R)$ and $e \leq a$ and $f \leq a$, multiplication of any sum of products of e,f by a does not change this sum of products. Hence $e + f - ef - fe + \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (ef)^n \leq a$. Thus

$$\sup\{e, f\} = e + f - ef - fe + \cdots + e(fe)^{n-1} + f(ef)^{n-1} - (ef)^n.$$

Denote by $s = \sup\{e, f\}$. Since $e, f \leq s$ taking $a = s$ above, multiplication of any sum of products of e,f by s does not change this sum of products. Therefore, $s^2 = s$ and so s is an idempotent.

Case 2. We show that if $(ef)^n e = (fe)^n f$, then for the idempotents e, f, the greatest lower bound exists and

$$\inf\{e, f\} = (ef)^n e = (fe)^n f,$$

respectively, the least upper bound exists and

$$\sup\{e, f\} = e + f - ef - fe + efe + fef - \cdots - (ef)^n - (fe)^n + (ef)^n e.$$

The verifications are analogous. However, now the hypothesis is not necessary in order to check $e(fe)^n \leq e$, or $(fe)^n f \leq f$, respectively, $e \leq e + f - ef - fe + efe + fef - \cdots - (ef)^n - (fe)^n + (ef)^n e$, or $f \leq e + f - ef - fe + efe + fef - \cdots - (ef)^n - (fe)^n + (fe)^n f$. □

A converse of the previous theorem also holds.

Indeed, in a similar vein, some arguments in its proof may be refined and give the following proposition (we skip the details).

Proposition 4. *Let* $e, f \in \mathrm{Id}(R)$. (a) *There exists* $\inf\{e, f\} = (ef)^n$ *if and only if* $(ef)^n = (ef)^n e = (fe)^n f$. (b) *There exists* $\sup\{e, f\} = e + f - \cdots - (ef)^n$ *if and only if* $(fe)^n = (ef)^n e = (fe)^n f$. *A similar statement holds for* $\inf\{e, f\} = (ef)^n e$ *and* $\sup\{e, f\} = e + f - \cdots - (ef)^n - (fe)^n + (ef)^n e$.

Corollary 5. *Let* $e, f \in \mathrm{Id}(R)$. *Then* $\inf\{e, f\}$ *exists and equals* $(ef)^n$, *and* $\sup\{e, f\}$ *exists and equals* $e + f - \cdots - (ef)^n$ *if and only if* $(ef)^n = (fe)^n$. *A similar statement holds for* $\inf\{e, f\} = (ef)^n e$ *and* $\sup\{e, f\} = e + f - \cdots - (ef)^n - (fe)^n + (ef)^n e$.

Proposition 6. *In the poset* $(\mathrm{Id}(R), \leq)$, *for two idempotents* e, f, $\inf\{e, f\} = (ef)^n$ (*or* $(ef)^n e$) *and* $\sup\{e, f\} = e + f - \cdots - (ef)^n$ (*respectively,* $e + f - \cdots - (ef)^n - (fe)^n + (ef)^n e$), *with suitable positive integers* n *depending on the idempotents* e, f *if and only if* e *and* f *are generalized commuting idempotents.*

We can finally give the following characterization.

Theorem 7. *In the poset* $(\mathrm{Id}(R), \leq)$, *two idempotents* e, f *have a greatest lower bound and a least upper bound with* $\inf\{e, f\} \in \langle e, f\rangle_s$ *if and only if the idempotents* e *and* f *are generalized commuting.*

Proof. By the previous proposition, the condition is sufficient. To prove it is also necessary, suppose (we have listed above the only four possibilities), say, $i = \inf\{e, f\} = (ef)^n$. By the inf definition (and partial order $\leq$), $if = fi = ie = ei = i$. Therefore, $(ef)^n e = ie = i = fi = f(ef)^n$, and we have generalized commutativity. The other three cases are analogous. □

Corollary 8. *If idempotents in a ring* R *commute, then* $(\mathrm{Id}(R), \leq)$ *forms a lattice with respect to* $\inf\{e, f\} = ef$ *and* $\sup\{e, f\} = e + f - ef$. *More, this is a Boolean algebra.*

Corollary 9. *Let* e *and* f *be orthogonal idempotents in an arbitrary ring. Then there is* $\inf\{e, f\} = 0$, *and there is* $\sup\{e, f\} = e + f$.

Actually, a direct proof of this last corollary is an easy exercise.

Corollary 10. $B(R) = Z(R) \cap \mathrm{Id}(R)$ *forms a Boolean algebra.*

It should be also noticed that generalized commutativity of idempotents is not sufficient for

$$\inf\{e, f'\} = 0 \Longleftrightarrow \inf\{e, f\} = e,$$

equivalence which holds in Boolean algebras.

Passage to the complementary idempotent does not even preserve the "commuting index" (i.e., $(ef)^n = (fe)^n$ does not imply $(ef')^n = (f'e)^n$).

To check these two last claims, the example already given beginning of this section can be used.

3. RINGS WITH ONLY GENERALIZED IDEMPOTENTS ARE ABELIAN

It it well-known (and somehow surprising) that if (and only if) idempotents in a ring R commute, then the ring is Abelian (i.e., all idempotents are central). As we will see below, even less is sufficient in order to have $B(R) = \mathrm{Id}(R)$.

Now, since generally

$$e, f \in Z(R) \Longrightarrow e, f \text{ commute } \Longrightarrow e, f \text{ generalized commute},$$

but examples (e.g., our example in the previous section) show that none of these implications can be reversed, one would expect a ring having *only* generalized idempotents to have a lattice of idempotents which should be a genuine generalization of the Boolean algebra $B(R)$.

In the next result we show that (unfortunately) it is not.

Proposition 11. *For a ring R the following conditions are equivalent*:

(a) *R is Abelian (i.e., all idempotents are central, or equivalently, $B(R) = \mathrm{Id}(R)$)*;
(b) *in R idempotent and nilpotent elements commute*;
(c) *in R idempotents commute*;
(d) *every idempotent e commutes with all the idempotents which are isomorphic to e*;
(e) *R has only generalized commuting idempotents*;
(f) *every idempotent e generalized commutes with all the idempotents which are isomorphic to e.*

Proof. Since most of these results are known (see Exercises 12.7 and 22.3A [2]), we prove only the generalized commuting statement (f) $\Longrightarrow$ (a).

Suppose the idempotent e is not central. Then there is $r \in R$ such that $er \neq re$, or equivalently, $er\bar{e} \neq 0$ or $\bar{e}re \neq 0$. In the first case, consider the idempotent $f = e + er\bar{e}$, which is different from e. It can be checked that $ef = f$ and $fe = e$ (see also Exercise 21.4 [2]), so these are isomorphic idempotents. Therefore, $e = fe = efe = (fe)^2 = \cdots$ and $f = ef = fef = (ef)^2 = \cdots$, and e is not generalized commuting with f.

In the second case, if $\bar{e}re \neq 0$, one deals similarly with the idempotent $g = e + \bar{e}re$. □

Remark. For the proof of (c) $\Longrightarrow$ (a), the nilpotent elements $er\bar{e}$ respectively $\bar{e}re$ (indeed, $(er\bar{e})^2 = (\bar{e}re)^2 = 0$) are used. Even extending the definition of generalized commutativity to arbitrary elements in a ring (including the nilpotent ones), $n = er\bar{e}$ (or $\bar{e}re$) is no more suitable for a "generalized" statement like (c): indeed, $0 = ne = ene = nen = \cdots$, so now e and n are generalized commuting.

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