# "Everything" lifts modulo nil ideals

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Let $R$ be a ring and $I$ an ideal of $R$. For a given ring theoretic property $\mathcal{P}$, we say that *elements* with property $\mathcal{P}$ *lift modulo* $I$ if for every $\overline{a} \in \overline{R} := R/I$ with property $\mathcal{P}$ (in $R/I$) there exists an element $a'$ with property $\mathcal{P}$ (in $R$) such that $\overline{a'} = \overline{a}$. Equivalently, $a' \in a + I$ or $a' - a \in I$.

An ideal $I$ is called *nil* if it consists only of nilpotent elements (i.e. $I \subseteq N(R)$).

About some *important* types of elements in rings, we have the following elementary

**Proposition 1** *Idempotents, units and nilpotents, all lift modulo nil ideals.*

**Proof.** As for *idempotents*, see **21.28**, p.319 in T. Y. Lam's book (A First Course in Noncommutative Rings).

As for *units*: assume $\overline{a} \in U(\overline{R})$. There exists $\overline{c} \in \overline{R}$ such that $\overline{ac} = \overline{ca} = \overline{1}$ and so $a'c' - 1, c'a' - 1 \in I$, denoting $a'$, $c'$ some representatives in the cosets $\overline{a}$, $\overline{c}$ respectively. Hence $a'c', c'a' \in 1 + I \subseteq 1 + N(R) \subseteq U(R)$, so $a'$ (and $c'$) are indeed units, as desired.

[The units in $1 + N(R)$ are called *unipotents*].

As for *nilpotents*: suppose $\overline{a} \in N(\overline{R})$. Then $\overline{a}^m = \overline{0}$ for some positive integer $m$ and so $(a')^m \in I$ for some representative $a'$ in $\overline{a}$. Since $I$ is nil, $[(a')^m]^n = 0$ for some positive integer $n$, while $a'$ is indeed nilpotent, as desired. ■

**Remarks**. 1) In the above proof for units, we have used the property: if $ab, ba \in U(R)$ implies $a, b$ are units.

We just recall (see T. Y. Lam, Exercises in classical Ring Theory, **Ex.1.4**):

If $ab \in U(R)$, $a$ and $b$ may not be units (unless the ring is Dedekind finite).

However, (b) if $a$ is left-invertible and not a right 0-divisor, then $a \in U(R)$.

Now if both $ab, ba \in U(R)$, the above conditions are fulfilled [$ba \in U(R)$ implies $a$ is left invertible and $ab \in U(R)$ implies $a$ is not a right 0-divisor] and $a, b \in U(R)$.

2) When using any of the above liftings modulo nil ideals, it is customarily (in order to simplify the writing) to say: if $\overline{a}$ is idempotent (or unit, or nilpotent) we can supose $a$ is idempotent (or unit, or nilpotent), respectively.