Print from page 2

Some temporary chat

This is the notes as they are on 29 November 2008. I will update this document quite frequently. There are another two or three chapters to be added.

This material covers the first three or four few lectures. There is also quite a lot of material that is not done in the lectures. Of this extra material you should choose those parts which you think will help you the most.

There are many exercises. You should not try to do them all immediately. Again choose the ones that you think will help you the most. Solutions to exercises will appear about one week after the relevant material is done in the lectures.

The material in theses chapter should not change (unless I find some typos or a silly mistake). The preamble will change. The Contents pages and Index will be updated. I may change the Introduction, and add to the Bibliography. (In fact, at the moment the Index and Bibliography are in the wrong position, but that won't be corrected until the course is finished.)

I suggest you run off a copy from page 2 (this page). That will put the page numbers on the correct side.

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Or you can google 'harold simmons' and I come at near the top. There are other Harold Simmons in this world, but if you remember I am not a Texan then you should get to me.

My web page contains a version of these notes, and other stuff not related to category theory that you may find useful.

Category Theory by Magic

Harold Simmons

A short introduction to the basics of category theory

This draft will be updated as the course proceeds. This version was compiled 29 November 2008

Contents

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Introduction

This book is an introduction to Category Theory and is specifically designed for the beginning postgraduate student in Mathematics. It assumes you, the reader, has a decent understanding of undergraduate mathematics, but does not require any specialized knowledge in any depth.

Of course, this does not mean that the book is unsuitable for a more advanced reader. If you don't know any category theory and want to learn some, then this is the book to start with. You may find some parts a bit slow or laboured, but you will know how to skip over these.

Mathematics

Access

Grid

Instruction

Collarboration

is a cooperative of several Departments/Schools of Mathematics in several Universities. It puts on courses for their combined body of postgraduate students. The lectures are broadcast from various centres, and the whole package of courses runs throughout the academic year.

One of these courses is a 10 hour introduction to category theory. This book is designed to complement the lectures. (It is up to the audience to compliment the lecturer.)

I am well aware that many postgraduates have to study in small groups, or even in isolation. I have written this book with that in mind. Thus it is not written in the style of a reference book, rather I try to lead you, the reader, into understanding the ideas. This, of course, may become a little grating once the understanding is achieved. (I hope it doesn't become a big grating, because then you might fall through it.)

As someone once said, mathematics is not a spectator sport. You can only learn mathematics by doing it. Accordingly I have included some 170 [*at last count -- 29 November 2008*] or so exercises. These are of varying difficulty and interest, but they do provide something to get your teeth into.

Category theory is used in several parts of mathematics, often as a simplifying tool. By expressing the material in categorical language some of the technical details that are not immediately relevant can be surpressed, to be uncovered later when they are needed. Such material can provide interesting examples of category theory at work. Unfortunately, most of these examples are not suitable for a first course in category theory, simply because you will probably not know the relavant background material. With this in mind I have based many of the examples and exercises in two rather simple parts of mathematics, the study of posets (and presets) and the study of monoid sets.

Each partially ordered set (and each pre-ordered set) is a rather pathetic example of a topological space. However, many topological constructions can be mimicked with posets without having to carry around the topological baggage.

A monoid is like a group without inverses. It is a ring without the additive structure. A monoid set is like at module over a ring without addition. Many module constructions can be mimicked using monoid sets.

The nice thing about these two topics is that, from a categorical perspective, there is nothing very sophisticated going on. Even if you don't know the notions it is easy to learn the basics quite quickly.

Many of the examples and exercises are taken from one or other of these topics. Of course, I give all the relevent definitions
[*Do we want two separate chapters*].
There are also illustrations from other parts of mathematics, but if you don't know the relevent background then you can skip these.

There is nothing worse than an exercises with no hint of a solution, or even an indication of its difficulty. With this in mind I have provided a fair number of solutions.

This book consists of [*sort out*]four chapters. Each chapter is divided into several sections, and some of these are divided into blocks (subsections). The Contents pages indicates this global structure.

Items such as Definitions, Examples, Theorems, . . . are numbered consecutively throughout each chapter. Thus X.y refers to the y$^{\text{th}}$ item in Chapter X.

At the end of each section or block there is a collection of exercises. These are numbered consecutively throughout that section. Thus Exercise X.y.z occurs in Chapter X, Section y, and is the z$^{\text{th}}$ one in the section. This is the case even when a section is divided into blocks.

Solutions to the exercises are given at the end of the book.

> At the time of writing – 29 November 2008– some solutions are missing. These will be inserted in due course.

Every now and then you will see a few words IN THIS FONT. This indicates a notion not yet defined. It may be defined later when more details are required, or it may not be defined if the notion does not play an essential part in this book. However, the notion is used in more advanced category theory.

I have included a list of various texts which you should know about and may find useful. I will not comment on these for I am not too impressed by some of them, but you should at least be aware of them.
[*At the moment this bibliography is immediately after this introduction. It may increase as the course develops.*]

Also every now and then you will find a remark *in this ugly font*. Each such remark indicates one of two things.

It may be a reminder to me that something needs to be done. Something needs to be re-done, or perhaps something extra needs to be inserted.

It may be a reference to more advanced material not yet written. The parent 10 hour course is not long enough to contain all the basic category theory that you should know. I plan to write an account of this material, probably as a second volume. But that won't be done in 2007.

This is the first draft of the book, and there are bound to be parts that need to be re-done, or material I have not yet included.

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Part I

Development

1
Categories

This chapter gives the definition of 'category' in Section 1.1, and follows that by four sections devoted entirely to examples of categories of various kinds.

If you have never met the notion of a category before, you should quite quickly read through Definition 1.1 and then go to Section 1.2. There you will find some examples of categories that you are familar with, although you may not have recognized the categorical structure before. In this way you will begin to see what Definition 1.1 is getting at. After that you can move around the chapter as you see fit.

Remember that it is probably better not to start at page 3 (this page) and read each word, sentence, paragraph, ..., in turn. Jump about a bit. If there is something you don't understand, or don't see the point of, then leave it for a while and come back to it later.

Life isn't linear, but written words are.

1.1 Categories defined

This section contains the definition of 'category', follows that with a few bits and pieces, and concludes with a discussion of some examples. No examples are looked at in detail, that is done in the remaining four sections. Section 1.2 contains a collection of simpler examples, some of which you will know already. You might want to dip into that section as you read this section. In the first instance you should find a couple of examples that you already know. As you become familiar with the categorical ideas you should look at some of the more complicated examples given in the later sections.

1.1 DEFINITION. A category $\boldsymbol{C}$ consists of

- a collection Obj of entities called objects
- a collection Arw of entities called arrows
- two assignments $\quad Arw \underset{target}{\overset{source}{\rightrightarrows}} Obj$
- an assignment $\quad Obj \xrightarrow{id} Arw$
- a partial composition $\quad Arw \times Arw \longrightarrow Arw$

where this data is subjected to certain restrictions as described below. ■

Before we look at the restriction on this data let's fix some notation.

- We let $A, B, C, \ldots$ range over objects.

- We let $f, g, h, \dots$ range over arrows.

This convention isn't always used. For instance, sometimes $a, b, c, \dots$ range over objects, and $\alpha, \beta, \gamma, \dots$ or $\theta, \phi, \psi, \dots$ range over arrows. The notation used depends on what is convenient at the time and what is the custom in the topic under discussion. Here we will take the above convention as the norm, but from time to time we will use other notations.

There are two assignments

source *target*

each of which attaches an object to an arrow. We write

$$A \xrightarrow{\ f\ } B$$

to indicate that f is an arrow with source A and target B. This is a small example of a diagram. Later we will see some slightly bigger ones.

This terminology isn't universally accepted. Sometimes combinations of

A	$\xrightarrow{\quad f \quad}$	B
source	arrow	target
domain	morphism	codomain
	map	

are used. Certainly morphisms (such as group morphisms) and maps (such as continuous maps) usually are examples of arrows in some category. However, it is better to use 'arrow' for the abstract notion, and so distinguish between the general and the particular.

The word 'domain' already has other meanings in mathematics. Why bother with this and 'codomain' when there are two perfectly good words that capture the idea quite neatly.

You will also see

$$f : A \longrightarrow B$$

used to name the arrow above. This notation helps to spread the confusion between arrows and functions.

All three of the notations

$$A \xrightarrow{\ \boldsymbol{id}_A \quad id_A \quad 1_A\ } A$$

are used for the identity arrow assigned to the object A. We will tend to use $\boldsymbol{id}_A$. Notice that the source and the target of $\boldsymbol{id}_A$ are both the parent object A. Quite often when there is not much danger of confusion $\boldsymbol{id}$ is written for $\boldsymbol{id}_A$. You will also find in the literature that some people write 'A' for the arrow $\boldsymbol{id}_A$. This is a notation so ridiculous that it should be laughed at in the street.

Certain pairs of arrows are compatible for composition to form another arrow. Two arrows

$$A \xrightarrow{\ f\ } B_1 \qquad\qquad B_2 \xrightarrow{\ g\ } C$$

are composible, in that order, precisely when B_1 and B_2 are the same object, and then an arrow

$$A \longrightarrow C$$

is formed. For arrows

$$A \xrightarrow{\;\;f\;\;} B \xrightarrow{\;\;g\;\;} C$$

both of the notations

$$A \xrightarrow{\;g \circ f \quad gf\;} C$$

are used for the composite arrow. Read this as

$$g \text{ after } f$$

and be careful with the order of composition. Here we will write $g \circ f$ for the composite.

Composition of arrows is associative as far as it can be.

For three arrows

$$A \xrightarrow{\;\;f\;\;} B \xrightarrow{\;\;g\;\;} C \xrightarrow{\;\;h\;\;} D$$

various composites are possible, as follows.

$$\begin{array}{l}
A \xrightarrow{\qquad\qquad\qquad (h \circ g) \circ f \qquad\qquad\qquad} D \\
A \xrightarrow{\;f\;} B \xrightarrow{\qquad\qquad h \circ g \qquad\qquad} D \\
A \xrightarrow{\;f\;} B \xrightarrow{\qquad g \qquad} C \xrightarrow{\;h\;} D \\
A \xrightarrow{\qquad\qquad g \circ f \qquad\qquad} C \xrightarrow{\;h\;} D \\
A \xrightarrow[\qquad\qquad\qquad h \circ (g \circ f) \qquad\qquad\qquad]{} D
\end{array}$$

It is required that the two extreme arrows are equal

$$(h \circ g) \circ f = h \circ (g \circ f)$$

and we usually write

$$h \circ g \circ f$$

for this composite. This is the first of the axioms restricting the data.

The second axiom says that identity arrows are just that. Consider

$$A \xrightarrow{\;\boldsymbol{id}_A\;} A \xrightarrow{\;\;f\;\;} B \xrightarrow{\;\boldsymbol{id}_B\;} B$$

an arbitrary arrow and the two compatible identity arrows. Then

$$\boldsymbol{id}_B \circ f = f = f \circ \boldsymbol{id}_A$$

must hold.

Given two objects A and B in an arbitrary category $\boldsymbol{C}$, there may be no arrows from A to B, or there may be many. We write

$$\boldsymbol{C}[A, B] \quad \text{or} \quad \boldsymbol{C}(A, B)$$

for the collection of all such arrows. For historical reasons this is usually called the

hom-set

from A to B, although

arrow-class

would be better. Some people insist that $\boldsymbol{C}[A, B]$ should be a set, not a class. Here such niceties will not trouble us too much.

As usual, there are some variants of this notation. We often write

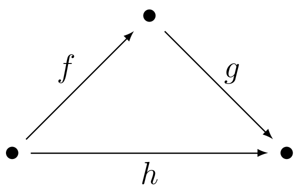
$$[A, B] \quad \text{for} \quad \boldsymbol{C}[A, B]$$

especially when it is clear which category $\boldsymbol{C}$ is intended. Some people write

$$\mathrm{Hom}_{\boldsymbol{C}}[A, B]$$

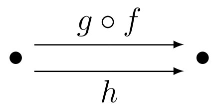
for this hom-set, but that is a waste of space. Whenever possible don't hide an important feature of a notation as a subscript.

We have seen above one very small diagram. Composition gives us a slightly larger one. Consider three arrows



arrange in a triangle, as shown. Here we haven't given each object a name, because we don't need to. However, the notation does *not* mean that the three objects are the same.

For this small diagram, the triangle, the composite $g \circ f$ exists to give us a parallel pair



of arrows across the bottom of the triangle. These two arrows may or may not be the same. When they are

$$h = g \circ f$$

we say the triangle commutes. We look at some more commuting diagrams in Section 2.1, and other examples occur throughout the book.

Examples of categories

In the remaining sections of this chapter we look at a fair selection of examples of categories. Roughly speaking these particular examples are of four kinds.

The first collection is listed in Table 1.1. These all have a similar nature and are examples of the most common kind of category we meet in practice. In each an object is a structured set, a set furnished with some extra gadgetry. An arrow between two objects is a function between the carrying sets where the function 'respects' the carried structure.

Category	Objects	Arrows
Set	sets	total functions
Pfn	sets	partial functions
$\boldsymbol{Set}_\perp$	pointed sets	point preserving functions
RelH	sets with a relation	relation respecting functions
Sgp	semigroups	morphism
Mon	monoids	morphism
CMon	commutative monoids	morphism
Grp	groups	morphism
AGrp	abelian groups	morphism
Rng	rings	morphism
CRng	commutative rings	morphism
Pre	pre-ordered sets	monotone maps
Pos	posets	monotone maps
Sup	complete posets	$\bigvee$-preserving momotone functions
Join	posets with all finitary joins	$\vee$-preserving monotone functions
Inf	complete posets	$\bigwedge$-preserving momotone functions
Meet	posets with all finitary meets	$\wedge$-preserving monotone functions
Top	topological spaces	continuous maps
$\boldsymbol{Top}_\star$	pointed topological spaces	point preserving continuous maps
$\boldsymbol{Top}^{\text{open}}$	topological spaces	continuous open maps
$\boldsymbol{Vect}_K$	vectors spaces over a given field K	linear transformations
Set-R	sets with a right action from a given monoid R	action preserving functions
R-***Set***	sets with a left action from a given monoid R	action preserving functions
Mod-R	right R-modules over a ring R	morphisms
R-***Mod***	left R-modules over a ring R	morphisms

Table 1.1: Categories of structured sets and structure preserving functions

Arrow composition is then just function composition. We look at the details of a couple of these categories in Section 1.2.

Some categories listed in Table 1.1 are not defined in this chapter. Some are used later to illustrate various facets of category theory, in which case each is defined when it first appears. Some categories are listed but not used in this book, but you should be able to fill in the details when you need to.

These simple examples tend to give the impression that in any category an object is a structured set and an arrow is a function of a certain kind. This is a false impression, and

Category	Objects	Arrows
$\boldsymbol{RelA}$	sets	binary relations
$\boldsymbol{Pos}^{\dashv}$	posets	poset adjunctions
$\boldsymbol{Pos}^{pp}$	posets	projection embedding pairs
$\widehat{S}$	presheaves on a given poset S	natural transformations
$\widehat{\boldsymbol{C}}$	presheaves on a given category $\boldsymbol{C}$	natural transformations
$\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$	chain complexes	
	Find some more	
	Put in categories of diagrams	

Table 1.2: More complicated categories

in Section 1.3 we look at some examples to illustrate this. In particular, these examples show that an arrow need not be a function (of the kind you first thought of).

Section 1.4 contains a couple of examples to show that the objects of a category can have a rather complicated internal structure, and the arrows are just as complicated. Both these examples are important in various parts of mathematics, but you shouldn't worry if you can not get to grips with them immediately.

Table 1.2 lists some of these more complicated examples looked at in Sections 1.3 and 1.4.

Finally in Section 1.5 we look at two very simple kinds of categories. These examples could be given now, but in some ways it is better if we leave them for a while.

Exercises

1.1.1 Observe the sets and functions do form a category $\boldsymbol{Set}$.

1.1.2 Can you see that each poset is a category, and each monoid is a category? Read that again.

1.2 Categories of structured sets

The categories we first meet usually have a rather simple structure. Each object is a structured set

$$(A, \cdots)$$

a set furnished with some extra gadgetry, and each arrow

$$(A, \cdots) \longrightarrow (B, \cdots)$$

is a (total) function

$$f : A \longrightarrow B$$

between the two carrying sets which respects the carried structure (in some appropriate sense).

More often than not these structured sets are ‘algebras’. Thus the furnishings carried by A are a selection of nominated elements, and a selection of nominated operations on A. These operations are usually binary or singulary, but other arities do occur.

You have already met

$$\textbf{\textit{Grp}} \qquad \textbf{\textit{Rng}} \qquad \textbf{\textit{Vect}}_K$$

but you may not have realized that each of these is a category. You should make sure that you understand the workings of each of these as a category of ‘algebras’. You may have to puzzle a bit over $\textbf{\textit{Vect}}_K$, but later we look at some more general examples of this nature, and that should help clear away the mist.

To help with the general idea, in the first part of this section we look at the category ***Mon*** of monoids. This has all the typical properties of an ‘algebraic’ category. You may not have met monoids before, so this example will serve as an introduction.

Monoids are quite important in category theory. They can tell us quite a lot about the structure of a particular category. Also, they can be used to illustrate many facets of category theory.

The exercises for the first part of this section look at several other categories of structured sets, some of which are not ‘algebraic’ in this intuitive sense. One of these

$$\textbf{\textit{Top}}$$

is particularly important, and you should make sure you understand it.

1.2 EXAMPLE. A monoid is a structure

$$(R,\ \star,\ 1)$$

where R is a set, $\star$ is a binary operation on R (usually written as an infix), 1 is a nominated element of R, and where

$$(r \star s) \star t = r \star (s \star t) \qquad 1 \star r = r = r \star 1$$

for all $r, s, t \in R$. In other words, the operation is associative and the nominated element is a unit for the operation.

Monoids are sometimes referred to as unital semigroups, or even semigroups. (Strictly speaking a semigroup is a structure $(R,\ \star)$ where $\star$ is an associative binary operation on the set R. Such a structure need not have a unit.)

Usually we omit the operation symbol and write

$$rs \quad \text{for} \quad r \star s$$

but for the time being we will stick to the official notation.

A monoid morphism

$$R \xrightarrow{\ \phi\ } S$$

between two monoids is a function that respects the furnishings, that is

$$\phi(r \star s) = \phi(r) \star \phi(s) \qquad \phi(1) = 1$$

for all $r, s \in R$. (Notice that here we have overloaded the operation symbol and the unit symbol. That shouldn't cause a problem here, but every now and then it is a good idea to distinguish between the source and target furnishings.)

It is routine to check that for two morphisms

$$R \xrightarrow{\ \phi\ } S \xrightarrow{\ \psi\ } T$$

between monoids the function composite

$$R \xrightarrow{\ \psi \circ \phi\ } T$$

is a morphism.

This gives us the category ***Mon*** of monoids (as objects) and monoid morphisms (as arrows). The verification of the axioms is almost trivial. Given a monoid R the identity arrow

$$R \xrightarrow{\ \boldsymbol{id}_R\ } R$$

is just the identity function on R viewed as a morphism. ■

As suggested above many categories fit into this form. Each object is a structured set, and each arrow (usually called a morphism or a map) is a structure respecting function. Almost all of the categories in Table 1.1 fit into this kind, but one or two don't.

In a sense the study of monoids is the study of composition in the miniature. There is a corresponding study of comparison in the miniature. That is the topic of the next example.

1.3 EXAMPLE. A pre-order $\leq$ on a set S is a binary relation that is both reflexive and transitive. (Sometimes a pre-order is called a quasi-order.) A partial order is a pre-order that is also anti-symmetric.

A

preset poset

is a set S furnished with a

pre-order partial order

respectively. Thus each poset is a preset, but not conversely.

When comparing two such structures

$$(R, \leq_R) \qquad (S, \leq_S)$$

we use the carrying sets R and S to refer to the structures and write $\leq$ for both the carried comparisons. Rarely does this cause any confusion, but when it does we are a bit more careful with the notation.

Given a pair R, S of presets a monotone map

$$R \xrightarrow{\ f\ } S$$

is a function, as indicated, such that

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in R$. It is routine to check that for two monotone maps

$$R \xrightarrow{\ f\ } S \xrightarrow{\ g\ } T$$

between presets the function composition $g \circ f$ is also monotone.

This gives us two categories

$$\boldsymbol{Pre} \qquad \boldsymbol{Pos}$$

where the objects are

$$\text{presets} \qquad \text{posets}$$

respectively, and in both cases the arrows are the monotone maps. Each identity arrow is the corresponding identity function viewed as a monotone map. ■

Consider a pair R and S of posets. Each is a preset, so we have the two collections of arrows

$$\boldsymbol{Pre}[R, S] \qquad \boldsymbol{Pos}[R, S]$$

in the categories. A few moment's thought shows that, as functions, these two are the same set. Technically, this shows that ***Pos*** is a FULL SUBCATEGORY OF ***Pre***.

The study of monoids is the study of composition in the miniature.

The study of presets is the study of comparison in the miniature.

What should we do to study these two notion together and in the large? Category theory!

In a sense every category is an amalgam of certain monoids and presets, and that is a good enough reason why we should always keep these two simple notions in mind.

From the examples we have seen so far it is easy to get the impression that certain things always happen. The next example shows that some categories can be awkward (and sometimes cantankerous).

1.4 EXAMPLE. We enlarge the category ***Set*** of sets and total functions to the category ***Pfn*** of sets and partial functions.

The objects of ***Pfn*** are just sets

$$A, B, C, \ldots$$

as in ***Set***. However, an arrow

$$A \xrightarrow{\ f\ } B$$

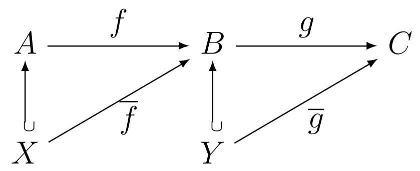
is a *partial* function from A to B. In other words, an arrow is a total function

$$\begin{array}{ccc} A & & B \\ \uparrow & \nearrow_{f} & \\ X & & \end{array}$$

from a subset X of the source A. (This is an example where the use of the word 'domain' for source can be confusing. The set X is the domain of definition of the partial function.) Notice that we need to distinguish between the total function $\overline{f}$ and the arrow f it determines. The notation has been chosen to emphasize that distinction.

We wish to show that these objects and arrows form a category ***Pfn***. To do that we must first produce a composition of arrows. (The naming of categories is a curious thing. Category theory emphasizes that although the objects are important, the arrows that compare objects are more important. Therefore the arrows should provide the name of any particular category. For historical reasons this rarely happens. Here I have used the arrows to name the category. I am not sure if this particular category has a standard name.)

Consider a pair of partial functions.

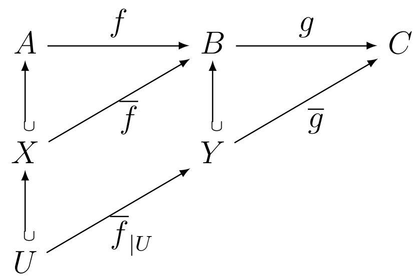


How might we compose these? We somehow want to stick $\overline{f}$ and $\overline{g}$ together, but these functions are not composition compatible.

We extract a subset $U \subseteq A$ by

$$a \in U \Longleftrightarrow a \in X \text{ and } \overline{f}(a) \in Y$$

(for $a \in A$). Trivially, $\overline{f}$ is defined on the whole of U, so we may restrict $\overline{f}$ to U.



Now we do have composition compatible functions. Thus we take

$$g \circ f$$

to be that arrow (partial function) determined by

$$\overline{g \circ f} = \overline{g} \circ \overline{f}_{|U}$$

to produce a composition of arrows in ***Pfn***.

Notice here how the symbol '$\circ$' is overloaded. On the right it is the standard composition of total functions. On the left it is the defined operation on partial functions. If at first you find this confusing then write '$\bullet$' for the defined operation. Thus

$$\overline{g \bullet f} = \overline{g} \circ \overline{f}_{|U}$$

is its definition.

Of course, there is still some work to be done. For instance, we need to show that this composition of arrows is associative. That is left as an exercise. ■

Once we get used to it the step from ***Set*** to ***Pfn*** is not so great. An arrow is still a function, and we just have to take a little more care with composition.

We began this section by looking at the category ***Mon*** of monoids. We conclude by looking at two categories attached to each monoid.

1.5 EXAMPLE. Let R be a fixed, but arbitrary, monoid. A

left right

R-set is a set A together with an action

$$\begin{array}{ccc} R, A \longrightarrow A & \qquad & A, R \longrightarrow A \\ r, a \longmapsto ra & & a, r \longmapsto ar \end{array}$$

where

$$\begin{array}{ccc} s(ra) = (sr)a & \qquad & (ar)s = a(rs) \\ 1a = a & & a = a1 \end{array}$$

for each $a \in A$ and $r, s \in R$. Of course, here the two definition are given in parallel.

These R-sets are the objects of two categories

R-***Set*** ***Set***-R

with left R-sets on the left and right R-sets on the right.

Given two R-sets A and B of the same handedness, a morphism

$$A \xrightarrow{\ f\ } B$$

is a function f such that

$$f(ra) = rf(a) \qquad f(ar) = f(a)r$$

for each $a \in A$ and $r \in R$. These are the arrows of the two categories. ■

This may look a quite simple example but it is useful. Many facets of category theory can be illustrated with these categories. We use them quite a lot in this book.

There are also module categories in miniature. We can replace the monoid R by a ring and replace each set A by an abelian group. This gives the categories

R-***Mod*** ***Mod***-R

of left and right modules over R, respectively. These categories have quite a bit more associated structure, and we won't delve into that so much in this book.

Exercises

1.2.1 The category ***Pno*** described in this exercise may look to be less than exciting, but it has an important role to play in mathematics. (This was originally discovered by Dedekind without the categorical paraphernalia.)

The objects of ***Pno*** are the structures (A, α, a) where A is a set, $\alpha : A \longrightarrow A$ is a function, and $a \in A$ is a nominated element. Given two such structures a morphism

$$(A, \alpha, a) \xrightarrow{f} (B, \beta, b)$$

is a function $f : A \longrightarrow B$ which preserves the structure in the sense that

$$f \circ \alpha = \beta \circ f \qquad f(a) = b$$

hold.

(a) Verify that ***Pno*** is a category.

(b) Show that $(\mathbb{N}, \mathrm{succ}, 0)$ is a ***Pno***-object (where succ is the successor function).

(c) Show that for each ***Pno***-object (A, α, a) there is a unique arrow

$$(\mathbb{N}, \mathrm{succ}, 0) \longrightarrow (A, \alpha, a)$$

and describe the behaviour of the carrying function.

1.2.2 Consider pairs (A, X) where A is a set and $X \subseteq A$. For two such pairs a morphism

$$(A, X) \xrightarrow{f} (B, Y)$$

is a function $f : A \longrightarrow B$ that respects the selected subsets, that is

$$f(x) \in Y$$

for each $x \in X$.

Show that such pairs and morphisms form a category.

We won't give this category a name, but we will refer to at as the category of sets with a distinguished subset.

1.2.3 Consider pairs (A, R) where A is a set and $R \subseteq A \times A$ is a binary relation on A. Show that these pairs are the objects of a category. You must find a sensible notion of morphism for such pairs.

1.2.4 A topological space $(S, \mathcal{O}S)$ is a set S furnished with a certain family $\mathcal{O}S$ of subsets of S (called the open sets of the space). This family is required to contain both $\emptyset$ and S, be closed under $\cap$ (binary intersection), and be closed under $\bigcup$ (arbitrary unions).

A continuous map

$$(S, \mathcal{O}S) \xrightarrow{\phi} (T, \mathcal{O}T)$$

between two such spaces is a function

$$\phi : S \longrightarrow T$$

such that

$$\phi^{\leftarrow}(V) \in \mathcal{O}S$$

for each $V \in \mathcal{O}T$. Here $\phi^{\leftarrow}$ is the inverse image map given by

$$x \in \phi^{\leftarrow}(V) \Longleftrightarrow \phi(x) \in V$$

for each $V \in \mathcal{O}T$ and $x \in S$.

Show that topological space and continuous maps form a category ***Top***.

1.2.5 Let A be an arbitrary object of an arbitrary category $\boldsymbol{C}$. Show that $\boldsymbol{C}[A, A]$ is a monoid under composition.

1.2.6 Fill in the details missing from the description of $\boldsymbol{Pfn}$. In particular, you should show that composition of partial functions is associative.

1.2.7 A pointed set is a set S with a nominated element which we usually write as $\perp$. An arrow

$$S \xrightarrow{\phi} T$$

between two such pointed sets is a function $\phi : S \longrightarrow T$ which respects the nominated points, that is $\phi(\perp) = \perp$.

Almost trivially pointed sets with these arrows form a category $\boldsymbol{Set}_\perp$.

Try to show that $\boldsymbol{Set}_\perp$ and $\boldsymbol{Pfn}$ are 'essentially the same' category.

1.2.8 Verify that for each monoid R both

$$R\text{-}\boldsymbol{Set} \qquad \boldsymbol{Set}\text{-}R$$

are categories.

Can you see how each is a category of structured sets?

1.3 An arrow need not be a function

Strictly speaking the header of this section should be

An arrow need not be a single function in the way you first think of it

but you could fall asleep reading that. In this section we look first at a couple of examples to show that arrows may not be the simple kind of things we have seen so far. Then we look at a couple of general constructions for turning an old category into a new one.

In the next example an arrow is still a function, but not where you might expect it to be.

1.6 EXAMPLE. The objects of this category are the finite sets. An arrow

$$A \xrightarrow{f} B$$

is a function

$$f : A \times B \longrightarrow \mathbb{R}$$

(with no imposed conditions). For each pair

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of arrows we define

$$g \circ f : A \times C \longrightarrow \mathbb{R}$$

by

$$(g \circ f)(a, c) = \sum \big\{ f(a, y) g(y, c) \mid y \in B \big\}$$

for $a \in A, b \in B$. With a little work we see that this produces a category. ■

Notice that we did not give this category a name. That is because in the general scheme of things it is not that important. It is merely an example to illustrate that an arrow need not be a function in the way you might expect it to be.

1.7 EXAMPLE. We have seen that the category ***Set*** of sets and functions can be extended to ***Pfn*** by adding more arrows but keeping the same objects. There is also a different extension to the category ***RelA***.

Again the objects of ***RelA*** are just sets. However, a ***RelA***-arrow

$$A \xrightarrow{F} B$$

is a subset $F \subseteq B \times A$ which we can think of as a relation from A to B. In other words, ***RelA***$[A, B]$ is just this set of all such relations from A to B. You should note the way the source and target have been set up. This is not a mistake, nor is it to be perverse. It leads to a neater description of the category.

Before we can claim this is a category we must define the composition of these arrows, and then check that the axioms are satisfied.

Consider an arrow F as above, so $F \subseteq B \times A$. For $a \in A$ and $b \in B$ we write bFa for $(b, a) \in F$. For two composible arrows

$$A \xrightarrow{F} B \xrightarrow{G} C$$

we defined the composition $G \circ F$ by

$$c(G \circ F)a \Longleftrightarrow (\exists b \in B)[cGbFa]$$

for $a \in A, b \in B$. Thus we show that a is $G \circ F$ related to c by passing through a common element $b \in B$.

It is now straight forward to check that this composition is associative, and that the equality relation on a set gives the identity arrow.

These two categories ***Set*** and ***RelA*** are connected in a certain way (which will be explained in more detail later). There is a canonical way

$$A \xrightarrow{f} B \qquad \longmapsto \qquad A \xrightarrow{\Gamma(f)} B$$

of converting a ***Set***-arrow into a ***RelA***-arrow with the same source and target. We simply take the graph of the function, that is we let

$$b\,\Gamma(f)\,a \Longleftrightarrow b = f(a)$$

for $a \in A, b \in B$. ■

The next example is important in itself, and also provides a miniature version of a central notion of category theory, that of an ADJUNCTION.

1.8 EXAMPLE. We modify the category ***Pos*** of posets, described in Example 1.3, to produce a new category ***Pos***$^{\dashv}$.

As with ***Pos***, the object of ***Pos***$^{\dashv}$ are posets, but the arrows are different.

Given a pair S, T of posets, an adjunction from S to T is a pair of monotone maps

$$S \underset{g}{\overset{f}{\rightleftarrows}} T$$

such that

$$f(a) \leq b \iff a \leq g(b)$$

for all $a \in S$ and $b \in T$. We call

f the left adjoint $\qquad$ g the right adjoint

of the pair, and sometimes write

$$f \dashv g$$

to indicate an adjunction. Sometimes using two letters can be a bit wasteful. So we write

$$S \underset{f_*}{\overset{f^*}{\rightleftarrows}} T$$

to indicate an adjunction $f^* \dashv f_*$. Sometimes a harpoon arrow

$$S \xrightharpoondown{f^* \dashv f_*} T$$

is used to indicate an adjunction. As here, by convention, an adjunction points in the direction of its left component. Thus S is the source and T is the target. (You are warned that in some of the older literature this convention hadn't yet been established. Also, some of the current ignoscenti only think they know what they are doing.)

Poset adjunction are the arrows of $\boldsymbol{Pos}^{\dashv}$.

This gives us the object and arrows of $\boldsymbol{Pos}^{\dashv}$, but we still have some work to do before we know we have a category.

Consider a pair of adjunctions.

$$R \xrightharpoondown{f^* \dashv f_*} S \xrightharpoondown{g^* \dashv g_*} T$$

which ought to be composible. How should the composite

$$R \xrightharpoondown{(g^* \dashv g_*) \circ (f^* \dashv f_*)} T$$

be formed? The two left hand components are monotone maps that compose to give a monotone map. Similarly the two right hand components are monotone maps that compose to give a monotone map. Thus we have a pair of monotone maps

$$R \underset{f_* \circ g_*}{\overset{g^* \circ f^*}{\rightleftarrows}} T$$

going in opposite directions. We check that this is an adjunction and take that as the composite

Almost trivially, this composition is associative, and so we do obtain a category. ■

It is not so surprising that any given monotone map may or may not have a left adjoint, and it may or may not have a right adjoint. It can have neither, and it can have one without the other. What is a little surprising is that it can have both adjoints where these are not the same. In fact, arbitrarily longs strings of adjoints can be produced. A particularly simple example of this is an important gadget in homotopy theory.

Once we get used to working with categories we find that old categories can be used to produce new categories. Let's look at a few examples of this.

1.9 EXAMPLE. Consider categories $\boldsymbol{C}$ and $\boldsymbol{D}$. To help us distinguish between these let us write

$A, B, C \dots$ for objects of $\boldsymbol{C}$ $\quad$ $f, g, h \dots$ for arrows of $\boldsymbol{C}$

$R, S, T \dots$ for objects of $\boldsymbol{D}$ $\quad$ $\theta, \phi, \psi \dots$ for arrows of $\boldsymbol{D}$

respectively. We form a new category, the product

$$\boldsymbol{C} \times \boldsymbol{D}$$

of $\boldsymbol{C}$ and $\boldsymbol{D}$ as follows.

Each new object is an ordered pair

$$(A, R)$$

of old objects, an object A from $\boldsymbol{C}$ and and object R from $\boldsymbol{D}$. A new arrow

$$(A, R) \longrightarrow (B, S)$$

is a pair of old arrows

$$A \xrightarrow{\ f\ } B \qquad\qquad R \xrightarrow{\ \theta\ } S$$

from the given categories. For composible new arrows

$$(A, R) \xrightarrow{(f, \theta)} (B, S \xrightarrow{(g, \phi)} (C, T)$$

the composite

$$(A, R) \xrightarrow{(g \circ f, \phi \circ \theta)} (C, T)$$

is formed using composition in the old categories.

Almost trivially, this does give a category. ■

That's not the most exciting example you have ever seen, is it? The construction is useful because it can help to show that seemingly different notions are instances of the same notion.

Here is a a more interesting construction.

1.10 EXAMPLE. It is possible to form a new category where the new objects are the arrows of an old category $\boldsymbol{C}$. This is sometimes called the arrow category of $\boldsymbol{C}$.

Consider the small graph

$$(\downarrow) \qquad \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}$$

with two nodes, here labelled 0 and 1, and with one edge.

Let $\boldsymbol{C}$ be an arbitrary category. We can use $(\downarrow)$ to convert $\boldsymbol{C}$ into a new category

$$\boldsymbol{C}^{\downarrow}$$

the category of $(\downarrow)$-diagrams in $\boldsymbol{C}$. We think of $(\downarrow)$ as a template for diagrams in $\boldsymbol{C}$, and these diagrams are the objects of $\boldsymbol{C}^{\downarrow}$. Thus a new object is a pair of old objects

$$\begin{array}{c} A_0 \\ \downarrow \alpha \\ A_1 \end{array}$$

and an old arrow between them. Given two such new objects a new arrow

$$\begin{array}{ccc} A_0 & & B_0 \\ \alpha & \xrightarrow{\;f\;} & \beta \\ A_1 & & B_1 \end{array}$$

is a pair of old arrows

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ \downarrow \alpha & & \downarrow \beta \\ A_1 & \xrightarrow[f_1]{} & B_1 \end{array} \qquad f_1 \circ \alpha = \beta \circ f_0$$

such that the square commutes. Composition of new arrows is performed in the obvious way, we compose the two component old arrows. You should check that this does give a category. ■

This is a simple example of a much more general construction, that of a FUNCTOR CATEGORY. We look at this once we know what a FUNCTOR is. A couple more simple examples of this construction are given in the exercises.

The idea of the previous example is to view *all* the arrows of the old category as the objects of the new category. Sometimes we want to do a similar thing but using only *some* old arrows.

1.11 EXAMPLE. Given a category $\boldsymbol{C}$ and an object S we form the two slice categories

$$(\boldsymbol{C} \downarrow S) \qquad\qquad (S \downarrow \boldsymbol{C})$$

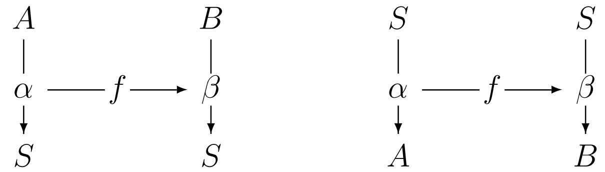
of objects

over | under

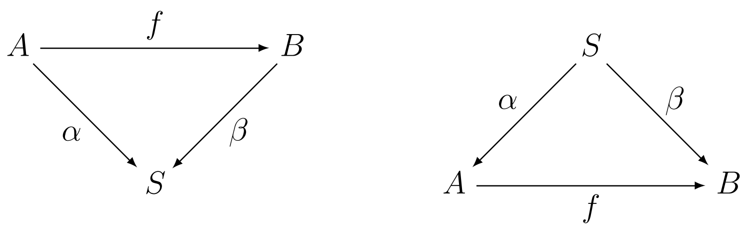
S. Each object of the new category is an arrow

to S | from S

of $\boldsymbol{C}$. An arrow of the new category



is an arrow of $\boldsymbol{C}$



for which the indicated triangle commutes. Composition of the new arrows is obtained from composition of arrows in $\boldsymbol{C}$ ■

As with Example 1.10 this construction is a particular case of a more general construction, that of a COMMA CATEGORY. Before we can explain that we need to understand the notion of a FUNCTOR.

Exercises

1.3.1 Consider the strictly positive integers $1, 2, 3, \cdots$ as object. For two such integers m, n let an arrow

$$n \longrightarrow m$$

be an $m \times n$ matrix A (with real entries). Given two compatible matrices

$$n \xrightarrow{\quad B \quad} k \qquad\qquad k \xrightarrow{\quad A \quad} m$$

let the composite

$$n \xrightarrow{\quad A \circ B \quad} m$$

be the matrix product AB.

Show that this gives a category.

Can you show that this example is a bit of a cheat?

1.3.2 A **directed graph**, or simply a **graph** for short, is a pair

$$(N, E)$$

of sets together with a pair of assignments

$$E \underset{\tau}{\overset{\sigma}{\rightrightarrows}} V$$

(as with a category). Each member of N is a **node**, and each member of E is an **edge**. For each edge $e \in E$ we call

$$\sigma(e) \qquad \tau(e)$$

the source node and the target node of e, and we write

$$a \xrightarrow{e} b$$

to indicate that $\sigma(e) = a$ and $\tau(e) = b$. In general there are no other conditions on these edges and nodes. In particular, there is no notion of composing edges. Notice that (modulo the size) each category is a graph.

A **graph morphism**

$$(N, E) \xrightarrow{f} (M, F)$$

is a pair of functions

$$N \xrightarrow{f_0} M \qquad\qquad E \xrightarrow{f_1} F$$

such that

$$\sigma \circ f_1 = f_0 \circ \sigma \qquad \tau \circ f_1 = f_0 \circ \tau$$

hold. Of course, here there are two different source and two different target assignments.

Show that, with the appropriate notion of composition, graphs and their morphisms form a category.

(There is a common notation whereby the collections of objects and arrows of a category $\boldsymbol{C}$ are $\boldsymbol{C}_0$ and $\boldsymbol{C}_1$, respectively. Some of this notation as been taken over here. There are extensions of the notion of a category, to those of a 2-category, a 3-category, 4-category,..., an ω-category, in which there are collections $\boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \ldots$ with interacting properties. Believe me, you don't want to know about these just yet.)

1.3.3 Consider any pair of categories $\boldsymbol{A}$ and $\boldsymbol{S}$. We form a new category. The objects are pairs

$$(A, R)$$

where A is an $\boldsymbol{A}$-object and R is a $\boldsymbol{S}$-object. An arrow

$$(A, R) \xrightarrow{(f, \phi)} (B, S)$$

is a pair of arrows

$$A \xrightarrow{f} B \qquad\qquad R \xleftarrow{\phi} S$$

from the two component categories where the $\boldsymbol{S}$-arrow goes backwards.

Show that with the obvious commposition this does form a category.

1.3.4 As in Exercise 1.2.8, each monoid R gives us a category $\boldsymbol{Set}$-R of (right) R-sets. We can also vary R to produce a larger category.

Each object is a pair

$$(A, R)$$

where R is a monoid and A is an R-set. Each arrow

$$(A, R) \xrightarrow{(f, \phi)} (B, S)$$

is a pair

$$A \xrightarrow{\quad f \quad} B \qquad\qquad R \xleftarrow{\quad \phi \quad} S$$

where ϕ is a monoid morphism and f is a function with

$$f(a\phi(s)) = f(a)s$$

for each $a \in A$ and $s \in S$.

Using the obvious composition, show that this does give a category.

1.3.5 Consider the category $\boldsymbol{RelA}$ of Example 1.7.

Show that the defined composition is associative, and hence it is a category.

Show also that

$$\Gamma(g \circ f) = \Gamma(g) \circ \Gamma(f)$$

for each pair of composable $\boldsymbol{Set}$-arrows.

(This result more or less shows that Γ is a COVARIANT FUNCTOR from $\boldsymbol{Set}$ to $\boldsymbol{RelA}$. This notion is discussed in the Chapter 3.)

1.3.6 Consider any pair

$$S \underset{g}{\overset{f}{\rightleftarrows}} T$$

of $\boldsymbol{Pos}$-arrows.

(a) Show that $f \dashv g$ precisely when both $\boldsymbol{id}_S \leq g \circ f$ and $f \circ g \leq \boldsymbol{id}_T$, where the two comparisons are pointwise.

(b) Show that if $f \dashv g$ then

$$f \circ g \circ f = f \qquad g \circ f \circ g = g$$

and hence $g \circ f$ is a closure operation on A and $f \circ g$ is a coclosure operation on B.

1.3.7 Posets and certain adjoint pairs form another category $\boldsymbol{Pos}^{pp}$.

The objects of $\boldsymbol{Pos}^{pp}$ are again just posets. A $\boldsymbol{Pos}^{pp}$-arrow

$$A \xrightarrow{(f, g)} B$$

is a $\boldsymbol{Pos}^{\dashv}$-arrow

$$A \xrightarrow{\; f \dashv g \;} B$$

for which $g \circ f = \boldsymbol{id}_A$. The arrows of $\boldsymbol{Pos}^{pp}$ are sometimes called projection pairs.

Show that these projection pairs are closed under composition, and hence $\boldsymbol{Pos}^{pp}$ is a category.

(You see here a useful little trick. It is sometimes helpful to draw arrows in different, but related, categories in a different way. Thus here we have

$$\begin{array}{ll} \boldsymbol{Pos} & \longrightarrow \\ \boldsymbol{Pos}^{\dashv} & \rightharpoondown \\ \boldsymbol{Pos}^{pp} & \Longrightarrow \end{array}$$

for the three different kinds of arrows.)

1.3.8 Consider the ordered sets $\mathbb{Z}$ and $\mathbb{R}$ as posets, and let

$$\mathbb{Z} \xrightarrow{\ \iota\ } \mathbb{R}$$

be the insertion.

(a) Show there are (unique) maps

$$\mathbb{R} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \mathbb{Z}$$

such that

$$\mathbb{Z} \xrightharpoondown{\ \iota \dashv \rho\ } \mathbb{R} \xrightharpoondown{\ \lambda \dashv \iota\ } \mathbb{Z}$$

are adjunctions.

(c) Show also that this composite is $\boldsymbol{id}_{\mathbb{Z}}$ in $\boldsymbol{Pos}^{\dashv}$ and the other composite, on $\mathbb{R}$, is idempotent.

(d) Show that $\iota \dashv \rho$ is a $\boldsymbol{Pos}^{pp}$-arrow, but $\lambda \dashv \iota$ is not.

1.3.9 For a poset S let $\mathcal{L}S$ be the poset of lower sections under inclusion. (A lower section of S is a subset $X \subseteq S$ such that

$$y \leq x \in X \Longrightarrow y \in X$$

for all $x, y \in S$.)

(a) For a monotone map

$$T \xrightarrow{\ \phi\ } S$$

between posets, show that setting $f = \phi^{\leftarrow}$ (the inverse image map) produces a monotone map

$$\mathcal{L}T \xleftarrow{\ f = \phi^{\leftarrow}\ } \mathcal{L}S$$

in the opposite direction.

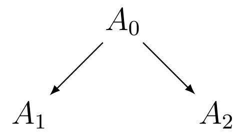
(b) Show that f has both a left adjoint and a right adjoint

$$f^{\sharp} \dashv f \dashv f_{\flat}$$

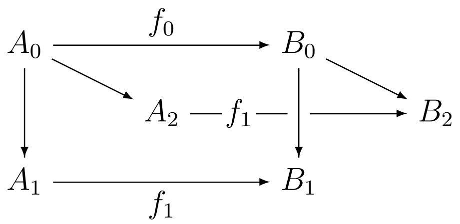
where, in general, these are different.

1.3.10 Let $\boldsymbol{C}$ be an arbitrary category. In Example 1.10 we used $(\downarrow)$ as a template to obtain a category $\boldsymbol{C}^{\downarrow}$ of certain diagrams from $\boldsymbol{C}$. The same idea can be used with other templates.

A wedge in a category $\boldsymbol{C}$ is a pair or arrows



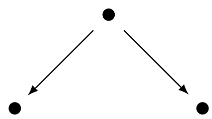
as shown. A wedge morphism



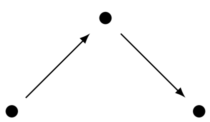
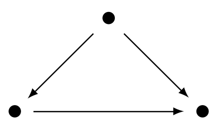
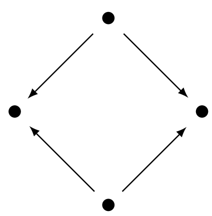
is a triple of arrows, as indicated, which make the two associated squares commute.

(a) Show that wedges and wedge morphisms form a category.

(b) This wedge example uses



as the template. Play around with other templates to produce other examples of categories. For example, consider each of

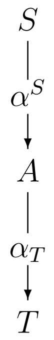
and worry about which cells are required to commute.

1.3.11 Let 1 and 2 be the 1-element set and the 2-element set, respectively. Describe the categories

$$(\boldsymbol{Set} \downarrow 1) \quad (1 \downarrow \boldsymbol{Set}) \quad (\boldsymbol{Set} \downarrow 2) \quad (2 \downarrow \boldsymbol{Set})$$

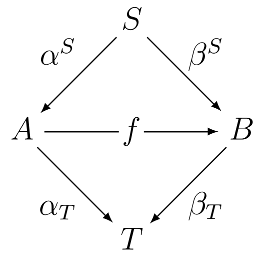
and show that you have met two of them already together with near relatives of the other two.

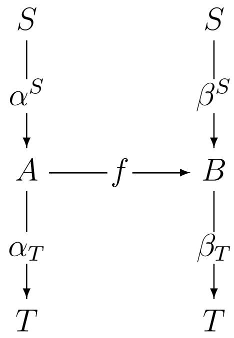
1.3.12 Given a category $\boldsymbol{C}$ and two object S, T we form



$$(S \downarrow \boldsymbol{C} \downarrow T)$$

the butty category between S and T. Each object of the new category is an object A of $\boldsymbol{C}$ together with a pair of arrows from S and to T. An arrow of the new category is an arrow f of $\boldsymbol{C}$ to make the two triangles commute.





(a) Show that with the appropriate notion of composaition this gives a category.

(b) Can you show that for an appropriate parent category $\boldsymbol{C}$ both the slice categories

$$(\boldsymbol{C} \downarrow T) \qquad (S \downarrow \boldsymbol{C})$$

are instance of the butty construction?

1.4 More complicated categories

From the examples we have seen so far you might conclude that category theory is making a bit of a fuss. It is true that objects are not just structured sets and arrows are not just functions, but the examples seem to suggest that we don't move too far from those ideas. Of course, as yet far we have seen only comparatively simple examples of categories.

One of the original aims of category theory was to organize and analyse what we now see as rather complicated categories. The simpler examples came along later.

In this section we look at a couple of examples of the more complicated kind of category. You probably won't understand these at a first reading, but you should give it a go. You should come back to these examples as you learn more about category theory.

1.12 EXAMPLE. Let S be any partially ordered set. We describe the category $\widehat{S}$ of PRESHEAVES ON S. There is a more general notion where S is replaced by an arbitrary category, but we save that for later.

We may think of $\widehat{S}$ as the category of 'sets developing over S'. At first sight the structure of $\widehat{S}$ looks quite complicated, but you will soon get used to it.

We think of S as a store of indexes $i, j, k, \ldots$ partially ordered

$$j \leq i$$

to form a poset.

A presheaf on S is an S-indexed family of sets

$$\mathsf{A} \qquad \big(A(i) \mid i \in S\big)$$

together with a family of connecting functions

$$\mathcal{A} \qquad A(i) \xrightarrow{A(j,i)} A(j)$$

one for each comparison $j \leq i$. Note these functions progress *down* the poset. These functions have to fit together in a coherent fashion. Thus

$$A(i,i) = \boldsymbol{id}_{A(i)}$$

for each index $i \in S$, and the triangle

$$\begin{array}{ccc} A(i) & \xrightarrow{A(k,i)} & A(k) \\ {\scriptstyle A(j,i)} \searrow & & \nearrow {\scriptstyle A(k,j)} \\ & A(j) & \end{array} \qquad A(k,j) \circ A(j,i) = A(k,i)$$

commutes for all $k \leq j \leq i$. These are the objects of $\widehat{S}$. (Take note of the way the connecting functions are indexed. Many people still make a mess of this.)

An arrow

$$\mathsf{A} \xrightarrow{f} \mathsf{B}$$

between two presheaves is an S-indexed family of functions

$$A(i) \xrightarrow{f_i} B(i)$$

such that

$$\begin{array}{ccc} A(i) & \xrightarrow{f_i} & B(i) \\ {\scriptstyle A(j,i)} \downarrow & j \leq i & \downarrow {\scriptstyle B(j,i)} \\ A(j) & \xrightarrow[f_j]{} & B(j) \end{array} \qquad f_j \circ A(j,i) = B(j,i) \circ f_i$$

commutes for all comparisons $j \leq i$ in S.

These arrows are composed in the obvious way, we compose the corresponding functions at each index. Of course, we have to show that the resulting squares commute, and that this composition is associative, but that is straight forward. ■

In this example we used a poset S to index the constructed category $\widehat{S}$. There is also a more general construction which converts an arbitrary category $\boldsymbol{C}$ into the category $\widehat{\boldsymbol{C}}$ of presheaves of $\boldsymbol{C}$. We look at that briefly in Section 3.5. Such presheaf categories occur in many places, and are not always recognized as such. Believe it or not ***Set***-R and ***Mod***-R are two such categories.

The next example is important in homology.

1.13 EXAMPLE. Let R be a fixed ring and consider the category

$$\boldsymbol{Mod}\text{-}R$$

of right R-modules. If you are not yet happy with ***Mod***-R then you can replaced it by the category ***AGrp*** of abelian groups. Also, if you are feeling confident then you can replace ***Mod***-R be any ABELIAN CATEGORY.

We are going to construct a new category

$$\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$$

out of the objects and arrows of ***Mod***-R.

A chain complex, or sometimes just a complex, over R is a $\mathbb{Z}$ indexed family

$$\mathsf{A} \qquad \cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

of objects and connecting arrows taken from ***Mod***-R. These arrows have to satisfy a certain condition which we come to in a moment.

$$\begin{array}{c} \vdots \\ \downarrow \\ A_{n+1} \\ \alpha_{n+1}\downarrow \\ A_n \\ \alpha_n\downarrow \\ A_{n-1} \\ \downarrow \\ \vdots \end{array}$$

Note the indexing of the objects. The indexes become smaller as we move along the chain to the right. For this reason it is sometimes convenient to think of the chain as progressing downwards. However, for obvious reasons, a chain is rarely printed in this form.

Also, it is customary to write $d_\bullet$ for each connecting arrow $\alpha_\bullet$ but as we will see, in the first instance that can be confusing.

The value of each index is important. Thus if we re-index by moving each object 1-step along, then we get a different complex. In particular, the object A_0 plays a special role in the complex. If we change the indexing then that role is give to a different object, and so we have a different complex.

The connecting arrows $\alpha_\bullet$ must interact in a simple way.

Given objects B and C of ***Mod***-R the zero arrow

$$B \xrightarrow{\;0\;} C$$

sends each element of B to the zero element of C. (Strictly speaking, we should label each zero arrow with its source and target, but that gets a bit cumbersome.) Consecutive connecting arrows in the complex are required to compose to 0, that is

$$\alpha_n \circ \alpha_{n+1} = 0$$

for each $n \in \mathbb{Z}$.

Each complex A is an object of the new category $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$. Let's call it an Object to distinguish it from an object of ***Mod***-R.

Given two Objects A and B, complexes from ***Mod***-R, what is an Arrow

$$\mathsf{A} \xrightarrow{\;\mathsf{f}\;} \mathsf{B}$$

in $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$? It is an indexed family of arrows of $\boldsymbol{Mod}\text{-}R$

$$\begin{array}{ccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} \\ \alpha_{n+1}\downarrow & & \downarrow\beta_{n+1} \\ A_n & \xrightarrow{f_n} & B_n \\ \alpha_n\downarrow & & \downarrow\beta_n \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} \end{array}$$

such that at each step the corresponding square commutes, that is

$$f_n \circ \alpha_{n+1} = \beta_{n+1} \circ f_{n+1}$$

for each $n \in \mathbb{Z}$. (This is why we choose to write α rather than the customary d for the connecting arrows.)

The structure of $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$ is now more or less obvious.

Given a pair

$$\mathsf{A} \xrightarrow{\mathsf{f}} \mathsf{B} \xrightarrow{\mathsf{g}} \mathsf{C}$$

of Arrows, we have commuting squares

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \alpha_{n+1}\downarrow & & \downarrow\beta_{n+1} & & \downarrow\gamma_{n+1} \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \end{array} \qquad \begin{array}{ccc} A_{n+1} & \xrightarrow{g_{n+1}\circ f_{n+1}} & C_{n+1} \\ \alpha_{n+1}\downarrow & & \downarrow\gamma_{n+1} \\ A_n & \xrightarrow{g_n\circ f_n} & C_n \end{array}$$

as on the left, and these horizontal components of these compose in $\boldsymbol{Mod}\text{-}R$ to give commuting squares as on the right. All these composites in $\boldsymbol{Mod}\text{-}R$ provide the composite

$$\mathsf{A} \xrightarrow{\mathsf{g}\circ\mathsf{f}} \mathsf{C}$$

in $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$.

Verifying that $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$ is a category is now straight forward. It is a simple exercise in diagram chasing which we look at in Section 2.1. ■

The gadget $\boldsymbol{Ch}(\boldsymbol{Mod}\text{-}R)$ is central to homology theory. But what is the point of setting it up as a category? It is because some of its properties can be analysed by arrow-theoretic methods without getting inside the internal structure of its Objects and Arrows. This is beyond the scope of this book, but not that far beyond.

Exercises

1.4.1 Try to get to grips with Example 1.12. To help with this consider the particular cases where S is a 2-element set partially ordered in the two different ways.

1.4.2 Try to get to grips with Example 1.13. To help with this consider the complexes A where only A_{-1}, A_0, A_1 are non-trivial.

1.5 Two simple categories and a bonus

As we are going to see in a moment, every monoid is a category with a simple object structure, and ever preset is a category with a simple arrow structure. Every category is a certain kind of amalgam of monoids and presets. Thus whenever you meet a new categorical notion it is worth trying it out on monoids and presets. Sometimes this gives a little bit of insight and sometimes not.

1.14 EXAMPLE. Each monoid $(R, \cdot, 1)$ can be viewed as a category with just one object. It doesn't matter what this object is, and it doesn't have any internal structure. Let's use

$$\bigstar$$

for this symbolic object. Don't confuse this with the monoid R.

For each $r \in R$ there is an arrow

$$\bigstar \xrightarrow{\ r\ } \bigstar$$

and again this has no internal structure. In other words the arrows of the category are the elements of R.

Composition of arrows is just the carried operation of R.

$$\begin{array}{ccccc} & & \bigstar & & \\ & \overset{r}{\nearrow} & & \overset{s}{\searrow} & \\ \bigstar & & \xrightarrow[s \circ r = sr]{} & & \bigstar \end{array}$$

The identity arrow

$$\boldsymbol{id}_{\bigstar} = 1$$

is just the unit of R.

This construction does produce a category because the operation on R is associative and 1 is a unit. ■

On its own this example is rather trite, but later we will add to it to illustrate several aspects of category theory.

1.15 EXAMPLE. Each pre-ordered set $(S, \leq)$ can be viewed as a category.

The objects are the elements

$$i, j, k, \ldots$$

of S.

Given a pair of objects i, j there is an arrow

$$i \longrightarrow j$$

precisely when $i \leq j$. Thus between any two objects there is at most one arrow. The existence of the arrow indicates a comparison between the objects. It is sometimes convenient to write

$$i \xrightarrow{\ (j,i)\ } j$$

for this arrow.

We have

$$\boldsymbol{id}_i = (i, i) \quad \text{since} \quad i \le i$$
$$(k, j) \circ (j, i) = (k, i) \quad \text{since} \quad i \le j \le k \Rightarrow i \le k$$

so the construction does give a category. ■

Again this example looks rather feeble, but again we will add to it later to produce more interesting structures. In fact, we have already used a preset as a category. Perhaps you can spot where we did this.

In Section 1.3 and 1.4 we saw various ways of producing a new category out of old categories. There is one very simple example of such a construction. This could have been done earlier, but we have saved it until last (at least for this chapter).

1.16 EXAMPLE. Each category $\boldsymbol{C}$ is a collection of objects and a collection of arrows with certain properties. In particular, each arrow

$$A \xrightarrow{\ f\ } B$$

has an assigned source and an assigned target. A formal trick converts $\boldsymbol{C}$ into another category $\boldsymbol{C}^{\mathrm{op}}$ called the opposite of $\boldsymbol{C}$. This category $\boldsymbol{C}^{\mathrm{op}}$ has the same objects as $\boldsymbol{C}$. Each arrow f of $\boldsymbol{C}$, as above, is turned into its formal dual

$$B \xrightarrow{\ f^{\mathrm{op}}\ } A$$

to produce an arrow of $\boldsymbol{C}^{\mathrm{op}}$. The formal composition of these formal arrows is defined by

$$f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$$

for each composible pair

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of arrows from the parent category $\boldsymbol{C}$.

A routine exercise (which you should go through at least once in your life) shows that $\boldsymbol{C}^{\mathrm{op}}$ is a category. ■

The process $f \longmapsto f^{\mathrm{op}}$ we doesn't actually do anything to the arrows. We merely decide that the words 'source' and 'target' should mean their exact opposites. Thus the change is merely formal rather than actual.

This trick shows there is a lot of duality in category theory. Notions often come in dual pairs

dog god

where a dog of a category $\boldsymbol{C}$ is nothing more than a god of its opposite $\boldsymbol{C}^{\mathrm{op}}$. We will see many example of this.

Sometimes the opposite category $\boldsymbol{C}^{\mathrm{op}}$ has properties rather different to the parent $\boldsymbol{C}$. For instance $\boldsymbol{Set}^{\mathrm{op}}$ is isomorphic to the category of complete, atomic boolean algebra and complete morphisms. As a simpler version of this the opposite of the category of finite sets is the category of finite boolean algebras. (Both of these observations are instances of Stone duality.)

Exercises

1.5.1 (a) Let R and S be monoids viewed as categories. What is the product category?
(a) Let R and S be presets viewed as categories. What is the product category?

1.5.2 Let S be a preset viewed as a category.
For an arbitrary element $s \in S$ what are the slice categories $(S \downarrow s)$ and $(s \downarrow S)$?
For an arbitrary elements $s, t \in S$ what is the butty category $(s \downarrow S \downarrow t)$? Be careful.

1.5.3 Each poset S is a category. What is the opposite S^{op}?
Each monoid R is a category. What is the opposite R^{op}?

1.5.4 Give a short and precise description of the category constructed in Exercise 1.3.3.

[*There are 28 exercises in this chapter*]

2
Basic gadgetry

As the title says, in this chapter we describe some of the basic gadgets of category theory. We meet notions such as

diagram
monic   epic
split monic   split epic
isomorphism
initial   final
wedge
product   coproduct
equalizer   coequalizer
pullback   pushout
universal solution

some of which are discussed only informally.

All of these notions are important, and have to be put somewhere in the book. It is more convenient to have them together in one place, and here seems the 'logical' place to put them.

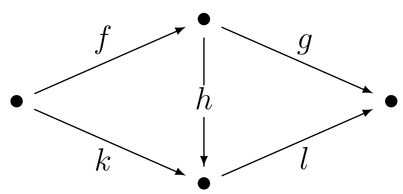
However, that does not mean you should plod through this chapter section by section. I suggest you get a rough idea of the notions involved, and then go to Chapter 3 (which discusses more important ideas). You can come back to this chapter several times to fill in the missing details.

2.1 Diagram chasing

As in many part of mathematics, in category theory we sometimes have to show that two things are equal. We don't often, or even ever, have to show that two objects are the same, but we often have to show that two arrows are equal. The main technique for doing that is diagram chasing.

Roughly speaking, a diagram in a category is a collection of objects together with a collection of arrows between these objects.

For instance, the following diagram



has four objects (unnamed) and five arrows f, g, h, k, l. There are also five composite arrows

$$g \circ f \quad h \circ f \quad l \circ h \circ f \quad l \circ h \quad l \circ k$$

and some of these may be equal.

This diagram has three cells; the left hand triangle, the right hand triangle, and the outer rectangle (lozenge). Some of these cells may commute.

- The left hand triangle commutes if $h \circ f = k$.
- The right hand triangle commutes if $l \circ h = g$.
- The outer cell commutes if $g \circ f = l \circ k$.

Roughly speaking a diagram chase is a process by which we show that a particular cell commutes knowing that other cells commute and using certain other properties of the diagram.

2.1 EXAMPLE. For the diagram above, if the two triangles commute then the outside cell commutes. We are given that

$$h \circ f = k \qquad l \circ h = g$$

and we must show that

$$g \circ f = l \circ k$$

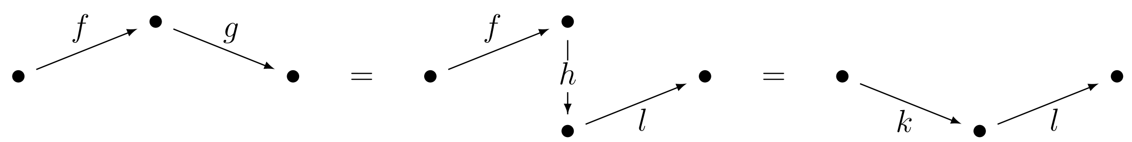
holds.

We can do that by equational reasoning. Thus

$$g \circ f = (l \circ h) \circ f = l \circ (h \circ f) = l \circ k$$

is a more or less trivial calculation.

However, it is more common to do this by chasing round the diagram and noting that certain composites are equal. Thus



is what we trace out with our pencil and think whilst we are doing it. ■

Since this example is so trivial it doesn't matter which method we use. However, for larger diagrams the chase is often easier to explain when we are talking to someone. This is a bit unfortunate since no-one has yet devised a method of writing down a diagram chase in an efficient manner.

Even with this simple diagram there are other problems which might arise.

2.2 EXAMPLE. Consider the small diagram above.

(a) If the outer cell commutes then neither of the two triangles need commute. This is because we could replace h by some other arrow without altering f, g, k, l.

(b) If the outer cell commutes and the right hand triangle commutes, then the left hand triangle need not commute. It is not hard to find an appropriate example in ***Set***. Simply let l collapse a lot of elements to the same element.

(c) If the outer cell commutes, the right hand triangle commutes, *and* l is MONIC, then the left hand triangle also commutes. We deal with this in the next section.

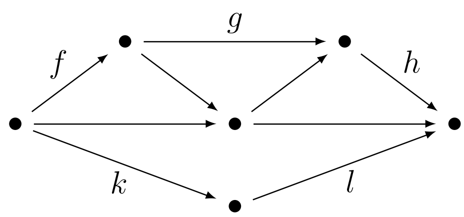
(b) If the outer cell commutes and the left hand triangle commutes, then the right hand triangle need not commute. It is not hard to find an appropriate example in ***Set***. Simply let the range of f be a small part of its target.

(e) If the outer cell commutes, the left hand triangle commutes, *and* f is EPIC, then the right hand triangle also commutes. We deal with this in the next section. ■

In due course we will take part in many diagram chases. For now I leave you with a couple of simple exercises and one that you might think is a bit devilish.

Exercises

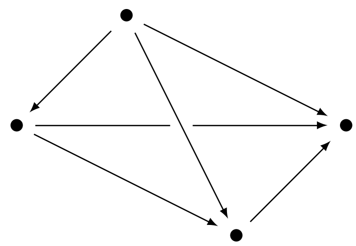
2.1.1 Consider the following diagram.



Show that if the four inner triangles commute, then so does the outer cell.

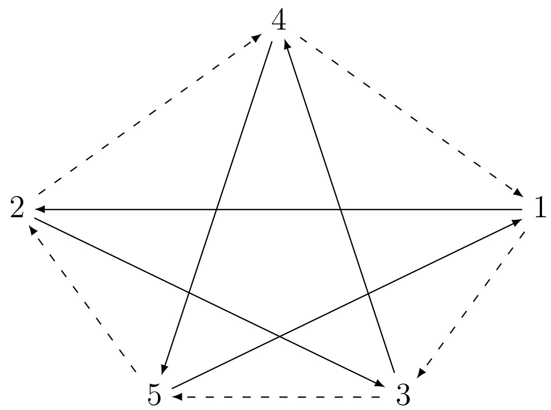
Write down the argument in the form of equational reasoning and in a pictorial form of the diagram chase.

2.1.2 Consider the triangular pyramid of arrows.



Given that the three other faces commute, show that the back face also commutes.

2.1.3 Consider a pentagram inscribed in a pentagon.



Suppose that the triangles

$$123 \quad 345 \quad 512 \quad 234 \quad 451$$

commute. Show that a trip twice round the pentagram is equivalent to a trip once round the pentagon.[1]

2.2 Monics and epics

It is clear, as they say, that in the great categorical menagerie all arrows have equal status, but some have more status than others. In this section we look at some of these special arrows.

2.3 DEFINITION. In a category an arrow

$$B \xrightarrow{\;m\;} A \qquad\qquad A \xrightarrow{\;e\;} B$$

is, respectively,

monic epic

if for each parallel pair of arrows

$$X \underset{g}{\overset{f}{\rightrightarrows}} B \qquad\qquad B \underset{g}{\overset{f}{\rightrightarrows}} X$$

we have

$$m \circ f = m \circ g \Longrightarrow f = g \qquad\qquad f \circ e = g \circ e \Longrightarrow x = y$$

as appropriate. ■

What are we getting at here? The following example gives the precursors of monics and epics, but you mustn't read too much into it. Later we will see that it can suggest quite a false story.

2.4 EXAMPLE. Consider one of the categories of structured sets. Thus each arrow is (carried by) a total function between the carriers of the two objects.

(m) If

$$B \xrightarrow{\;m\;} A$$

is injective as a function then it is monic as an arrow. To see this suppose

$$m \circ f = m \circ g$$

for some parallel pair

$$X \underset{g}{\overset{f}{\rightrightarrows}} B$$

[1] It isn't well know that a pentagram can bring you good luck. All you have to do is hop round it twice without once thinking of a frog.

of arrows. We require $f = g$. Thus, since f and g are total functions it suffices to show

$$f(x) = g(x)$$

for each $x \in X$.

We have

$$m(f(x)) = (m \circ f)(x) = (m \circ g)(x) = m(g(x))$$

for each such x. But m is injective, that is

$$m(b_1) = m(b_2) \Longrightarrow b_1 = b_2$$

for $b_1, b_2 \in B$, to give the required result.

(e) If

$$A \xrightarrow{\quad e \quad} B$$

is surjective as a function then it is epic as an arrow. To see this suppose

$$f \circ e = g \circ e$$

for some parallel pair

$$B \underset{g}{\overset{f}{\rightrightarrows}} X$$

of arrows. We require $f = g$, that is

$$f(b) = g(b)$$

for each $b \in B$.

Consider any such $b \in B$. Since e is surjective we have $b = e(a)$ for some $a \in A$. But now

$$f(b) = f(e(a)) = (f \circ e)(a) = (g \circ e)(a) = g(e(a)) = g(b)$$

to give the required result. ■

These examples show that

$$\text{injective} \Longrightarrow \text{monic} \qquad \text{surjective} \Longrightarrow \text{epic}$$

for appropriately nice categories. However, you are warned. Even in nice categories these implications can be far from equivalences.

There are several quite common categories of structured sets in which an epic arrow need not be surjective. Roughly speaking an arrow

$$A \xrightarrow{\quad e \quad} B$$

is epic if the range $e[A]$ of e is a 'large part' of B. Exercises 2.2.5, 2.2.6, and 2.2.8 give some examples of this. Of course, in many (most) categories the notions of 'injective arrow' and 'surjective arrow' don't make sense.

Monics and epics are those arrows that can be cancelled on one side or the other. If an arrow has a 1-sided inverse than it can be cancelled on the appropriate side. This gives us special classes of monics and epics.

2.5 DEFINITION. A pair of arrows

$$B \xrightarrow{\ s\ } A \qquad\qquad A \xrightarrow{\ r\ } B$$

such that

$$r \circ s = \boldsymbol{id}_B$$

are a

section retraction

respectively (as indicated by the initial letter). ■

It is not too hard to show that each section is monic and each retraction is epic. For this reason each such arrow is often referred to has a

split monic split epic

respectively. In some ways this is better terminology.

As we said at the beginning of Section 2.1 rarely do we need to show that two objects of a category are the same. But we often have to show they are isomorphic.

2.6 DEFINITION. A couple of arrows

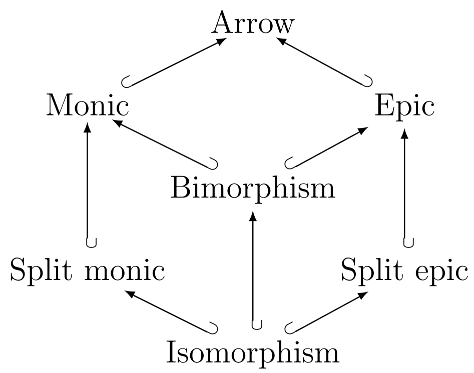
$$B \xrightarrow{\ g\ } A \qquad\qquad A \xrightarrow{\ f\ } B$$

such that

$$g \circ f = \boldsymbol{id}_A \qquad\qquad f \circ g = \boldsymbol{id}_B$$

form an inverse pair of isomorphisms and each component is an isomorphism. ■

An arrow is an isomorphism if it has a 2-sided inverse, and hence each isomorphism is both a split monic and a split epic. This gives us a short hierarchy of kinds of arrows.



It is easy to see that an arrow that is both a split monic and a split epic is automatically an isomorphism (and there is a stronger result). However, an arrow that is both monic and epic need not be an isomorphism.

2.7 DEFINITION. A bimorphism is an arrow that is both monic and epic.

Each isomorphism is a bimorphism, but there can be bimorphisms which are not isomorphisms.

A category is balanced if each bimorphism is an isomorphism. ■

Monic and epics often have a role to play in a diagram chase. There are a couple of exercises to illustrate this.

Exercises

2.2.1 (a) Show that

$$\text{section} \Longrightarrow \text{monic} \qquad \text{retraction} \Longrightarrow \text{epic}$$
$$\text{section+epic} \Longrightarrow \text{iso} \qquad \text{retraction+monic} \Longrightarrow \text{iso}$$

that is show that if an arrow satisfies the hypothesis then it satisfies the conclusion.

(b) Show that if arrows

$$B \xrightarrow{\ g\ } A \xrightarrow{\ f\ } B \xrightarrow{\ h\ } A$$

satisfy

$$h \circ f = \boldsymbol{id}_A \qquad f \circ g = \boldsymbol{id}_B$$

then $g = h$, and each arrow is an isomorphism.

2.2.2 Consider a preset as a category.

(a) Show that every arrow is a bimorphism.

(b) When is the preset balanced?

2.2.3 Consider a monoid as a category.

Which of the elements (when viewed as arrows) are monic, epic, a retraction, a section, iso?

What is a balanced monoid?

2.2.4 Consider a composible pair of arrows.

$$\bullet \xrightarrow{\ m\ } \bullet \xrightarrow{\ n\ } \bullet$$

Show that if both m and n are monic, then so is the composite $n \circ m$.

Show that if the composite $n \circ m$ is monic, then so is m.

Find an example where the composite $n \circ m$ is monic but n is not.

State the corresponding results for epics.

Obtain similar results (where possible) for the other classes of arrows discussed in this section.

2.2.5 Consider the category ***Mon*** of monoids, and view $\mathbb{N}$ and $\mathbb{Z}$ as additively written monoids. Show that the insertion

$$\mathbb{N} \xhookrightarrow{\ e\ } \mathbb{Z}$$

is epic.

2.2.6 Consider the category ***Rng*** of rings. Show that the insertion

$$\mathbb{Z} \xhookrightarrow{\ e\ } \mathbb{Q}$$

is epic.

2.2.7 (a) Let $\boldsymbol{C}$ be a category of structured sets. Suppose $\boldsymbol{C}$ has a particular object S which has a special element $\star$ (usually not part of the official furnishings) such that for each object A and element $a \in A$, there is a unique arrow

$$S \xrightarrow{\alpha} A$$

with $\alpha(\star) = a$.

(This is a particular instance of a more general notion called a selector, or sometimes a generator.)

Show that in $\boldsymbol{C}$ each monic is injective.

(b) Show that in

$$\boldsymbol{Set},\ \boldsymbol{Pos},\ \boldsymbol{Top},\ \boldsymbol{Mon},\ \boldsymbol{Grp},\ \boldsymbol{Rng},\ \boldsymbol{Set}\text{-}R$$

each monic is injective.

2.2.8 (a) In $\boldsymbol{Top}$ an isomorphism is usually called something else. What is the name used?

Show that in $\boldsymbol{Top}$ each monic is injective.

Show that an arrow of $\boldsymbol{Top}$ that is bijective as a function need not be an isomorphism.

(b) Let $\boldsymbol{Top}_2$ be the category of hausdorff spaces and continuous maps.

Show that the insertion

$$\mathbb{Q} \xhookrightarrow{e} \mathbb{R}$$

is epic in this category.

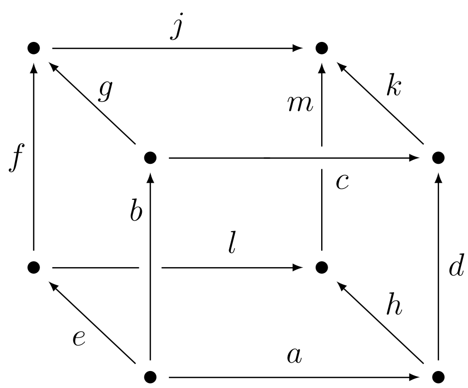
More generally, show that if

$$T \xrightarrow{\epsilon} S$$

is an arrow of $\boldsymbol{Top}_2$ where the range $\epsilon[T]$ is dense in the target S, then ϵ is epic.

If you are brave you can show that this result does not hold for $\boldsymbol{Top}_1$.

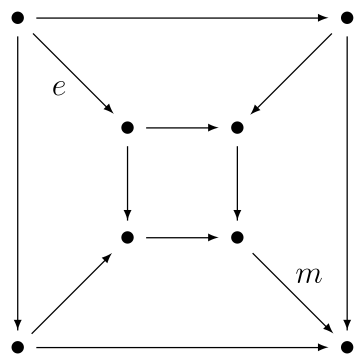
2.2.9 Consider the following cube of arrows $a, b, \ldots, l, m$.



(e) Show that if e is epic and if the other five faces commute, then the back face commutes.

(m) Show that if m is monic and if the other five faces commute, then the bottom face commutes.

2.2.10 Consider the diagram



and suppose the four trapeziums commute.

(a) Show that if the inner square commutes then so does the outer square.

(b) Conversely, show that if e is epic, m is monic, and the outer square commutes, then so does the inner square.

2.3 Simple limits and colimits

Limits and colimits (of the categorical kind) occur all over mathematics, and concrete examples of these notions were being used before category theory was invented. Different areas of mathematics tend to use different terminology, mainly because of historical reasons than natural cussedness, but that is not too important. It was one of the first achievements of category theory to codify and extract the essential content of these notions.

In the next four section we look at the basic examples of these notions, as given in Table 2.1. These examples can be set in a more general context, but we don't attempt that just yet. However, we can look at a particular case of the general notion which you already know about.

Let S be a poset viewed as a category. It is usual to think of the comparison as progressing upwards, that is $i \le j$ means that i is below j. However, to fit in with the categorical picture we think of the comparison as progressing to the right. Thus

$$i \le j \qquad\qquad i \longrightarrow j$$

mean the same thing.

Let X be a subset of S. A

left solution   right solution

for X is an element $a \in S$ such that

$$a \le x \qquad\qquad x \le a$$

for each $x \in X$. A

limit   colimit

is a 'best possible' solution on the appropriate side. In other words, it is a solution a such that

$$b \le a \qquad\qquad a \le b$$

for each solution b of the appropriate handedness.

You should recognize these notions under different names.

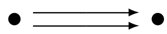
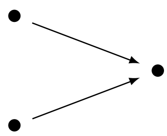
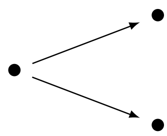
	Limit	Template	Colimit
(1)	final object		initial object
(2)	binary product		binary coproduct
(3)	equalizer		co-equalizer
(4)	pullback		
(5)			pushout

Table 2.1: Some simple limits and colimits

Exercises

2.3.1 What are these notions for a poset usually called?
What happens if X is empty?
What happens if X is a singleton?
What happens if X is a pair of elements?
What differences might occur if S is a preset?

2.4 Initial and final objects

In some categories some objects play special roles because they take up extreme positions.

2.8 DEFINITION. An object S of a category $\boldsymbol{C}$ is, respectively

initial final

if for each object A there is a *unique* arrow

$$S \longrightarrow A \qquad\qquad A \longrightarrow S$$

as indicated. Here the uniqueness is important.

Sometimes a final object is said to be terminal. ■

You may not know this terminology, but you already know some examples of these two notions.

2.9 EXAMPLE. (a) Consider the category ***Set*** of sets and let

$$\mathsf{1} = \{\star\}$$

be a singleton set. For each set A there is a unique arrow

$$A \longrightarrow \mathsf{1}$$

the function that collapses everything to $\star$. Thus $\mathsf{1}$ is a final object of ***Set***.

Let $\emptyset$ be the empty set. (What else can it be?) You will probably have to think about this, but for each set A there is a unique arrow

$$\emptyset \longrightarrow A$$

(since the function doesn't have any requirements that it must satisfy). Thus $\emptyset$ is an initial object of ***Set***.

(b) Consider the category ***AGrp*** of abelian groups. Let O be the trivial group. For each abelian group A, the group O is uniquely embedded in A, and there is a unique morphism

$$A \longrightarrow O$$

to O. Thus O is both initial and final in ***AGrp***. ■

A category ***C*** may or may not have an initial object. It may or may not have a final object. It can have one without the other. It can have both. If it has both then these objects may or may not be the same. An object that is both initial and final is often called a zero object.

It is easy to show that any two initial objects of a category are uniquely isomorphic. For this reason we usually speak of *the* initial object rather than an initial object. In the same way and two final objects are uniquely isomorphic, and we speak of *the* final object.

It common to let $\mathsf{1}$ be the final object of a category (assuming this does exists). Because of certain special cases where they arise quite naturally, an arrow

$$\mathsf{1} \xrightarrow{\ a\ } A$$

to an object A is a global element of A. For instance, in ***Set*** these pick out the members of a set in the usual sense. In more structured categories these can pick out a special kind of member of an object.

Exercises

2.4.1 Show that in a category any two initial objects are uniquely isomorphic. That is, if I, J are two initial objects, then there is a unique arrow $I \longrightarrow J$, and this is an isomorphism.

State and prove the dual result concerning final objects.

2.4.2 Suppose that I is initial in ***C***. Show that each ***C***-arrow of the form

$$A \longrightarrow I$$

is a retraction, and prove the corresponding result for final objects. Hence show that if $\boldsymbol{C}$ has both an initial object I and a final object F and there is an arrow

$$F \longrightarrow I$$

then I and F are isomorphic. In such a case we have a zero object.

2.4.3 Show that the category ***Pno*** has an interesting initial object but a boring final object. What are these objects?

2.4.4 Show that the category ***Grp*** of groups has both an initial and a final object, and these are the same.

Show that the category ***Rng*** of unital rings has both an initial and a final object, and these are not the same.

What about the categories ***Idm*** and ***Fld*** of integral domains and fields?

2.4.5 Show that for each set A there is a bijection between the elements of A and the ***Set***-arrows $1 \longrightarrow A$.

Show further that for each pair of ***Set***-arrows

$$A \xrightarrow{\ f\ } B \qquad\qquad 1 \longrightarrow A$$

where the second represents the element $a \in A$, the composite

$$1 \longrightarrow A \xrightarrow{\ f\ } B$$

represents the element $f(a) \in B$.

2.4.6 Let S be a poset and consider the category $\widehat{S}$ of presheaves over S (as described in Example 1.13).

(a) Show that this category has a final object 1.

(b) Show that for a presheaf $A = (\mathsf{A}, \mathcal{A})$ over S a global element $1 \longrightarrow A$ is a kind of choice function for the family $\mathcal{A}$. It 'threads' its way through the component sets $A(s)$. You should make precise this notion of 'thread'.

2.5 Products and coproducts

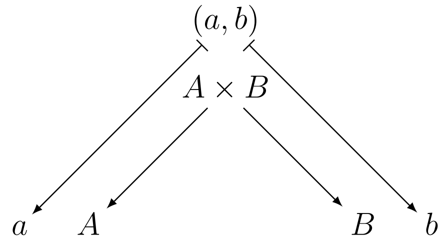
We all know how to form the cartesian product

$$A \times B$$

of two sets A and B, the set of all ordered pairs

$$(a, b)$$

for $a \in A$ and $b \in B$. We also know that often when A and B carry structures of a similar kind then the product $A \times B$ can be furnished with the same kind of structure. Groups, rings, topological spaces, ... provide examples of this construction. In these cases we find that the two projections



are arrows in the appropriate category.

There is also a dual process which is not so clear.

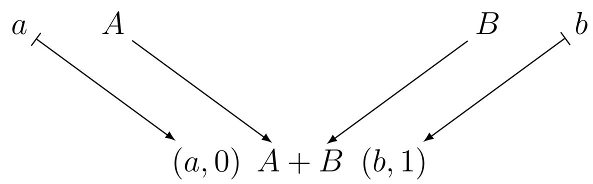
Given two sets A and B we can form the disjoint union (sum)

$$A + B$$

of the sets, a larger set that includes copies of A and B with minimal interference. Technically, we tag the elements of A and B to remember their origin, and take the union of the tagged versions of the sets.

$$A + B = \big(A \times \{0\}\big) \cup \big(B \times \{1\}\big)$$

We then find that the two embeddings



locate disjoint copies of the parent sets within the sum.

What about this dual process for structured sets?

Given two groups, or two rings, A and B, can we find a group or ring that includes copies of A and B with minimal interference? It can be done but we have to think a bit before we spot the construction. If you don't know how to do this then you should worry about it for a while.

What we can do here is look at a variant of this dual problem.

Given two *abelian* groups A and B we wish to find an *abelian* group that includes copies of A and B with minimal interference. This construction is much easier.

Let's suppose the two abelian groups are written multiplicatively. Thus

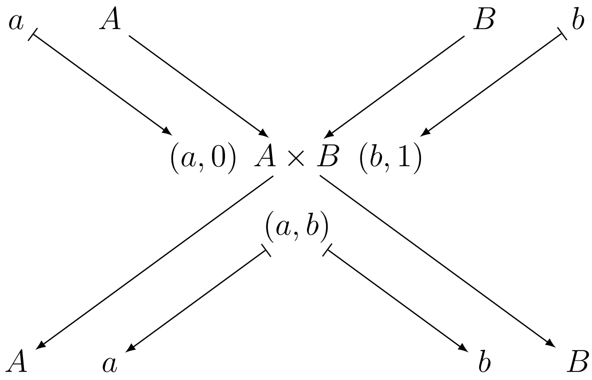
$$(A, \cdot, 1) \qquad (B, \cdot, 1)$$

are the two structures. Let

$$A \times B$$

be the cartesian product of these two groups. This, of course, is also an abelian group.

We have four morphisms.



The lower two are the projections. The upper two are the embeddings which solve our problem.

There is something going on here, isn't there? Category theory can help to explain this. In all cases we are looking for a universal solution to a particular kind of problem which comes in two forms, a left handed version and a right handed version.

For the remainder of this section we fix a category $\boldsymbol{C}$, and we fix a pair of objects A and B of $\boldsymbol{C}$. We place these as

$$A$$

$$B$$

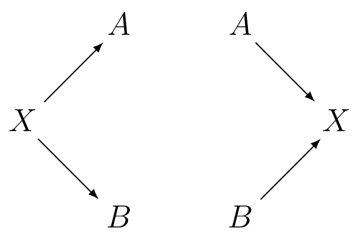
to help with various diagrams we draw. (Just why we do this will become clear when we look at more general constructions in Chapter 5.)

We are going to look at the left handed version and the right handed version of the problem in parallel. Thus each definition and result that we give is really two definitions or results in one. The left hand side gives the left version and the right hand side gives the right version (in the sense of 'dexterous' not 'correct').

2.10 DEFINITION. Given a pair A, B of objects of a category $\boldsymbol{C}$, a wedge

to   from

the pair A, B is an object X together with a pair of arrows



in the parent category $\boldsymbol{C}$. ■

Often a wedge of this kind is called a

cone   cocone

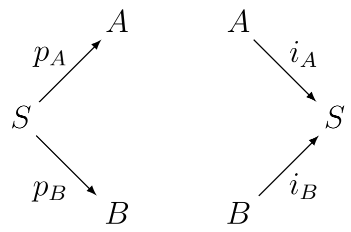
depending on which side of the pair it lies. However, it is hardly worth remembering which is which so we call both a wedge.

For any given pair there may be many wedges on one side or the other. We look for a 'best possible' wedge, one that is as 'near to' the pair as possible. Technically, we look for a universal wedge. You will probably need to read this next definition several times. Remember also that it is two definitions in one, so in the first instance concentrate on one side or other.

2.11 DEFINITION. Given a pair A, B of objects of a category $\boldsymbol{C}$, a

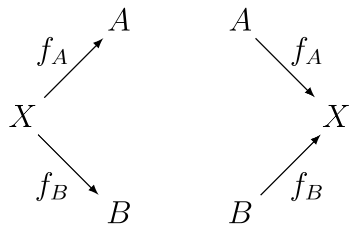
product coproduct

of the pair is a wedge



with the following universal property.

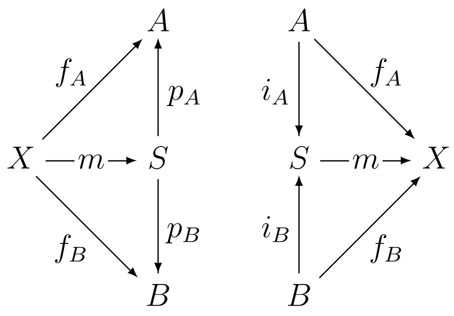
For each wedge



there is a *unique* arrow

$$X \xrightarrow{\ m\ } S \qquad\qquad S \xrightarrow{\ m\ } X$$

such that



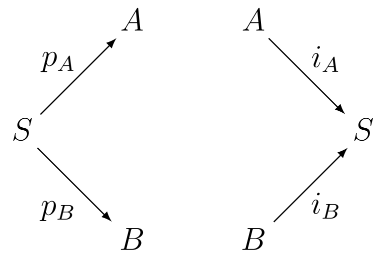
commutes. This arrow m is the mediating arrow (or mediator) for the wedge on X. ■

There are a couple of things about this definition that you should notice.

Firstly, a product or coproduct is not just an object. It is an object furnished with a pair of arrows.

Secondly, the mediator is unique for the given wedge on X. This has some important consequences.

2.12 LEMMA. *Let*



be a

product coproduct

wedge in the category $\boldsymbol{C}$. *Let*

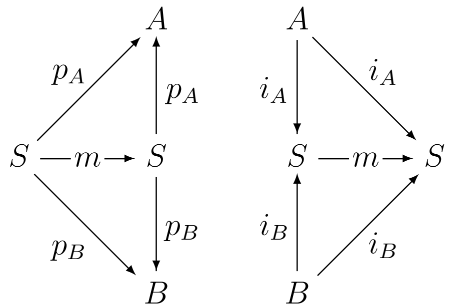
$$S \xrightarrow{\;k\;} S$$

be any endo-arrow of S *for which*

$$p_\bullet = p_\bullet \circ k \qquad k \circ i_\bullet = i_\bullet$$

where $\bullet$ *ranges over* $\{A, B\}$. *Then* $k = \boldsymbol{id}_S$.

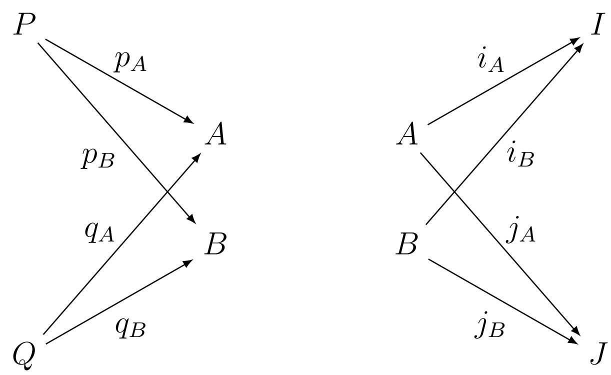
Proof. We consider the given wedge both as a special wedge and an arbitrary wedge. Thus there is a unique arrow, the mediator, such that



commutes. Since $\boldsymbol{id}_S$ makes these diagram commute we see that $m = \boldsymbol{id}_S$. But the given arrow k makes this diagram commute, and hence $k = \boldsymbol{id}_S$. ■

This result leads to the essential uniqueness of the universal solution.

2.13 LEMMA. *For object* A, B *in a category* $\boldsymbol{C}$ *let*



be a pair of

product	coproduct

wedges. Then

P, Q	I, J

are uniquely isomorphic over the wedges. In other words, there are unique arrows

$$\begin{array}{ccc} P & \qquad & I \\ f \downarrow \uparrow g & & f \downarrow \uparrow g \\ Q & & J \end{array}$$

such that

$$\begin{array}{llll} (1) & p_A = q_A \circ f \quad p_B = q_B \circ f & (3) & i_A = g \circ j_A \quad i_B = g \circ j_B \\ (2) & q_A = p_A \circ g \quad q_B = p_B \circ g & (4) & j_A = f \circ i_A \quad j_B = f \circ i_B \end{array}$$

and in particular f and g are an inverse pair of isomorphisms.

Proof. We look at the product, left hand, version and leave the coproduct version as an exercise.

The object Q and the pair q_A, q_B form a product wedge for A, B. The object P and the pair p_A, p_B form an arbitrary wedge for A, B. Thus there is a unique mediator f satisfying (1).

By reversing the roles of P and Q we see there is a unique mediator satisfying (2).

From (1, 2) we have

$$p_\bullet \circ g \circ f = q_\bullet \circ f = p_\bullet$$

where $\bullet$ is each of A and B. A use of Lemma 2.12 gives

$$g \circ f = \boldsymbol{id}_P$$

and

$$f \circ g = \boldsymbol{id}_Q$$

follows by a similar argument. ■

The left hand part of this result shows that if a pair of objects has a product then that gadget is essentially unique. Thus we often speak of *the* product of a pair. Similarly, from the right hand part of this result, we speak of *the* coproduct of a pair of objects.

Of course, in some categories not all products or coproducts exist. A pair of objects may have one of these gadgets without the other. The pair may have both, or it may have neither. The existence of products and coproducts in som eparticular categories is looked at in Exercises 2.5.1 and 2.5.2. For now we look at a simple result which relates the categorical notions to the concrete construction we discussed at the beginning of this section.

2.14 LEMMA. *Let A and B be a pair of sets. Then the*

cartesian product	*disjoint union*
$A \times B$	$A + B$

furnished with the canonical functions forms the

product *coproduct*

*of the pair in **Set**.*

Proof. We look at the right hand, coproduct, version and leave the left hand version as an exercise.

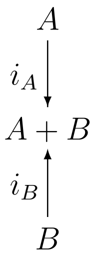
The elements of

$$A + B$$

are of two kinds

$$(a, 0) \text{ for } a \in A \qquad (b, 0) \text{ for } b \in B$$

where the tag 0 or 1 records the parent of the element. The canonical embeddings

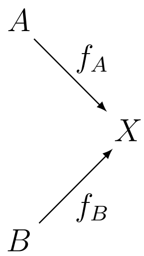


$$i_A(a) = (a, 0)$$

$$i_B(b) = (b, 1)$$

merely tag the input. We must show that these form a coproduct wedge.

Consider any wedge



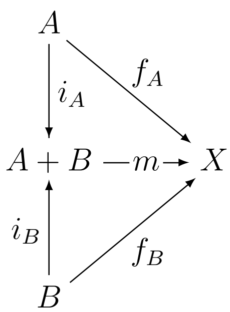
to some set X. We define

$$A + B \xrightarrow{\ m\ } X$$

by

$$m(a, 0) = f_A(a) \qquad m(b, 1) = f_B(b)$$

for $a \in A$ and $b \in B$. Trivially, the diagram



commutes. We must show that m is the only function that makes this diagram commute.

Consider any function

$$A + B \xrightarrow{\ h\ } X$$

with

$$h \circ i_A = f_A \qquad h \circ i_B = f_B$$

that is h makes the diagram commute. For $a \in A$ and $b \in B$ we have

$$\begin{aligned} h(a,0) &= h(i_A(a)) = (h \circ i_A)(a) = f_A(a) = m(a,0) \\ h(b,1) &= h(i_B(b)) = (h \circ i_B)(b) = f_B(b) = m(b,1) \end{aligned}$$

and hence $h = m$, as required. ■

We conclude this section with a few remarks on terminology and notation.

Strictly speaking the two notions we have described here are the

binary product   binary coproduct

respectively. There are more general notions that deal with an arbitrary number of objects, not just two.

The notations

$$A \times B \qquad A + B$$

$$A \sqcap B \qquad A \amalg B$$

are used for the object associated with the constructed wedge. Of course, a use of '×' does *not* mean that the object is constructed using a cartesian product.

In some categories the product and the coproduct produce the same object (but not the same structuring arrows). For such cases

$$A \oplus B$$

is a common notation, and this is often referred to as a biproduct. Yes, I really mean that.

Exercises

2.5.1 Choose a selection of the categories

Set, ***CMon***, ***Mon***, ***AGrp***, ***Grp***, ***CRng***, ***Rng***, ***Set***-R, ***Mod***-R, ***Pos***, ***Top***

and make sure you include ***Top***.

Show that each category has all binary products, and that each is given by a cartesian product with the obvious projections.

2.5.2 Show that each of

Set, ***Pos***, ***CMon***, ***AGrp***, ***Set***-R, ***Mod***-R, ***Top***

has all binary coproducts.

Can you spot any similarities between the various constructions?

2.5.3 Each poset is a category.
What is the product of two elements?
What is the coproduct of two elements?

2.5.4 Show that $\boldsymbol{Set}_\perp$ has all binary products and all binary coproducts.

2.5.5 Consider the category of sets with a distinguished subset.
Does this category have binary products.
Does it have binary coproducts.

2.5.6 Consider the category ***RelA*** of sets and relations of Example 1.7. Show that this has all binary products and coproducts and give a precise description of these. (The product is *not* given by a cartesian product.)

2.5.7 Let $\boldsymbol{C}$ be a category with a final object $\mathbb{1}$ and all binary products.
(a) Show that for each object A the three objects $\mathbb{1} \times A, A, A \times \mathbb{1}$ are isomorphic.
(b) Show that for each triple A, B, C of objects, the two objects

$$(A \times B) \times C \quad A \times (B \times C)$$

are isomorphic.

2.5.8 Let $\boldsymbol{C}$ be a category with all binary products and coproducts. For objects A, B, C let

$$L = A \times C + B \times C \qquad R = (A + B) \times C$$

to form two more objects.
Show there is an arrow

$$L \longrightarrow R$$

and find an example to show that there need not be an arrow $R \longrightarrow L$.

2.5.9 In the category ***AGrp*** of abelian groups the cartesian product of two objects implements both the product and the coproduct.
Does this work in ***Grp***?
Consider the cartesian product $A \times B$ of two abelian groups. This gives the coproduct of A and B in ***AGrp***.
Does this give the coproduct of A and B in ***Grp***?

2.6 Equalizers and coequalizers

When we first see their categorical definition, equalizers and coequalizers are not something we immediately relate to our previous mathematical experience. However, once we become familiar with the categorical idea we do begin to realize that we have seen particular instances of the notions.

There are two notions here, the left notion and the right notion. We develop the two version in parallel. For instance, the following definition is two definitions in one.

2.15 DEFINITION. Given a parallel pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

of arrows in a category $\boldsymbol{C}$, an arrow

$$X \xrightarrow{h} A \qquad\qquad B \xrightarrow{h} X$$

makes equal the parallel pair if

$$f \circ h = g \circ h \qquad\qquad h \circ f = h \circ g$$

that is, the two composite arrows

$$X \underset{g \circ h}{\overset{f \circ h}{\rightrightarrows}} B \qquad\qquad A \underset{h \circ g}{\overset{h \circ f}{\rightrightarrows}} X$$

agree. ■

Depending on the circumstances any given parallel pair could be made equal, on one side or the other, by many different arrows. We look for a 'best possible' coalescing arrow.

2.16 DEFINITION. Given a parallel pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

of arrows in a category $\boldsymbol{C}$,

an equalizer a coequalizer

is an arrow

$$S \xrightarrow{k} A \qquad\qquad B \xrightarrow{k} S$$

which makes equal f and g, and has the following universal property.

For each arrow

$$X \xrightarrow{h} A \qquad\qquad B \xrightarrow{h} X$$

which makes equal the parallel pair, there is a *unique* arrow

$$X \xrightarrow{m} S \qquad\qquad S \xrightarrow{m} x$$

such that

$$h = k \circ m \qquad\qquad h = m \circ k$$

holds. This arrow m is the mediating arrow (or mediator) for the arrow h. ■

Read this definition a couple of times and compare it with Definition 2.11. Later, in Chapter **??**, we will see that both notions are particular instances of a more general notion.

For now we develop the idea of Definition 2.16. We follow a path quite similar to that in Section 2.5. Here is the analogue of Lemma 2.12.

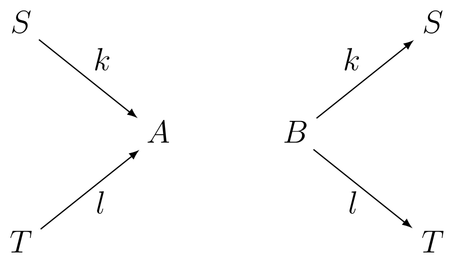
2.17 LEMMA. *Each equalizer is monic. Each coequalizer is epic.*

The proof of this is quite similar to that of Lemma 2.12, so we leave it as an exercise. We use the result to obtain the analogue of Lemma 2.13.

2.18 LEMMA. *For a parallel pair*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

of arrows in a category $\boldsymbol{C}$ *let*



be a pair of

equalizers *coequalizers*

respectively. Then S, T *are uniquely isomorphic over the wedges, In other words, there are unique arrows*

$$S \underset{n}{\overset{m}{\rightleftarrows}} T$$

such that

$$\begin{array}{llll} (1) & l = k \circ m & (3) & l = m \circ k \\ (2) & k = l \circ n & (4) & k = n \circ l \end{array}$$

and in particular m *and* n *are an inverse pair of isomorphisms.*

Proof. We look at the coequalizer, right hand, version and leave the equalizer version as an exercise.

The arrow l makes equal f and g. The arrow k is the coequalizer of f and g. Thus there is a unique mediator m satisfying (3).

By reversing the roles of l and k we see there is a unique mediator n satisfying (4).

From (3, 4) we have

$$n \circ m \circ k = n \circ l = k = \boldsymbol{id}_S \circ k$$

and hence

$$n \circ m = \boldsymbol{id}_S$$

since k is epic. Similarly

$$m \circ n = \boldsymbol{id}_T$$

to show that m and n are an inverse pair of isomorphisms. ■

The left hand part of this result shows that if a pair of arrows has an equalizer then that gadget is essentially unique. Thus we often speak of *the* equalizer of a pair. Similarly, from the right hand part of this result, we speak of *the* coequalizer of a pair of arrows.

Let's now look at a few examples.

Of course, any given pair of arrows need not have an equalizer, nor a coequalizer. In contrast to this for some categories these gadgets always exist.

2.19 EXAMPLE. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a parallel pair of functions, arrows in ***Set***. Let

$$S = \{a \in A \mid f(a) = g(a)\}$$

be the set of elements of A which f and g treat equally. Then the insertion

$$S \overset{i}{\hookrightarrow} A$$

is the equalizer of f and g.

To see that suppose the function

$$X \xrightarrow{h} A$$

makes equal f and g. For each $x \in X$ we have

$$f(h(x)) = (f \circ h)(x) = (g \circ h)(x) = g(h(x))$$

so that $h(x) \in S$, and hence the function

$$\begin{array}{ccc} X & \xrightarrow{m} & S \\ x & \longmapsto & h(x) \end{array}$$

is the required mediator. ■

A similar idea can be used in several other categories.

The category ***Set*** also has all coequalizers. To obtain these we combine two standard construction which, at first sight, seem to have little to do with category theory.

Almost certainly you will know the essential content of the following example, but you may not have seen it set out like this.

2.20 EXAMPLE. Let S be an arbitrary set, and let $\sim$ be an equivalence relation on S. This relation partitions S into blocks (equivalence classes). For each $s \in S$ let $[s]$ be the block in which s lives, and let

$$S/{\sim}$$

be the set of all such blocks. Let

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S/{\sim} \\ s & \longmapsto & [s] \end{array}$$

be the induced surjection.

Let

$$S \xrightarrow{h} X$$

be any function. The kernel of h is the relation $\approx$ on S given by

$$s_2 \approx s_2 \Longleftrightarrow h(s_1) = h(s_2)$$

for $s_1, s_2 \in S$. Trivially, this is an equivalence relation.

Now suppose $\approx$ includes $\sim$, that is

$$s_1 \sim s_2 \Longrightarrow h(s_1) = h(s_2)$$

for $s_1, s_2 \in S$. Under these conditions there is a commuting triangle

$$\begin{array}{ccc} S & \xrightarrow{h} & X \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle h^\sharp} & \\ & S/{\sim} & \end{array}$$

for some unique function $h^\sharp$. This function is given by

$$h^\sharp([s]) = h(s)$$

for $s \in S$. The only problem is to show that $h^\sharp$ is well-defined. ■

To produce a coequalizer we generate a certain equivalence relation.

2.21 EXAMPLE. Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a parallel pair of functions, arrows in ***Set***. Let $\leadsto$ be the relation on B given by

$$b_1 \leadsto b_2 \Longleftrightarrow (\exists a \in A)[b_1 = f(a) \text{ and } b_2 = g(a)]$$

for $b_1, b_2 \in B$. Let $\sim$ be the equivalence relation on B generated by $\leadsto$. (You might try to find an explicit description of $\sim$ sometime. It's not as easy as it looks.)

Using the construction of Example 2.20 we may check that the canonical quotient

$$B \longrightarrow B/{\sim}$$

is the coequalizer of the pair f, g (in ***Set***). ■

A modification of this construction can be used in several other categories.

Exercises

2.6.1 Prove Lemma 2.17, and complete the proof of Lemma 2.18.

2.6.2 Complete the proof of Example 2.19. In other words, show that the function m does make the relevant triangle commute, and it is the only function to make that triangle commute.

2.6.3 Let

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

be a parallel pair of morphisms between groups (written multiplicatively).

(a) Let

$$E = \{a \in A \mid f(a) = g(a)\}$$

and let S be the subgroup of A generated by E. Show that the insertion

$$S \hookrightarrow A$$

is the equalizer of f and g in ***Grp***.

(b) Let

$$F = \{f(a)g(a)^{-1} \mid a \in A\}$$

and let K be the normal subgroup generated by F. Show that the canonical quotient

$$B \longrightarrow B/K$$

is the coequalizer of f and g in ***Grp***.

2.6.4 Write down the details missing from Example 2.20. (None of these details are difficult, but you should at least list what is missing.)

2.6.5 Write down the details missing from Example 2.21.

2.6.6 Let

$$S \underset{\psi}{\overset{\phi}{\rightrightarrows}} T$$

be a parallel pair of continuous maps between topological spaces. Let

$$T \xrightarrow{\ \theta\ } T/{\sim}$$

be the coequalizer in ***Set*** of the pair of functions ϕ and ψ.

Show there is a suitable topology on $T/{\sim}$ for which θ becomes the coequalizer of the pair ϕ, ψ in ***Top***.

2.6.7 Consider the forgetful functor

$$\boldsymbol{Pre} \longleftarrow \boldsymbol{Pos}$$

from posets to presets. Eventually we will see that this exercise produces the left adjoint to this functor.

(a) Let S is a pre-ordered set and consider the relation $\sim$ on S given by

$$a \sim b \Longleftrightarrow a \leq b \leq a$$

(for $a, b \in S$).

Show that $\sim$ is an equivalence relation on the set S.

Show that S is a poset precisely when $\sim$ is equality.

(b) Let $S/\sim$ be the set of blocks of $\sim$ and let

$$S \xrightarrow{\ \eta\ } S/{\sim}$$

be the canonical quotient.

Show that

$$[a] \leq [b] \Longleftrightarrow a \leq b$$

for $a, b \in S$ produces a well-defined partial order on $S/{\sim}$.

Show that the function η is monotone.

(c) Consider any monotone map

$$S \xrightarrow{\ f\ } T$$

from the preset S to a poset T.

Show that

$$a \sim b \Longrightarrow f(a) = f(b)$$

for all $a, b \in S$.

Show there is a unique monotone map $f^\sharp$ such that the

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & S/{\sim} & \end{array}$$

triangle commutes.

2.6.8 Consider a diagram

$$\bullet \xrightarrow{\ e\ } \bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet$$

where e makes equal f and g. Suppose also there is a commuting diagram

$$\begin{array}{ccccc} \bullet & \xrightarrow{\ e\ } & \bullet & \xrightarrow{\ p\ } & \bullet \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} \\ \bullet & \xrightarrow[\ g\ ]{} & \bullet & \xrightarrow[\ q\ ]{} & \bullet \end{array}$$

where the bottom and top composites are identity arrows.

Show that e is the equalizer of f and g.

2.7 Pullbacks and pushouts

As I said in Section 2.3, the notions discussed in Sections 2.4, 2.5, 2.6, and this section are all particular cases of a more general notion, that of a

limit colimit

of a diagram. In this section we begin to use the terminology and the ideas behind this more general notion. This is not essential here, but it will help when we look at the more general notion in Chapter 5.

Each of the gadgets we are interested in is the universal solution of a certain problem posed by a diagram. For the simple gadgets of this chapter the shape of the diagram - the template - determines the name of the gadget. These templates are given in Table 3.1 alongside the name of the

left universal solution right universal solution

for that shape. (Don't worry if you can't see the template for row (1). It is there, but it's empty.)

The diagram for a

pullback pushout

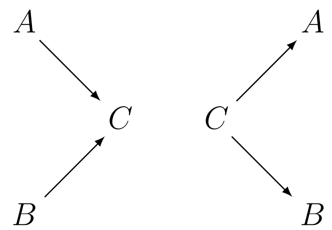
is a

left wedge right wedge

as shown in the table. Such a wedge poses a problem on the appropriate side.

As before, we develop the two notions in parallel. So each definition or result is two for the price of one. (And unlike Tescbury's it isn't just a con to make you buy more.)

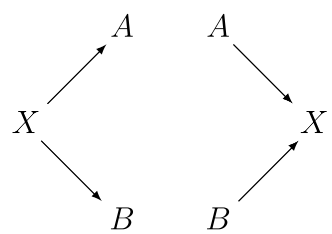
2.22 DEFINITION. Let



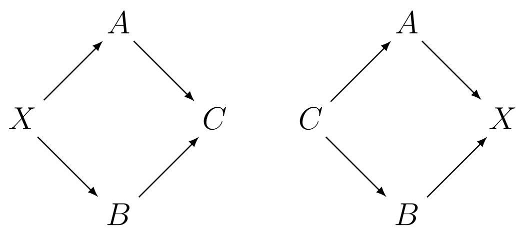
be a wedge in a category $\boldsymbol{C}$. A solution for the

left right

problem posed by the wedge is a wedge



(of the opposite handedness) such that the square

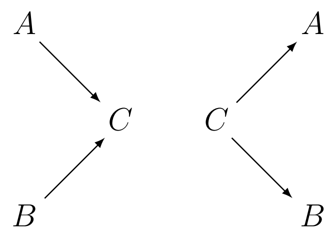


commutes. ■

Notice that we haven't given each arrow a name. We are beginning to work more and more in terms of diagrams, and we name an arrow only when it becomes necessary. (It is also the case that we need not name the objects, but let's not go that far just yet.)

The problem posed by a wedge can have many different solutions. We look for a 'best possible' solution.

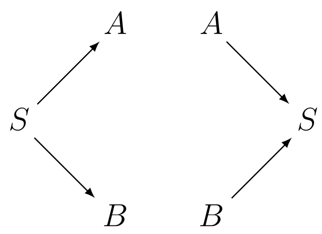
2.23 DEFINITION. Let



be a wedge in a category $\boldsymbol{C}$. A

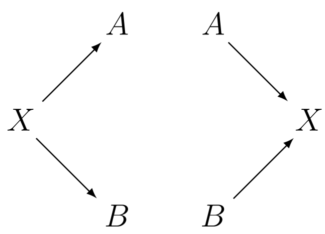
pullback pushout

for the wedge is a solution



with the following universal property.

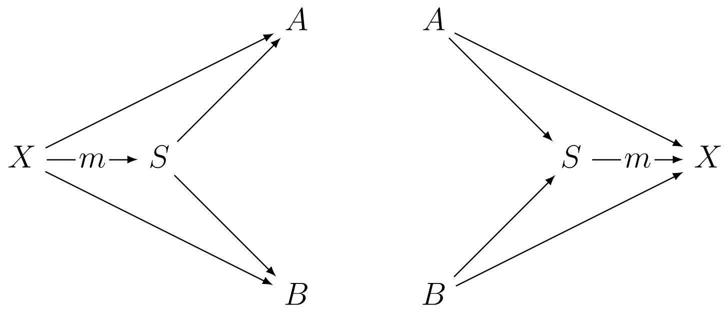
For each solution



there is a *unique* arrow

$$X \xrightarrow{\ m\ } S \qquad\qquad S \xrightarrow{\ m\ } X$$

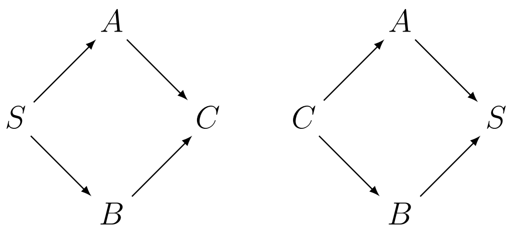
such that the diagram



commutes. This arrow m is the mediating arrow (or mediator) for the wedge on X. ■

As always, each universal solution is essentially unique. To prove this here we first obtain the analogue of Lemma 2.12.

2.24 LEMMA. *Let*



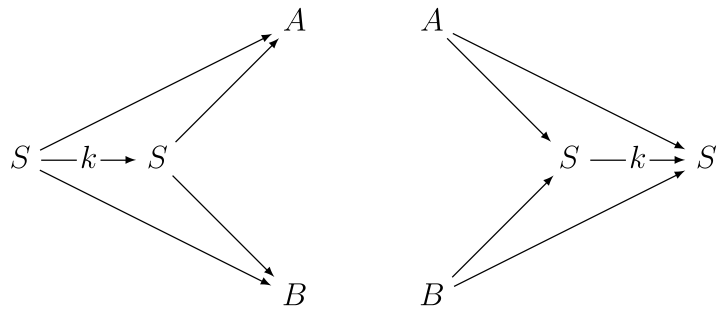
be a

pullback *pushout*

square in the category $\boldsymbol{C}$*. Let*

$$S \xrightarrow{\ k\ } S$$

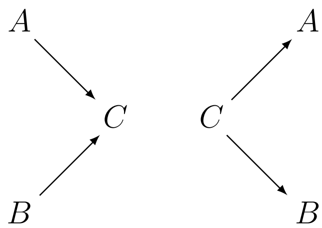
be any endo-arrow of S *for which the two triangles*



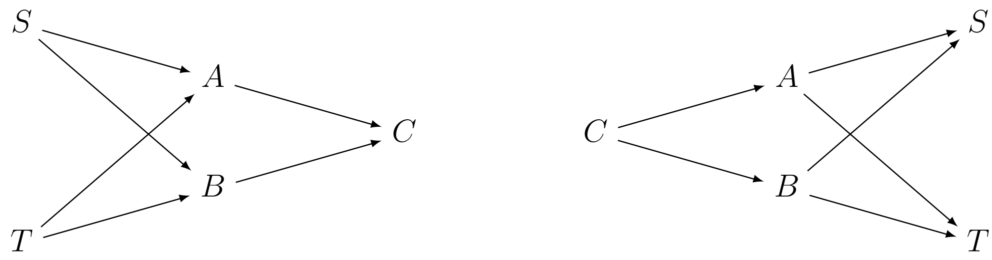
commute. Then $k = \boldsymbol{id}_S$.

You should be able to see the proof of this immediately. We use the result to obtain the analogue of Lemma 2.13.

2.25 LEMMA. *For a wedge*



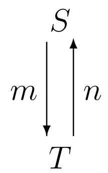
in a category $\boldsymbol{C}$*, let*



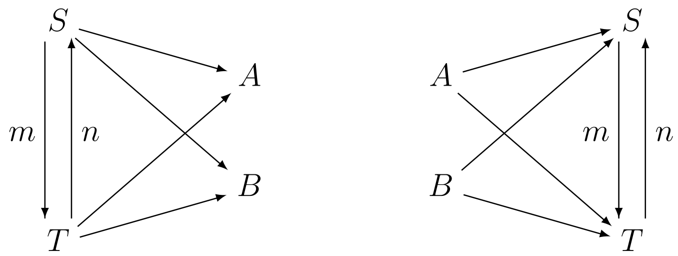
be a pair of

pullback *pushout*

squares. Then S, T *are uniquely isomorphic over the parent wedge. In other words, there are unique arrows*



such that all the triangles



commute. In particular, m *and* n *are an inverse pair of isomorphisms.*

Proof. We are given two solutions of the parent problem. Furthermore, each is a universal solution. Thus the associated mediators are the arrows m and n.

We now apply Lemma 2.24 to the two compounds

$$n \circ m \qquad m \circ n$$

to show these are

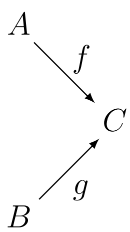
$$\boldsymbol{id}_S \qquad \boldsymbol{id}_T$$

respectively. ■

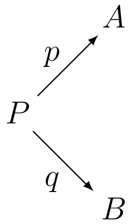
If you found this proof a little hard to follow, try labelling the arrows and re-work the argument using equational reasoning.

Let's now look at a couple of examples of these notions.

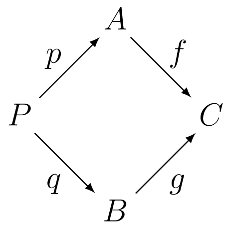
2.26 EXAMPLES. (a) Let



be a wedge of functions, arrows in ***Set***. Let



be the product wedge of the two sets A, B. Note that the square

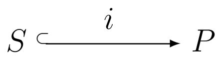


need not commute.

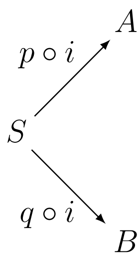
Let

$$S = \{z \in P \mid f(p(z)) = g(q(z))\}$$

be the set of elements of P which arrive at the same place no matter which route they take. Let



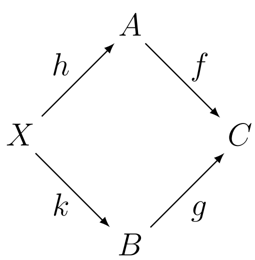
be the insertion of S in P. Then the wedge



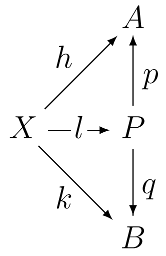
is the pullback of the parent wedge.

To see this observe first that, by construction, this wedge on S is a solution to the problem posed by the parent wedge.

Consider any solution



to the posed problem. Using the product property we have a commuting diagram



for some unique function l. For each $x \in X$ we have

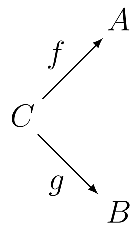
$$\begin{aligned} f(p(l(x)) &= (f \circ p \circ l)(x) \\ &= (f \circ h)(x) \\ &= (g \circ k)(x) \\ &= (g \circ q \circ l)(x) = g(q(l(x))) \end{aligned}$$

to show that $l(x) \in S$. We may now check that

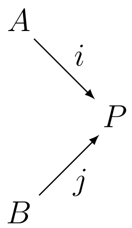
$$\begin{array}{ccc} X & \xrightarrow{\ m\ } & S \\ x & \longmapsto & l(x) \end{array}$$

is the required unique mediating arrow.

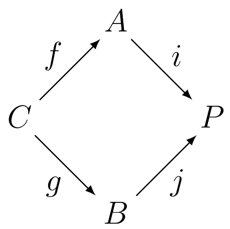
(b) Let



be a wedge of functions, arrows in ***Set***. Let



be the coproduct wedge of the two sets A, B. Note that the square



need not commute.

Let $\rightsquigarrow$ be the relation on P given by

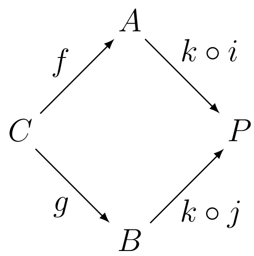
$$z_1 \rightsquigarrow z_2 \iff (\exists c \in C)[z_1 = (i \circ f)(c) \text{ and } z_2 = (j \circ g)(c)]$$

for $z_1, z_2 \in B$. Let $\sim$ be the equivalence relation on P generated by $\rightsquigarrow$. Let $S = P/{\sim}$ and let

$$P \xrightarrow{\ k\ } S$$

be the canonical quotient.

By construction the square



commutes, and so we do have a solution to the posed problem. We need to show it is a universal solution. This follows by a few simple calculations. ■

Did you spot anything about these two constructions?

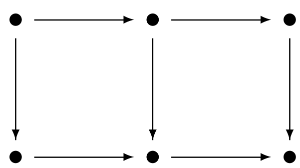
Exercises

2.7.1 (a) Suppose the category $\boldsymbol{C}$ has all binary products and all equalizers. Show that $\boldsymbol{C}$ has all pullbacks.

(b) Suppose the category $\boldsymbol{C}$ has all binary coproducts and all coequalizers. Show that $\boldsymbol{C}$ has all pushouts.

2.7.2 Let S be a poset which as a category has all pushouts. What does this mean lattice theoretically. (There is a lattice theoretic notion which precisely matches the categorical notion, but is rarely recognized as such.)

2.7.3 Consider the following commuting diagram



of two inner cells and one outer cell.

Show that if each of the two inner cells is a pullback, then so is the outer cell.

Show that if the outer cell and the right inner cell are pullbacks, then the left inner cell is a pullback.

Sort out the corresponding results for pushouts.

2.7.4 Show that monics are stable across pullbacks. In other words, show that if

$$\begin{array}{ccc} \bullet & \xrightarrow{\;k\;} & \bullet \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\;g\;]{} & \bullet \end{array}$$

is a pullback and f is monic, then h is monic.

2.7.5 Show that equalizers are stable across pullbacks. In other words, show that if

$$\begin{array}{ccc} \bullet & \xrightarrow{\;k\;} & \bullet \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[\;g\;]{} & \bullet \end{array}$$

is a pullback and f is is the equalizer of some pair, then h is the equalizer of some other pair.

2.8 Using the opposite category

In Sections 2.2 to 2.7 we have looked at six pairs of gadgets, a left version and a right version. By using the opposite category $\boldsymbol{C}^{\mathrm{op}}$ we can make precise this left-right symmetry, and halve the work.

Consider any arrow

$$\bullet \xrightarrow{\;\;f\;\;} \bullet$$

of a category $\boldsymbol{C}$. Then

$$\begin{array}{l} f \text{ is monic in } \boldsymbol{C} \Longleftrightarrow f^{\mathrm{op}} \text{ is epic in } \boldsymbol{C}^{\mathrm{op}} \\ f \text{ is epic in } \boldsymbol{C} \Longleftrightarrow f^{\mathrm{op}} \text{ is monic in } \boldsymbol{C}^{\mathrm{op}} \end{array}$$

to show that one of the notions immediately gives the other one.

Consider any object

$$K$$

of a category $\boldsymbol{C}$. Then

$$\begin{array}{l} K \text{ is final in } \boldsymbol{C} \Longleftrightarrow K \text{ is initial in } \boldsymbol{C}^{\mathrm{op}} \\ K \text{ is initial in } \boldsymbol{C} \Longleftrightarrow K \text{ is final in } \boldsymbol{C}^{\mathrm{op}} \end{array}$$

to show that one of the notions immediately gives the other one.

This duality is a useful little trick. It can help to save work. For instance, we have seen that each gadget discussed in this chapter is ‘essentially unique’, but in each case we only did half the proof. This is because the other half is the same argument carried out in the opposite category.

Exercises

2.8.1 Check that each of the gadgets of this chapter is the dual of a similar gadget in the opposite category.

There are 43 exercises in this chapter

3
Functors and natural transformations

Eilenberg and MacLane invented (discovered) category theory in the early 1940s. They were working on Čech cohomology and wanted to separate the routine manipulations from those with more specific content. It turned out that category theory is good at that. Hence its other name abstract nonsense which is not always used with affection.

Another part of their motivation was to try to explain why certain 'natural' construction are natural, and other constructions are not. Such 'natural' constructions are now called natural transformations, a term that was used informally at the time but now has a precise definition.

They observed that a natural transformation passes between two gadgets. These had to be made precise, and are now called functors.

In turn each functor passes between two gadgets, which are now called categories.

In other words, categories were invented to support functors, and these were invented to support natural transformations.

But why the somewhat curious terminology? This is explained on pages 29 and 30 of [6]

> ...the discovery of ideas as general as these is chiefly the willingness to make a brash or speculative abstraction, in this case supported by the pleasure of purloining words from philosophers: "Category" from Aristotle and Kant, "Functor" from Carnap ...

That, of course, is the bowdlerized version.

Most of the basic notions were set up in [4], and that paper is still worth reading.

3.1 Functors defined

The basic tenet of category theory is that whenever we conceive of a collection of 'objects' - things we don't want to take apart - we should, at the same time, decide how these 'objects' are to be compared. We then formalize a category. In other words, for any given category $\boldsymbol{C}$ we should think of the arrows of $\boldsymbol{C}$ as those gadgets which compare the objects. Furthermore, these arrows are just as important as, and sometimes more important than, the objects.

To stay true to this principle we must now ask a question. We have invented a collection of things called categories. How should categories be compared? Functors are the comparison gadgets.

3.1 DEFINITION. (Preliminary) Given a pair of categories

$$\boldsymbol{Src} \qquad \boldsymbol{Trg}$$

a functor

$$\begin{array}{ccc} \boldsymbol{Src} & \longrightarrow & \boldsymbol{Trg} \\ A & \longmapsto & FA \\ f & \longmapsto & F(f) \end{array}$$

consists of two assignments. One sends objects to objects, and the other sends arrows to arrows. ■

As here it is customary to use the same letter for both assignments. I find it helpful to use brackets in the arrow assignment but not in the object assignment.

Of course, there is more to a functor than just a pair of assignments. It is suppose to be a 'morphism of categories' in the sense that it must somehow respect the structure of the two categories.

What can that mean?

The first bit is that a functor F must preserve identity arrows. For each $\boldsymbol{Src}$ object A we must have

$$A \xrightarrow{\boldsymbol{id}_A} A \quad \longmapsto \quad FA \xrightarrow{\boldsymbol{id}_{FA}} FA$$

that is

$$F(\boldsymbol{id}_A) = \boldsymbol{id}_{FA}$$

in equational form.

The second bit is that a functor F must preserve composition of composible arrows. But here there could be a twist in the tale. Given an arrow

$$A \xrightarrow{f} B$$

in the source category $\boldsymbol{Src}$, the arrow $F(f)$ in the target category $\boldsymbol{Trg}$ must pass between the two objects FA and FB of $\boldsymbol{Trg}$. But there are two ways it might do that. It can preserve the direction or it can reverse the direction. This leads to two kinds of functors.

$$A \xrightarrow{f} B \quad \overset{\text{Covariant}}{\longmapsto} \quad FA \xrightarrow{F(f)} FB$$

$$A \xrightarrow{f} B \quad \overset{\text{Contravariant}}{\longmapsto} \quad FB \xrightarrow{F(f)} FA$$

For both kinds the source and target of an arrow are preserved as an unordered pair. For a covariant functor the direction of the arrow is always preserved, but for a contravariant functor the direction of the arrow is always reversed.

3.2 DEFINITION. (In full) Given a pair of categories

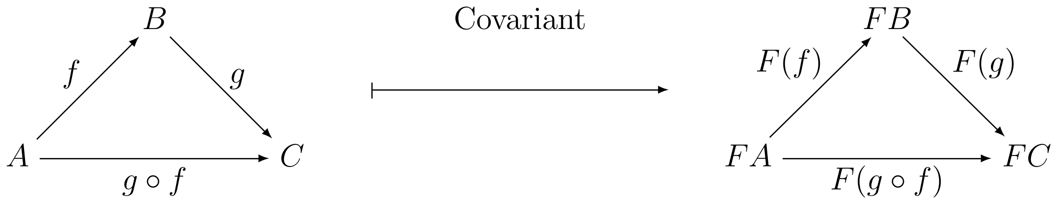
$$\boldsymbol{Src} \qquad \boldsymbol{Trg}$$

a functor

$$\begin{array}{rcl} \boldsymbol{Src} & \longrightarrow & \boldsymbol{Trg} \\ A & \longmapsto & FA \\ f & \longmapsto & F(f) \end{array}$$

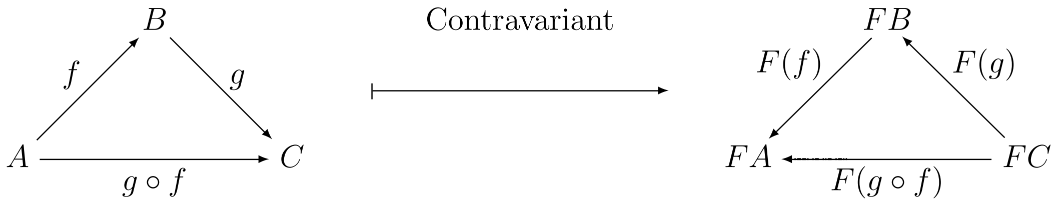
consists of two assignments. One sends objects to objects, and the other sends arrows to arrows.

(Co) For a covariant functor composition is preserved as follows.



$$F(g \circ f) = F(g) \circ F(f)$$

(Contra) For a contravariant functor composition is preserved as follows.



$$F(g \circ f) = F(f) \circ F(g)$$

For both kinds identity arrows are preserved in the sense that

$$F(\boldsymbol{id}_A) = \boldsymbol{id}_{FA}$$

for each object A. ■

Note that it doesn't make sense to say a functor is sometimes covariant and sometimes contravariant. A functor is either always covariant or always contravariant.

(There is a notion of a multi-functor with several input positions for objects. Such multi-functors can be covariant in some input positions and contravariant in the other input positions. The simplest example of this is the 2-placed hom-functor. We meet this in the next section.)

In the main we deal with covariant functors and refer to these as functors. Only when it is important do we specifically mention the variance of a functor.

Exercises

3.1.1 Consider a pair S, T of monoids viewed as categories.
What is a covariant functor from S to T?
What is a contravariant functor from S to T?

3.1.2 Consider a pair S, T of presets viewed as categories.
What is a covariant functor from S to T?
What is a contravariant functor from S to T?

3.1.3 Show that for each pair $\boldsymbol{Src}$ and $\boldsymbol{Trg}$ of categories, covariant functors

$$\boldsymbol{Src}^{\mathrm{op}} \longrightarrow \boldsymbol{Trg} \qquad\qquad \boldsymbol{Src} \longrightarrow \boldsymbol{Trg}^{\mathrm{op}}$$

are just contravariant functors from $\boldsymbol{Src}$ to $\boldsymbol{Trg}$.

3.1.4 Define the composite $G \circ F$ of two functors F and G (perhaps of different variance), and show that the result is a functor.

How does the variance of $G \circ F$ relate to that of F and G?

3.2 Some simple functors

In this section we look at some simple examples of functors. Most of these are chosen merely to illustrate the notion, but one or two are important in their own right.

Forgetful functors

Let $\boldsymbol{C}$ be any category of structured sets. Thus each object

$$(A, \cdots)$$

is a set furnished with some gadgetry, and each arrow

$$(A, \cdots) \longrightarrow (B, \cdots)$$

is a function between the two carrying sets. Arrow composition is just function composition.

Look at what we have here. We have a covariant functor

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

which sends each object to its carrying set, and each arrow to its carrying function. I know this is not very exciting, but the idea can help to clear up a bit of confusion from time to time.

This is an example of a forgetful functor. There are a few more given in Table 3.1. In each case something is forgotten (or ignored) as we pass from the source category to the to the target category. In the first batch some structure is forgotten. In the second batch some property is forgotten. In the third batch it is a mixture of structure and property that is forgotten.

All of these forgetful functors are covariant. Occasionally we meet a contravariant forgetful functor. Consider the functors

$$\boldsymbol{Pos}^{\dashv} \xrightarrow{\;L\;} \boldsymbol{Pos} \qquad\qquad \boldsymbol{Pos}^{\dashv} \xrightarrow{\;R\;} \boldsymbol{Pos}$$

which pick out the left and the right component of each arrow.

$\boldsymbol{Rng} \longrightarrow \boldsymbol{AGrp}$	forget the $\times$-structure
$\boldsymbol{Rng} \longrightarrow \boldsymbol{Mon}$	forget the $+$-structure
$\boldsymbol{Mod}\text{-}R \longrightarrow \boldsymbol{AGrp}$	forget the action
$\boldsymbol{Mod}\text{-}R \longrightarrow \boldsymbol{Set}\text{-}R$	forget the $+$-structure
$\boldsymbol{CMon} \longrightarrow \boldsymbol{Mon}$	For all three
$\boldsymbol{AGrp} \longrightarrow \boldsymbol{Grp}$	the commutative property
$\boldsymbol{CRng} \longrightarrow \boldsymbol{Rng}$	is forgotten
$\boldsymbol{Sup} \longrightarrow \boldsymbol{Join} \longrightarrow \boldsymbol{Pos}$	First forget arbitrary suprema but retain joins, and then forget these
$\boldsymbol{Inf} \longrightarrow \boldsymbol{Meet} \longrightarrow \boldsymbol{Pos}$	

Table 3.1: Some forgetful functors

Hom functors

The next two examples, one covariant and one contravariant, are very important. We will meet them many times in several guises.

Let $\boldsymbol{C}$ be an arbitrary category. Let K be an object of $\boldsymbol{C}$. For each object A we have an arrow set

$$LA = \boldsymbol{C}[K, A] \qquad RA = \boldsymbol{C}[A, K]$$

(unless, of course, this collection is too big to be a set). Thus we have a pair of object assignments

$$\begin{array}{ccc} \boldsymbol{C} & \longrightarrow & \boldsymbol{Set} \\ A & \longmapsto & LA \end{array} \qquad \begin{array}{ccc} \boldsymbol{C} & \longrightarrow & \boldsymbol{Set} \\ A & \longmapsto & RA \end{array}$$

to the category of sets. These are the object assignments of a pair of functors where each arrow

$$A \xrightarrow{\ f\ } B$$

of $\boldsymbol{C}$ is sent to

$$\begin{array}{ccc} \boldsymbol{C}[K, A] & \xrightarrow{L(f)} & \boldsymbol{C}[K, B] \\ r & \longmapsto & f \circ r \end{array} \qquad \begin{array}{ccc} \boldsymbol{C}[B, K] & \xrightarrow{R(f)} & \boldsymbol{C}[A, K] \\ l & \longmapsto & l \circ f \end{array}$$

that is

$$L(f)(r) = f \circ r \qquad R(f)(l) = l \circ f$$

respectively.

In this case we should check that we do have a pair of functors, and sort out the variance of each.

Consider a pair of arrows

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of $\boldsymbol{C}$. Consider also arrows

$$K \xrightarrow{\ r\ } A \qquad\qquad C \xrightarrow{\ l\ } K$$

that is

$$r \in LA \qquad\qquad l \in RC$$

respectively. Then

$$\begin{aligned}\big(L(g)\circ L(f)\big)(r) &= L(g)\big(L(f)(r)\big)\\ &= L(g)(f\circ r)\\ &= g\circ(f\circ r)\\ &= (g\circ f)\circ r = L(g\circ f)(r)\end{aligned} \qquad \begin{aligned}\big(R(f)\circ R(g)\big)(l) &= R(f)\big(R(g)(l)\big)\\ &= R(f)(l\circ g)\\ &= (l\circ g)\circ f\\ &= l\circ(g\circ f) = R(g\circ f)(l)\end{aligned}$$

to show that L is a covariant functor and R is a contravariant functor. We should also show that

$$L(\boldsymbol{id}_A) = \boldsymbol{id}_{LA} \qquad R(\boldsymbol{id}_A) = \boldsymbol{id}_{RA}$$

but that is more or less trivial.

Exercises

3.2.1 For an arbitrary category $\boldsymbol{C}$ consider the arrow category $\boldsymbol{C}^{\downarrow}$ of Example 1.10. Show there are three functors

$$\boldsymbol{C}^{\downarrow} \underset{T}{\overset{S}{\rightrightarrows}} \boldsymbol{C} \xrightarrow{\ \Delta\ } \boldsymbol{C}^{\downarrow}$$

between the categories. The functor Δ is often called the diagonal functor.

3.2.2 Let S be a poset viewed as a category. What is a contravariant functor

$$S \longrightarrow \boldsymbol{Set}$$

to $\boldsymbol{Set}$? You have seen this notion before.

3.2.3 Let R be a monoid viewed as a category. What is a covariant functor, and what is a contravariant functor

$$R \longrightarrow \boldsymbol{Set}$$

to $\boldsymbol{Set}$? Both these notions occur elsewhere in this book, but are described in a different way.

3.2.4 In Example 1.7 we looked at the graph $\Gamma(f)$ of a function (between sets). Show that this is the arrow assignment of a functor, and determine the variance of that functor.

3.2.5 In Example 1.9 we saw how to produce the product $\boldsymbol{C} \times \boldsymbol{D}$ of two categories. This enables us to think of 2-placed functors

$$\boldsymbol{C} \times \boldsymbol{D} \longrightarrow \boldsymbol{Trg}$$

with two input position. In particular, for a given category $\boldsymbol{C}$ we can view

$$\boldsymbol{C} \times \boldsymbol{C} \xrightarrow{\boldsymbol{C}[-,-]} \boldsymbol{Set}$$

as a 2-placed functor.

Think about this for a while, and sort out what it should mean.

3.3 Some less simple functors

In this section we look at some examples of functors with a bit more content, although none of them are very complicated. Some of the examples may look a bit contrived, but each one is a miniature version of something quite important.

3.3.1 Three power set functors

It may come as a bit of a surprise, but different functors can have the same object assignment. In this block we look at three endo-functors on $\boldsymbol{Set}$

$$\boldsymbol{Set} \longrightarrow \boldsymbol{Set}$$

where the object assignment of each

$$A \longmapsto \mathcal{P}A$$

sends a set A to its power set. Furthermore, two of these functors are covariant and one is contravariant.

It is common to use the same letter as the name of both the object assignment and the arrow assignment. Here we can't do that. We use

$$\boldsymbol{Set} \begin{array}{c} \xrightarrow{\ \exists\ } \\ \xleftarrow{\ \mathsf{I}\ } \\ \xrightarrow[\ \forall\ ]{} \end{array} \boldsymbol{Set}$$

as the three names, where the two outer ones are covariant and the central one is contravariant.

The stacking of the functors is significant, but that won't become clear for some time. Also the use of '∃' and '∀' may look a bit pretentious, but in a more general setting these functors really do have something to do with quantification. We will see just a hint of this shortly.

For each set A we have

$$\exists A = \mathcal{P}A \qquad \mathsf{I}A = \mathcal{P}A \qquad \forall A = \mathcal{P}A$$

as the three object assignments.

For the three arrows assignments consider any arrow of $\boldsymbol{Set}$

$$A \xrightarrow{\ f\ } B$$

in other words a function between the two sets. We require three functions

$$\mathcal{P}A \xrightarrow{\exists(f)} \mathcal{P}B \qquad \mathcal{P}A \xleftarrow{\mathsf{I}(f)} \mathcal{P}B \qquad \mathcal{P}A \xrightarrow{\forall(f)} \mathcal{P}B$$

where the central one reverses the direction. We set

$$\exists(f)(X) = f[X] \qquad \mathsf{I}(f)(Y) = f^{\leftarrow}(Y) \qquad \forall(f)(X) = f[X']'$$

for each $X \in \mathcal{P}A$ and $Y \in \mathcal{P}B$. Here $f[\,\cdot\,]$ indicates the direct image across f, and $f^{\leftarrow}(\,\cdot\,)$ indicates the inverse image across f. Notice that $\forall(f)$ uses the dual complement of the direct image (for $(\cdot)'$ indicates complementation).

We find that

$$b \in \exists(f)(X) \iff (\exists a \in A)[b = f(a)\ \&\ a \in X]$$
$$b \in \forall(f)(X) \iff (\forall a \in A)[b = f(a) \Rightarrow a \in X]$$

for all $X \in \mathcal{P}$ and $b \in B$. Notice how the description matches the name. We also have

$$a \in \mathsf{I}(f)(Y) \iff f(a) \in Y$$

for all $Y \in \mathcal{P}B$ and $a \in A$.

It is not immediately clear that these construction do give functors, so we must check that.

For functions

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

we must show that

$$\exists(g \circ f) = \exists(g) \circ \exists(f) \qquad \mathsf{I}(g \circ f) = \mathsf{I}(f) \circ \mathsf{I}(g) \qquad \forall(g \circ f) = \forall(g) \circ \forall(f)$$

hold, together with the identity requirements. This is not difficult, provided you take a bit of care.

Exercises

3.3.1 Consider the three constructions $\exists, \mathsf{I}, \forall$ on $\boldsymbol{Set}$. Show that each passes across composition in the required manner.

3.3.2 For each set A the power set $\mathcal{P}A$ is a poset under inclusion. Show that for each function

$$A \xrightarrow{\ f\ } B$$

the three functions

$$\mathcal{P}A \;\begin{array}{c} \xrightarrow{\exists(f)} \\ \xleftarrow{\mathsf{I}(f)} \\ \xrightarrow[\forall(f)]{} \end{array}\; \mathcal{P}B$$

form a double poset adjunction.

3.3.2 Spaces, presets, and posets

In this block we compare the category ***Top*** of topological spaces with the categories ***Pre*** and ***Pos***. We set up four functors.

$$\boldsymbol{Pre} \underset{\Downarrow}{\overset{\Uparrow}{\rightleftarrows}} \boldsymbol{Top} \underset{\Xi}{\overset{\mathcal{O}}{\rightrightarrows}} \boldsymbol{Pos}$$

The two on the left are covariant. They also form an ADJUNCTION, but we won't explain that until Chapter 4. The two on the right are contravariant. They are also NATURALLY ISOMORPHIC, and we explain that later in this chapter.

Before we begin you should remember that a topological space need not be hausdorff. The separation propertiers T_0 and T_1 play a minor role here.

We look first at the two covariant functors $\Uparrow$ and $\Downarrow$ on the left.

Consider an arbitrary preset A. An upper section of A is a subset $U \subseteq A$ such that

$$\left.\begin{array}{r} a \in U \\ a \le b \end{array}\right\} \Longrightarrow b \in U$$

for all $a, b \in A$. Let ΥA be the family of all upper sections of A. This is a topology on A, and is sometimes called the Alexandroff topology. (As a topology it is pretty pathetic, but it does have its uses.)

Let $\Uparrow A$ be the preset A viewed as a topological space, that is with ΥA as the carried topology. We think of $\Uparrow A$ as an upgrading of A. (It's getting above itself.) This gives the object assignment of one of the functors. The arrow assignment is more or less trivial.

For a topological space S with topology $\mathcal{O}S$, the specialization order of S is the comparison on S given by

$$r \le s \Longleftrightarrow (\forall U \in \mathcal{O}S)[r \in U \Longrightarrow s \in U] \Longleftrightarrow r \in s^-$$

for $r, s \in S$. Here s^- is the closure of $\{s\}$. This specialization order is a pre-order on S. (You might like to check that S is T_0 precisely when $\le$ is a partial ordering, and S is T_1 precisely when $\le$ is equality.)

Let $\Downarrow S$ be the space S viewed as a preset, that is with its specialization order as its carried comparison. We think of $\Downarrow S$ as a downgrading of S. (It isn't making enough of its talents.) This gives the object assignment of the other functor. The arrow assignment is more or less trivial.

The Exercises 3.3.3, 3.3.4, and 3.3.5 fill in some of the missing details.

Next we look at the two contravariant functors $\mathcal{O}$ and Ξ on the right.

For each space S let $\mathcal{O}S$ be its topology viewed as a poset under inclusion. For each continuous map

$$T \xrightarrow{\ \phi\ } S$$

between spaces let

$$\mathcal{O}S \xrightarrow{\ \mathcal{O}(\phi) = \phi^{\leftarrow}\ } \mathcal{O}T$$

be the inverse image function. Almost trivially, this is monotone.

This gives us one of the functors.

Let

$$\mathsf{2} = \{0, 1\}$$

be the 2-element set. Let

$$\mathcal{O}\mathsf{2} = \{\emptyset, \{1\}, \mathsf{2}\}$$

to produce a topology on $\mathsf{2}$. This is sometimes called Sierpinsky space. (Notice that $\mathcal{O}\mathsf{2}$ is just the Alexandroff topology on the poset $\{0 < 1\}$.)

For each space S let

$$\Xi S = \boldsymbol{Top}[S, \mathsf{2}]$$

be the set of continuous characters of S. These are partially ordered pointwise, that is

$$p \leq q \iff (\forall s \in S)[p(s) \leq q(s)]$$

for $p, q \in \Xi S$. For each continuous map

$$T \xrightarrow{\ \phi\ } S$$

between space let

$$\begin{array}{ccc} \Xi S & \xrightarrow{\ \Xi(\phi)\ } & \Xi T \\ p & \longmapsto & p \circ \phi \end{array}$$

for $p \in \Xi S$.

This gives us the other functor.

Of course, there are a few things to be checked. These are dealt with by Exercises 3.3.6 and 3.3.7.

Exercises

3.3.3 (a) For a preset A, what is the specialization order of $\Uparrow A$?

(b) For a space S, show that $\mathcal{O}S \subseteq \Upsilon{\Downarrow}S$.

3.3.4 (a) Show that a monotone function

$$A \xrightarrow{\ f\ } B$$

between presets is continuous relative to the two Alexandroff topologies.

Show that $\Uparrow$ is a functor.

(b) Show that a continuous map

$$S \xrightarrow{\ \phi\ } T$$

between spaces is monotone relative to the two specialization orders.

Show that $\Downarrow$ is a functor.

3.3.5 Let

$$\theta : A \longrightarrow S$$

be a function from a pre-ordered set to a topological space.

Show that θ is monotone (relative to $\Downarrow S$) precisely when it is continuous (relative to $\Uparrow A$).

Show there is a bijection between

$$\boldsymbol{Pre}[A, \Downarrow S] \qquad \boldsymbol{Top}[\Uparrow A, S]$$

for arbitrary A and S.

3.3.6 Show that for each continuous map

$$T \xrightarrow{\ \phi\ } S$$

between spaces, the function $\mathcal{O}(\phi)$ is monotone.

Show that $\mathcal{O}(\phi)$ passes across composition, and hence $\mathcal{O}$ is a functor.

Show that the function $\Xi(\phi)$ does convert continuous characters into continuous characters, and that $\Xi(\phi)$ is monotone.

Show that $\Xi(\phi)$ passes across composition, and hence Ξ is a functor.

Where have you seen some of these calculations before?

3.3.7 For an arbitrary space S and open set $U \in \mathcal{O}S$, let

$$\chi_S(U) : S \longrightarrow \mathsf{2}$$

be the characteristic function of U, that is

$$\chi_S(U)(s) = \begin{cases} 1 \text{ if } s \in U \\ 0 \text{ if } s \notin U \end{cases}$$

for $s \in S$.

Show that for each $U \in \mathcal{O}S$ the character $\chi_S(U)$ is continuous, and hence we have an assignment

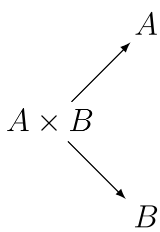
$$\mathcal{O}S \xrightarrow{\ \chi_S\ } \Xi S$$

between the two posets.

Show that χ_S is an isomorphism of posets. (This is more that showing χ_S is a monotone bijection.)

3.3.3 Functors from products

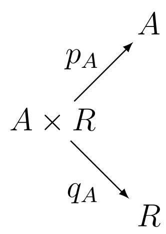
In Section 2.5 we defined the notion of a product of two objects A and B in a category $\boldsymbol{C}$. This consists of a wedge



with certain properties. As we saw in Section 2.5 the object $A \times B$ is unique *only* up to a certain canonical isomorphism. What happens if we decide to change some of these selected objects and modify the projections accordingly?

Suppose the category $\boldsymbol{C}$ has all binary products. For each pair A, B of objects suppose we select, in some way or other, a product wedge for that pair. This choice could be fairly haphazard, but it still produces a functor.

Let R be some fixed object of $\boldsymbol{C}$. For each object A let



be a selected product wedge. Let

$$FA = A \times R$$

so we have an object assignment

$$A \longmapsto FA$$

on $\boldsymbol{C}$. We show there is a corresponding arrow assignment

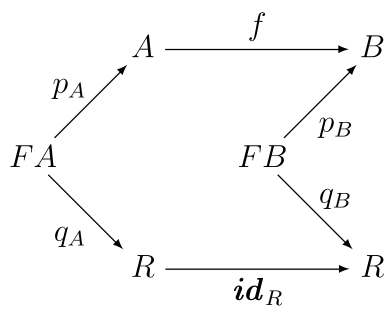
$$f \longmapsto F(f)$$

so that the pair of assignments form an endo-functor on $\boldsymbol{C}$.

Let

$$A \xrightarrow{\ f\ } B$$

be an arrow of $\boldsymbol{C}$. We have a diagram



using the selected product wedges. The product property gives an arrow

$$FA \xrightarrow{\ F(f)\ } FB$$

with certain properties. With a little bit of works we can check that this gives an endo-functor on $\boldsymbol{C}$.

Exercises

3.3.8 Using the mediating property for product wedges to complete the details of the functorial product construction.

Show also that the coproduct construction gives an endo-functor on the parent category.

3.3.9 If you are happy with the previous exercise, you can try this generalization.

Let $\boldsymbol{C}$ be a category with all binary products, and consider the assignment

$$\begin{array}{rcl} \boldsymbol{C} \times \boldsymbol{C} & \longrightarrow & \boldsymbol{C} \\ (A_1, A_2) & \longmapsto & A_1 \times A_2 \end{array}$$

which attaches a product object to each pair of objects.

Show that this fills out to a functor.

3.3.4 Comma category

In this block we use two functors to produce a new category from three old categories. This construction generalizes the two slice constructions of Example 1.11.

We start with three categories and two functors

$$\boldsymbol{U} \xrightarrow{\ U\ } \boldsymbol{C} \xleftarrow{\ L\ } \boldsymbol{L}$$

where we think of $\boldsymbol{U}$ as the upper component and $\boldsymbol{L}$ as the lower component. Using these we produce a category

$$(U \downarrow L)$$

sometimes called a comma category.

Each new object is a triple (conveniently written vertically)

$$\begin{array}{ccc} A_U & UA_U & A_U \in \boldsymbol{U} \\ & \vert & \\ \alpha & \alpha & \alpha \in \boldsymbol{C} \\ & \downarrow & \\ A_L & LA_L & A_L \in \boldsymbol{L} \end{array}$$

formed using an upper object A_U from $\boldsymbol{U}$, a lower object A_L from $\boldsymbol{L}$, and a connecting central arrow α from $\boldsymbol{C}$, as indicated.

The new arrows

$$\begin{array}{ccc} UA_U & & UB_U \\ \vert & & \vert \\ \alpha & \xrightarrow{\quad f \quad} & \beta \\ \downarrow & & \downarrow \\ LA_L & & LB_L \end{array}$$

are formed using an arrow f_U from $\boldsymbol{U}$ and and arrow f_L from $\boldsymbol{L}$

$$
\begin{array}{ccc}
A_U & \xrightarrow{\;f_U\;} & B_U \\
UA_U & \xrightarrow{\;U(f_U)\;} & UB_U \\
\Big\downarrow \alpha & & \Big\downarrow \beta \\
LA_L & \xrightarrow[\;L(f_L)\;]{} & LB_L \\
A_L & \xrightarrow[\;f_L\;]{} & B_L
\end{array}
$$

such that the square commutes.

You should check that this does produce a category and generlizes the two slice constructions.

The category used to be written

$$(\boldsymbol{U}, \boldsymbol{L})$$

which, believe it or not, is where the name comma category comes from.

Exercises

3.3.10 Fill in the details of the construction of the comma category.

3.3.11 (a) What is $(U \downarrow L)$ when both U and L are the identity endo-functor on $\boldsymbol{C}$?

(b) For an object S of $\boldsymbol{C}$ describe the slice categories $(\boldsymbol{C} \downarrow S)$ and $(S \downarrow \boldsymbol{C})$ as comma categories.

3.3.12 For convenience let $\boldsymbol{Com}$ be the comma category $(U \downarrow L)$. Construct three forgetful functors

$$\boldsymbol{Com} \longrightarrow \boldsymbol{U} \qquad \boldsymbol{Com} \longrightarrow \boldsymbol{C}^{\downarrow} \qquad \boldsymbol{Com} \longrightarrow \boldsymbol{L}$$

using the arrow category in the central one.

3.3.5 Other examples

Functors appear all over the place in mathematics. You have seen some before, perhaps without knowing it. Exercises 3.3.13 to 3.3.18 give a few more examples.

Exercises

3.3.13 For a group A let δA be the derived subgroup (generated by the commutators). In particular, $A/\delta A$ is an abelian group. Show that each of the two object assignments

$$A \longmapsto \delta A \qquad A \longmapsto A/\delta A$$

is part of a functor.

3.3.14 Let

$$S \xrightarrow{\ \phi\ } R$$

be a morphism between monoids. Using restriction of scalars we may view each (right) R-set A as an S-set. The S-action $\star$ is obtained from the R-action $\cdot$ by

$$a \star s = a \cdot \phi(s)$$

for each $a \in A$ and $s \in S$.

(a) Show that that this construction does convert the R-set A into an S-set.

(b) Show that the construction produces a functor

$$\boldsymbol{Set}\text{-}S \xleftarrow{\ \Phi\ } \boldsymbol{Set}\text{-}R$$

which is trivial on objects *and* arrows.

(c) Try generalizing this construction using rings and modules.

3.3.15 You will have to think clearly to do this exercise.

We form a new large category $\boldsymbol{MON}$. Each object of $\boldsymbol{MON}$ is a category $\boldsymbol{Set}\text{-}R$ for some monoid R. The arrows of $\boldsymbol{MON}$ are the functors between these categories.

Show that the construction of Exercise 3.3.14 produces a contravariant functor

$$\boldsymbol{Mon} \longrightarrow \boldsymbol{MON}$$

from the small to the large.

3.3.16 In Exercise 1.2.7 you were asked to show that the two categories $\boldsymbol{Set}_\perp$ and $\boldsymbol{Pfn}$ are 'essentially the same' category. Re-do that exercise to show there is an inverse pair of functors passing between the two categories.

3.3.17 Each preset S can be converted into a poset in a canonical fashion. We consider the relation $\sim$ on S given by

$$s_1 \sim s_2 \iff s_1 \leq s_2 \text{ and } s_2 \leq s_1$$

for $s_1, s_2 \in S$. Almost trivially this is an equivalence relation on S, and is equality precisely when S is a poset.

Let $S/{\sim}$ be the corresponding set of blocks $[s]$ for $s \in S$, and partially order $S/{\sim}$ by

$$[s_1] \leq [s_2] \iff s_2 \leq s_2$$

for $s_1, s_2 \in S$.

(a) Show that this construction of a poset $S/{\sim}$ is well-defined, and show that the canonical function

$$S \longrightarrow S/{\sim}$$

is monotone.

(b) Show that

$$S \longmapsto S/{\sim}$$

is the object part of a functor $\boldsymbol{Pre} \longrightarrow \boldsymbol{Pos}$.

3.3.18 This exercise makes precise the notion of 'freely generated by' in appropriate circumstances.

Suppose we have two categories ***Src*** and ***Trg*** and a forgetful functor between them. It is customary not to give such a functor a name, but here it will help if it does have one. Let

$$\boldsymbol{Src} \xleftarrow[¿]{} \boldsymbol{Trg}$$

be the forgetful functor. (Once you get used to the idea you can drop '¿'.)

Suppose to each object A of ***Src*** we attach a ***Trg***-object FA and an arrow

$$A \xrightarrow{\eta_A} (¿ \circ F)A$$

of ***Src*** with the following universal property.

For each ***Src***-arrow

$$A \xrightarrow{f} ¿S$$

where S is a ***Trg***-object, there is a unique ***Trg***-arrow

$$FA \xrightarrow{f^\sharp} S$$

such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & ¿S \\ & {\scriptstyle \eta_A} \searrow \quad \nearrow {\scriptstyle ¿(f^\sharp)} & \\ & (¿ \circ F)A & \end{array}$$

commutes in ***Src***.

(a) Show that

$$A \longmapsto FA$$

is the object assignment of a functor $\boldsymbol{Src} \longrightarrow \boldsymbol{Trg}$.

(b) Show that for each ***Src***-object A and ***Trg***-object S the assignment

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Src}[A, ¿S] & \longrightarrow & \boldsymbol{Trg}[FA, S] \end{array}$$

is a bijection, and describe its inverse.

(c) Where have you seen this construction before?

3.4 Natural transformations defined

As we have seen, each arrow of a category compares two objects, and each functor compares two categories. Next we will see how each natural transformation compares two functors.

How might we compare two functors F and G? Surely we want them to pass between the same two categories

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{Trg}$$

in the same direction. It also seems reasonable to insist that they have the same variance, either both are covariant or both are contravariant.

Given these condition, how might we compare F and G?

Consider an arbitrary object A of $\boldsymbol{Src}$. The two functors pass this to two objects FA and GA of $\boldsymbol{Trg}$. We compare these objects in $\boldsymbol{Trg}$. Thus we look for an arrow

$$FA \xrightarrow{\tau_A} GA$$

of $\boldsymbol{Trg}$. We do this for each object A of $\boldsymbol{Src}$.

3.3 DEFINITION. (Preliminary) Given a parallel pair

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{Trg}$$

of functors of the same variance, a natural transformation

$$F \xrightarrow{\tau} G$$

is a family of arrows of $\boldsymbol{Trg}$

$$FA \xrightarrow{\tau_A} GA$$

indexed by the objects A of $\boldsymbol{Src}$. ■

Notice that each component arrow τ_A passes in the same direction, from F to G in this case.

Of course, there is more to a natural transformation than just an indexed family of arrows. The selected arrow τ_A is required to be natural for variation of A. This is where we have to take note of the common variance of F and G.

3.4 DEFINITION. (In full) Given a parallel pair

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{Trg}$$

of functors of the same variance, a natural transformation

$$F \xrightarrow{\tau} G$$

is a family of arrows of ***Trg***

$$FA \xrightarrow{\tau_A} GA$$

indexed by the objects A of ***Src***, and such that for each arrow

$$A \xrightarrow{f} B$$

of ***Src*** the appropriate square in ***Trg*** commutes

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ F(f)\downarrow & \text{covariant} & \downarrow G(f) \\ FB & \xrightarrow[\tau_B]{} & GB \end{array} \qquad \begin{array}{c} A \\ \downarrow f \\ B \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ F(f)\uparrow & \text{contravariant} & \uparrow G(f) \\ FB & \xrightarrow[\tau_B]{} & GB \end{array}$$

depending on the common variance of F and G. ■

This is quite a short definition, but it has some subtleties. This will become clear as we look at various examples in the next section.

A natural transformation is a comparison of functors. We refine the notion to make precise the idea of two functors being 'essentially the same'.

3.5 DEFINITION. A natural isomorphism between two functors F and G is a natural transformation

$$F \xrightarrow{\tau} G$$

such that for each source arrow A the selected arrow

$$FA \xrightarrow{\tau_A} GA$$

is an isomorphism in the target category. ■

Sometimes two functors that are naturally isomorphic are said to be naturally equivalent.

Exercises

3.4.1 (a) Consider the small graph $(\downarrow)$ as described in Example 1.10. We may view this as a very small category with two objects and three arrows. The two identity arrows have been omitted from the picture.

Show that the objects of $\boldsymbol{C}^{\downarrow}$ are essentially the covariant functors

$$(\downarrow) \longrightarrow \boldsymbol{C}$$

and the arrows of $\boldsymbol{C}^{\downarrow}$ are the natural transformations between these functors.

(b) Show that each of the categories of Exercise 1.3.10 is the category of functors

$$\nabla \longrightarrow \boldsymbol{C}$$

and natural transformations for some appropriate template category ∇.

(c) Can you see a generalization of this idea?

3.4.2 For an arbitrary poset S consider the category $\widehat{S}$ of presheaves on S, as defined in Example 1.12.

Describe this as a category of functors and natural transformations.

3.4.3 Let R be a monoid viewed as a category. In Exercise 3.2.3 you located the functors

$$R \longrightarrow \boldsymbol{Set}$$

of both variance. Now locate the natural transformations between these functors.

3.4.4 Let

$$F \xrightarrow{\ \tau\ } G$$

be a natural isomorphism between two functors. For each source object A the arrow

$$FA \xrightarrow{\ \tau_A\ } GA$$

has an inverse

$$FA \xleftarrow[\ \sigma_A\ ]{} GA$$

in the target category. Show that the family σ of arrows is a natural transformation.

3.5 Examples of natural transformations

In this section we look at several examples of natural transformation. Some of these build on earlier examples of functors. The exercises give further details and examples.

As a first example let's have a look at some natural transformations between hom-functors.

3.6 EXAMPLE. Let $\boldsymbol{C}$ be an arbitrary category, and let K and L be arbitrary objects of $\boldsymbol{C}$. These give hom-functors

$$[K, -] \qquad [L, -]$$

from $\boldsymbol{C}$ to $\boldsymbol{Set}$.

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

As is customary, here we can omit the name of the parent category on the hom-functors. Let

$$L \xrightarrow{\ \phi\ } K$$

be an arbitrary arrow of $\boldsymbol{C}$. For each object A of $\boldsymbol{C}$ we have an assignment

$$\begin{array}{ccc} [K, A] & \xrightarrow{\ \tau_A\ } & [L, A] \\ l & \longmapsto & l \circ \phi \end{array}$$

given by composition in $\boldsymbol{C}$. We show these functions form a natural transformation between the two functors.

To do that we must consider an arbitrary arrow f of $\boldsymbol{C}$, as on the left

$$
\begin{array}{ccccc}
 & l & \longmapsto & & l \circ \phi \\
A & \big\downarrow & [K,A] \xrightarrow[\tau_A]{- \circ \phi} [L,A] & & \big\downarrow \\
f \big\downarrow & & f \circ - \big\downarrow \qquad\qquad \big\downarrow f \circ - & & \\
B & & [K,B] \xrightarrow[- \circ \phi]{\tau_B} [L,B] & & f \circ (l \circ \phi) \\
 & f \circ l & \longmapsto & (f \circ l) \circ \phi &
\end{array}
$$

and verify that the inner square commutes. To do that we take an arbitrary member

$$K \xrightarrow{\ l\ } A$$

of the top left hand corner of the square and track it both ways to the bottom right hand corner. We required that both paths give the same member of $[L, B]$. Thus we require

$$(f \circ l) \circ \phi = f \circ (l \circ \phi)$$

which is immediate. ■

You should now work out the corresponding result for the contravariant hom-functors. Exercises 3.5.1 and 3.5.2 deal with this and a more involved version.

By inspecting the construction of the natural transformation τ of Example 3.6 we see that it is completely determined by one particular output

$$\phi = \tau_K(\boldsymbol{id}_K)$$

of one particular component of τ. There is more to this.

3.7 EXAMPLE. Let $\boldsymbol{C}$ be an arbitrary category, let

$$\boldsymbol{C} \xrightarrow{\ F\ } \boldsymbol{Set}$$

be an arbitrary functor to $\boldsymbol{Set}$. Let K be an arbitrary object of $\boldsymbol{C}$. What can a natural transformations

$$[K, -] \longrightarrow F$$

look like? We show that they are in bijective correspondence with the elements of the set FK.

(a) Consider first any element $k \in FK$. This gives us a family of functions

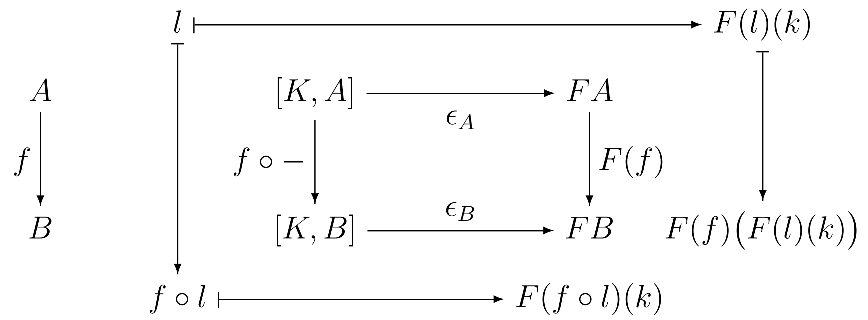
$$
\begin{array}{rcl}
[K, A] & \xrightarrow{\ \epsilon_A\ } & FA \\
l & \longmapsto & F(l)(k)
\end{array}
$$

indexed by the objects A of $\boldsymbol{C}$. This function ϵ_A is 'evaluation at k'.

We check that the family

$$[K, -] \xrightarrow{\epsilon} F$$

is a natural transformation. To do that we must consider an arbitrary arrow f of $\boldsymbol{C}$, as on the left



and verify that the inner square commutes. To do that we take an arbitrary member

$$K \xrightarrow{l} A$$

of the top left hand corner of the square and track both ways to the bottom right hand corner. We required that both paths give the same member of FB. Thus we require

$$F(f \circ l)(k) = F(f)\big(F(l)(k)\big)$$

for arbitrary f and l. But F is a covariant functor. so

$$F(f)\big(F(l)(k)\big) = \big(F(f) \circ F(l)\big)(k) = F(f \circ l)(k)$$

to give the required result.

(b) This gives us many examples of natural transformations from $[K, -]$ to F. Are there any more? In fact, we can show that every such natural transformation is determined by a unique element of FK.

Consider an arbitrary natural transformation

$$[K, -] \xrightarrow{\tau} F$$

look at the K-component

$$[K, K] \xrightarrow{\tau_K} FK$$

and set

$$k = \tau_K(\boldsymbol{id}_K)$$

to produce $k \in FK$. We show that τ is 'evaluation at k'.

Consider an arbitrary object A of $\boldsymbol{C}$ and an arbitrary member

$$K \xrightarrow{l} A$$

of $[K, A]$. We show that

$$\tau_A(l) = F(l)(k)$$

holds.

To do that we remember that the square

$$\begin{array}{ccc} [K,K] & \xrightarrow{\tau_K} & FK \\ {\scriptstyle l \circ -}\downarrow & & \downarrow{\scriptstyle F(l)} \\ [K,A] & \xrightarrow[\tau_A]{} & FA \end{array}$$

commutes. By tracking the member $\boldsymbol{id}_K$ of the top left hand corner we obtain the required result. ■

When the natural transformation

$$[K, -] \xrightarrow{\epsilon} F$$

induced by $k \in FK$ is a natural isomorphism, we say the pair (K, k) is a pointwise representation of F. We say F is representable when it has at least one pointwise representation.

There is, of course, a contravariant version of this example, and in a way that is more important.

Let $\boldsymbol{C}$ be an arbitrary category. A presheaf on $\boldsymbol{C}$ is a contravariant $\boldsymbol{Set}$-valued functor.

$$\boldsymbol{C} \xrightarrow{F} \boldsymbol{Set}$$

Such presheaves F and G are compared via natural transformations.

$$G \xrightarrow{\tau} F$$

These presheaves, as objects, and natural transformations, as arrows, form a category $\widehat{\boldsymbol{C}}$, the Yoneda completion of $\boldsymbol{C}$. That need not concern us here.

Each object A of $\boldsymbol{C}$ gives a presheaf on $\boldsymbol{C}$

$$\widehat{A} = \boldsymbol{C}[-, A]$$

the contravariant hom-functor. These are sometimes called the representable presheaves.

Let A be a fixed object of $\boldsymbol{C}$, and let F be a fixed presheaf on $\boldsymbol{C}$. The basic Yoneda result characterizes the natural transformations

$$\widehat{A} \longrightarrow F$$

from the representable to the arbitrary. They are essentially the elements of the set FA. See Exercises 3.5.4 to 3.5.6. You might also want to have another look at Exercises 3.4.2 and 3.4.3.

In Block 3.3.1 we set up three endo-functors

$$\exists \quad \mathsf{I} \quad \forall$$

on $\boldsymbol{Set}$. There are several natural transformations associated with these.

3.8 EXAMPLE. Let $\boldsymbol{Id}$ be the identity endo-functor on $\boldsymbol{Set}$. Thus

$$\boldsymbol{Id}A = A \qquad \boldsymbol{Id}(f) = f$$

for each set A and function f. We set up two natural transformations

$$\boldsymbol{Id} \xrightarrow{\ \eta^{\exists}\ } \exists \qquad\qquad \boldsymbol{Id} \xrightarrow{\ \eta^{\forall}\ } \forall$$

using the two indicated covariant endo-functors on $\boldsymbol{Set}$. Thus for each set A we require a pair a functions

$$A \xrightarrow{\ \eta^{\exists}_A\ } \mathcal{P}A \qquad\qquad A \xrightarrow{\ \eta^{\forall}_A\ } \mathcal{P}A$$

with appropriate properties. Notice that here we have omitted '$\boldsymbol{Id}$'. This should not cause too much confusion.

We must produce $\eta^{\exists}$ and $\eta^{\forall}$ so that for each function f the two squares

$$\begin{array}{ccccc} A & \xrightarrow{\ \eta^{\exists}_A\ } & \mathcal{P}A & & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \exists(f)} & & \downarrow \\ B & \xrightarrow[\ \eta^{\exists}_B\ ]{} & \mathcal{P}B & & f[X] \end{array} \qquad\qquad \begin{array}{ccccc} A & \xrightarrow{\ \eta^{\forall}_A\ } & \mathcal{P}A & & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \forall(f)} & & \downarrow \\ B & \xrightarrow[\ \eta^{\forall}_B\ ]{} & \mathcal{P}B & & f[X']' \end{array}$$

commute. For each $a \in A$ each of

$$\eta^{\exists}_A(a) \qquad \eta^{\forall}_A(a)$$

must be a certain subset of A. Given the other requirements, there isn't much choice. ■

The two natural transformations $\eta^{\exists}$ and $\eta^{\forall}$ of this last example don't look very interesting. However, in a more general setting they are quite important. We ***may*** return to this in ***a supplementary chapter***.

Let's now look at the contravariant power set functor. In the next example we set up a natural isomorphism which again doesn't look very exiting. However, the idea has many important refinements.

3.9 EXAMPLE. The inverse image functor I on $\boldsymbol{Set}$ is contravariant, and is really a hom-functor in disguise.

The set

$$\mathsf{2} = \{0, 1\}$$

induces a hom-functor $[-, \mathsf{2}]$ on $\boldsymbol{Set}$. Thus we have two endo-functors on $\boldsymbol{Set}$

$$\boldsymbol{Set} \begin{array}{c} \xrightarrow{\ \ \mathsf{I}\ \ } \\ \xrightarrow[\ [-,\mathsf{2}]\ ]{} \end{array} \boldsymbol{Set}$$

both of which are contravariant. We show that these two functors are naturally isomorphic.

To do that we recall that for any set A its subsets are in bijective correspondence with the characteristic functions on A. Thus, for each $X \in \mathcal{P}A$ we let

$$\chi_A(X) : A \longrightarrow \mathsf{2}$$

be given by

$$\chi_A(X)(a) = \begin{cases} 1 \text{ if } a \in X \\ 0 \text{ if } a \notin X \end{cases}$$

for $a \in A$. The assignment

$$\begin{array}{rcl} \mathcal{P}A & \xrightarrow{\ \chi_A\ } & [A, \mathsf{2}] \\ X & \longmapsto & \chi_A(X) \end{array}$$

is a bijection. We show this is natural for variation of A. To do that we must show that the inner square commutes

$$\begin{array}{ccccccc} & X & \longmapsto & & & p = \chi_A(X) \\ A & \big\downarrow & \mathcal{P}A & \xrightarrow{\ \chi_A\ } & [A, \mathsf{2}] & \big\downarrow \\ f \big\uparrow & & f^{\leftarrow} \big\downarrow & & \big\downarrow - \circ f & \\ B & & \mathcal{P}B & \xrightarrow[\ \chi_B\ ]{} & [B, \mathsf{2}] & p \circ f \\ & f^{\leftarrow}(X) & \longmapsto & & q & \end{array}$$

for an arbitrary arrow f, as on the left. Observe the contravariance here.

We track an arbitrary member $X \in \mathcal{P}A$ of the top left hand corner both ways to the bottom right hand corner. This gives us two members q and $p \circ f$ of $[B, \mathsf{2}]$. A simple calculation shows that these are the same function. ■

Any two compatible contravariant functors can be composed to produce a covariant functor. That often happens when we produce a 'representation' of an algebra of some kind. Let's look at a miniature version of that.

3.10 EXAMPLE. The inverse image functor I on Set can be composed with itself

$$\Pi = \mathsf{I} \circ \mathsf{I}$$

to produce a covariant endo-functor on ***Set***. Thus for each set A we have

$$\Pi A = \mathcal{P}^2 A$$

the second power set of A, the family of all collections of subsets of A. To describe the behaviour of Π on functions it is convenient to fix some notation.

Each function f gives us two other functions

$$A \xrightarrow{\ \ f\ \ } B$$

$$\mathcal{P}A \xleftarrow{\ \mathsf{I}(f) = f^{\leftarrow}\ } \mathcal{P}A$$

$$\mathcal{P}^2 A \xrightarrow{\ \Pi(f)\ } \mathcal{P}^2 B$$

where the central one goes in the opposite direction. For the pair of sets

$$A \qquad B$$

we let

$$\begin{array}{cc} x \in A & y \in B \\ X \in \mathcal{P}A & Y \in \mathcal{P}B \\ \mathcal{X} \in \mathcal{P}^2A & \mathcal{Y} \in \mathcal{P}^2B \end{array}$$

be typical members of the indicated sets. We have

$$x \in \mathsf{I}(f)(Y) \iff f(x) \in Y$$

for each $Y \in \mathcal{P}B$ and $x \in A$. This gives

$$Y \in \Pi(f)(\mathcal{X}) \iff \mathsf{I}(f)(Y) \in \mathcal{X} \iff f^{\leftarrow}(Y) \in \mathcal{X}$$

for $\mathcal{X} \in \mathcal{P}^2A$ and $Y \in \mathcal{P}B$.

For each set A let

$$A \xrightarrow{\ \eta_A\ } \mathcal{P}A$$

be the function given by

$$X \in \eta_A(x) \iff x \in X$$

for x, X as above. We show that

$$\boldsymbol{Id} \xrightarrow{\ \eta\ } \Pi$$

is a natural transformation.

We must show that the inner square commutes

$$\begin{array}{ccccc} x & \longmapsto & & & \eta_A(x) \\ \Big\downarrow & A & \xrightarrow{\ \eta_A\ } & \mathcal{P}^2A & \Big\downarrow \\ & {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle \Pi(f)} & \\ & B & \xrightarrow[\ \eta_B\ ]{} & \mathcal{P}^2B & \Pi(f)(\eta_A(x)) \\ f(x) & \longmapsto & & \eta_B(f(x)) & \end{array}$$

for each function f as indicated on the left of the square. To do that, as usual we take an arbitrary member x of the top left hand component and track it both ways to the bottom right hand component. Thus

$$\eta_B(f(x)) = \Pi(f)(\eta_A(x))$$

is the problem. This can be verified by a simple calculation. ■

As final example we look at one of the motivating 'natural' constructions and its 'unnatural' mate.

3.11 EXAMPLE. Let K be a (commutative) field, and let $\boldsymbol{Vect}_K$ be the category of vector spaces over K, or K-spaces for short.

Each K-space V is an abelian group, written additively, and furnished with an action

$$\begin{array}{rcl} K, V & \longrightarrow & V \\ r, a & \longmapsto & ra \end{array}$$

satisfying the usual axioms. This is a left action, but since K is commutative the difference between left and right hardly matters.

These K-spaces are the objects of $\boldsymbol{Vect}_K$, and the arrows are the corresponding linear transformations.

The field K is itself a K-space. Thus for an arbitrary K-space V we may form the hom-set

$$V^* = \boldsymbol{Vect}_K[V, K]$$

in $\boldsymbol{Vect}_K$. It turns out that we can furnish V^* as a K-space to produce the dual space of V. In fact, $(\cdot)^*$ is a contravariant endo-functor on $\boldsymbol{Vect}_K$. It is an enriched hom-functor.

We wish to investigate the interaction between a parent K-space V and its dual space V^*. To do that it helps if we fix some notation and terminology.

We let

$$\begin{array}{rl} r, s, t, \ldots & \text{range over scalars, the members of } K \\ a, b, c, \ldots & \text{range over vectors, the members of } V \\ \alpha, \beta, \gamma, \ldots & \text{range over } \textit{name} \text{ , the members of } V^* \end{array}$$

and we let

$$f, g, h, l \ldots$$

range over linear transformations between K-spaces.

The elements of V^* are those functions

$$\alpha : V \longrightarrow K$$

such that

$$\alpha(0) = 0$$
$$\alpha(a + b) = \alpha(a) + \alpha(b)$$
$$\alpha(ra) = r\alpha(a)$$

for all $a, b \in V$ and $r \in K$. We add these pointwise and this, with the obvious zero, furnishes V^* as an abelian group.

The action

$$\begin{array}{rcl} K, V^* & \longrightarrow & V^* \\ r, \alpha & \longmapsto & r\alpha \end{array}$$

is given by

$$(r\alpha)(a) = r(\alpha(a))$$

for $r \in K, \alpha \in V^*$, and $a \in V$. This converts V^* into a K-space.

Each finite dimensional K-space V is uniquely determined up to isomorphism. If V has dimension $n \geq 0$ then the isomorphisms

$$K^n \longrightarrow V$$

are in bijective correspondence with the bases of V. The crucial fact, which we won't prove here but which you should look up sometime, is as follows.

> Let V be a finite dimensional K-space. Then the dual space V^* is finite dimensional with the same dimension. In particular $V \cong V^*$.

This suggest a problem.

Let V be finite dimensional. There is at least one isomorphism

$$V \longrightarrow V^*$$

but is there a canonical one? To set up such an isomorphism we must first select a base for V, and then the resulting isomorphism is hardly canonical.

Now we come to the puzzling bit. Or what used to be the puzzling bit.

Each K-space V has a dual space V^* which itself has a dual space V^{**}. This is the second dual of V. We know that $(\cdot)^{**}$ is a covariant endo-functor on $\boldsymbol{Vect}_K$ (because it is the composite of two contravariant endo-functors). Furthermore, it is easy to exhibit members of V^{**}.

For each $a \in V$ let

$$a^\frown : V^* \longrightarrow K$$

be the function given by

$$a^\frown(\alpha) = \alpha(a)$$

for $\alpha \in V^*$.

We can now check three facts.

(1) For each $a \in V$ the functions $a^\frown$ is a member of V^{**}.

(2) For each K-space V the assignment

$$V \xrightarrow{(\cdot)^\frown} V^{**}$$

is a linear transformation which is injective.

(3) The whole family of assignments $(\cdot)^\frown$is a natural transformation.

From this we see that for each finite dimensional K-space V the assignment of (2) is a canonical isomorphism, independent of any choice of base. ■

You can see what the puzzle was. Why is it that the second dual seems to have a 'natural' behaviour whereas the first dual doesn't? Category theory helped to explain that.

We will meet many more functors and natural transformations. Some of these are quite complicated. The exercises give some hints of what can happen.

Exercises

3.5.1 Consider arbitrary objects K, L of an arbitrary category $\boldsymbol{C}$. Show how a natural transformation

$$[-, L] \xrightarrow{\tau} [-, K]$$

can be induced by an arrow between K and L.

3.5.2 Let $\boldsymbol{C}$ be an arbitrary category and let

$$Q \xrightarrow{\ p\ } P$$

be an arbitrary arrow of $\boldsymbol{C}$. Let

$$R \xrightarrow{\ s\ } S$$

be an arbitrary function (between sets). For each object A of $\boldsymbol{C}$ let

$$FA = \boldsymbol{Set}[\boldsymbol{C}[A, P], R] \qquad GA = \boldsymbol{Set}[\boldsymbol{C}[A, Q], S]$$

using hom-sets in the two different categories.

(a) Show that each of

$$A \longmapsto FA \qquad\qquad A \longmapsto GA$$

is the object assignment of a functor

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

and determine the variance of each.

(b) Use the arrow p and function s to produce a natural transformation $F \longrightarrow G$.

3.5.3 Complete the details of Example 3.7.

3.5.4 Consider the notion of a presheaf as defined just after Example 3.7. Where have you seen examples of this before?

3.5.5 Consider an arbitrary category $\boldsymbol{C}$, and arbitrary presheaf F on $\boldsymbol{C}$, an arbitrary object A of $\boldsymbol{C}$, and an arbitrary element $a \in FA$.

For each set X consider the following assignment.

$$\begin{array}{rcl} \boldsymbol{C}[X, A] & \xrightarrow{\ \overline{a}_X\ } & FX \\ k & \longmapsto & F(k)(a) \end{array}$$

Check that this is a function. In other words, show that the output does live in FX. Show that the whole family $\overline{a}$ is a natural transformation.

3.5.6 Continuing with the notation of Exercise 3.5.5, consider an arbitrary natural transformation

$$\widehat{A} \xrightarrow{\ \tau\ } F$$

and let $a = \tau_A(\boldsymbol{id}_A)$.

Check that $a \in FA$, and show that $\tau = \overline{a}$.

3.5.7 Describe the two natural transformations $\eta^{\exists}$ and $\eta^{\forall}$ of Example 3.8.

3.5.8 Complete the calculation of Example 3.9.

3.5.9 (a) Do the calculation required to complete Example 3.10.

(b) By Example 3.9 the inverse image functor I is naturally isomorphic to the hom-functor $[-, \mathsf{2}]$. Thus Π is naturally isomorphic to the endo-functor with

$$A \longmapsto [[A, \mathsf{2}], \mathsf{2}]$$

as the object assignment.

Write down the arrow assignment and re-do Example 3.10 for this functor.

(c) Which version do you think is easier to understand?

3.5.10 Consider Example 3.11.

(a) Write down all the axioms needed to set up $\boldsymbol{Vect}_K$. (The axioms for a field, for an additive abelian group, for an action, and for a linear transformation.)

It is instructive not to overload the notation. In other words, distinguish between the various additions, and use a different visible infix for each multiplication.

(b) Verify that the dual space V^* of a K-space is itself a K-space. (You may now go back to the standard, overloaded, notation.)

(c) Show that $(\cdot)^*$ is a contravariant endo-functor on $\boldsymbol{Vect}_K$. In particular, you must decide how $(\cdot)^*$ behaves on arrows of $\boldsymbol{Vect}_K$.

3.5.11 Continuing with Example 3.11, verify the three facts (1, 2, 3).

3.5.12 Show that the two functors $\mathcal{O}$ and Ξ of Block 3.3.2 are natural isomorphic.

3.5.13 Consider the functors arising from the product construction, as described in Block 3.3.3 and Exercises 3.3.8 and 3.3.9. Show that the projection arrows form a natural transformations.

3.5.14 Let $\boldsymbol{C}$ be a category with all binary products. Let R, S be two objects and let

$$F = - \times R \qquad G = - \times S$$

to obtain two endo-functors on $\boldsymbol{C}$.

Show that each arrow

$$R \xrightarrow{\ \phi\ } S$$

of $\boldsymbol{C}$ induces a natural transformation

$$F \xrightarrow{\ \phi_\bullet\ } G$$

between these functors. This involves some serious diagram chasing.

3.5.15 Let $\boldsymbol{C}$ be a category with all binary products and coproducts, and let A, B, C be three arbitrary objects of $\boldsymbol{C}$. Let

$$L = A \times C + B \times C \qquad R = (A + B) \times C$$

to form two more objects.

Show that by fixing two of A, B, C, each of L and R is an endo-functor of $\boldsymbol{C}$, and there is a natural transformation $L \longrightarrow R$.

If you are brave you might try the 3-placed version of this, that is do not fix two of A, B, C.

3.5.16 Recall the difference between a monoid and a semigroup. (A semigroup need not have a unit.)

Given a semigroup A let

$$FA = A \cup \{\omega\}$$

where ω is a new element not in A. Let

$$A \xrightarrow{\ \iota\ } FA$$

be the insertion. Let $\star$ be the operation on FA given by

$$a \star b = ab \qquad a \star \omega = a = \omega \star a \qquad \omega \star \omega = \omega$$

for all $a, b \in A$.

(a) Show that $(FA, \star, \omega)$ is a monoid.

(b) Show that ι is a semigroup morphism.

(c) Show that for each semigroup morphism

$$A \xrightarrow{\ f\ } B$$

to a monoid B, there is a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle\iota}\searrow \quad \nearrow{\scriptstyle f^{\sharp}} & \\ & FA & \end{array}$$

for some unique monoid morphism $f^{\sharp}$.

(d) Show that F fills out to a functor. You will have to sort out its source and target.

(e) Show that ι is natural for variation of A. You will have to insert a couple of trivial functors.

(f) What happens if A already has a unit?

3.5.17 For an arbitrary set A think of A as an alphabet. Let FA be the set of all words on A, finite lists

$$\mathsf{a} = [a_1, \ldots, a_l]$$

for $a_1, \ldots, a_l \in A$. The empty word, with $l = 0$, is allowed.

(a) Show that FA is a monoid under concatenation.

(b) Show that F fills out to a functor. Make sure you write down the source and target.

(c) Show that the assignment

$$\begin{array}{rcl} A & \longrightarrow & FA \\ a & \longmapsto & [a] \end{array}$$

is a natural transformation. You will have to sort out the two functors it passes between.

(d) Show that FA is the free monoid on the sets A in a sense that you should make precise.

(e) What happens if A already carries a monoid structure?

3.5.18 Consider the two functors

$$A \longmapsto \delta A \qquad\qquad A \longmapsto A/\delta A$$

of Exercise 3.3.13 (where here only the object assignments have been given).

(a) Show that both the canonical embedding ι and the canonical quotient κ

$$\delta A \xrightarrow{\ \iota\ } A \qquad\qquad A \xrightarrow{\ \eta\ } A/\delta A$$

is natural for variation of A. You must describe explicitly the source and target for each functor.

(b) Show that for each morphism

$$A \xrightarrow{\ f\ } B$$

from an arbitrary group A to an abelian group B, there is a unique morphism

$$A/\delta A \xrightarrow{\ f^\sharp\ } B$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & A/\delta A & \end{array}$$

commutes.

3.5.19 Consider the 'freely generated by' construction of Exercise 3.3.18. Show that the family η of arrows is a natural transformation.

3.5.20 Let ∇ be an arbitrary category and think of this as a template. Let $\boldsymbol{C}$ be an arbitrary category. These combine to give another category $\boldsymbol{C}^\nabla$. The objects a of $\boldsymbol{C}^\nabla$ are the covariant functors

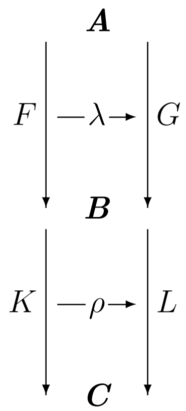
$$\nabla \longrightarrow \boldsymbol{C}$$

and the arrows are the natural transformations between these functors.

Show that this does give a category. The problem is to produce an appropriate method of composing natural transformations.

3.5.21 The composition used in Exercises 3.5.20 is know as the vertical composition of natural transformations. There is also a horizontal composition. (Don't ask what is vertical or horizontal about the two constructions.)

Consider three categories $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, four functors F, G, K, L, and two natural transformations λ, ρ, as shown.



Show that for each object A of $\boldsymbol{A}$ the following square commutes.

$$\begin{array}{ccc} (K\circ F)A & \xrightarrow{\rho_{FA}} & (L\circ F)A \\ {\scriptstyle K(\lambda_A)}\downarrow & & \downarrow{\scriptstyle L(\lambda_A)} \\ (K\circ G)A & \xrightarrow[\rho_{GA}]{} & (L\circ G)A \end{array}$$

Let

$$(K\circ F)A \xrightarrow{(\rho\star\lambda)_A} (L\circ G)A$$

be the diagonal of this square. Show that he whole family

$$(\rho\star\lambda)_\bullet$$

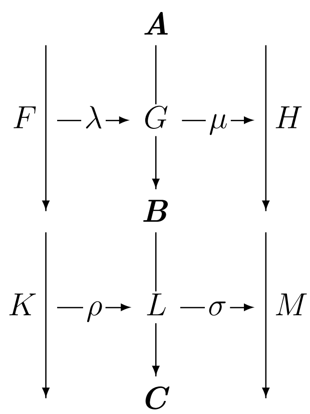
is natural.

This gives the horizontal composite of

$$(K\circ F) \xrightarrow{\rho\star\lambda} (L\circ G)$$

of the two natural transformations between the composite functors.

3.5.22 Consider three categories $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, six functors F, G, H, K, L, M, and four natural transformations $\lambda, \mu, \rho, \sigma$, as shown.



Using vertical and horizontal composition (as in Exercises 3.5.20 and 3.5.20) show that

$$(\sigma \star \mu) \circ (\rho \star \lambda) = (\sigma \circ \rho) \star (\mu \circ \lambda)$$

holds.

At the moment there are 53 exercises in this chapter.

4
Limits and colimits in general

In Chapter 2, Sections 2.3 – 2.7 we looked at some of the simpler examples of limits and colimits. These examples are brought together in Table 2.1 which is repeated here for convenience. In this chapter we generalize the idea.

Before we begin the details it is useful to outline the five steps we go through together with the associated notions for each step. After that we look at each step in more detail.

Template

This is the shape ∇ that a particular kind of diagram can have. It is a picture consisting of nodes (blobs) and edges (arrows). The central column of Table 4.1 lists a few of the simpler templates. Technically, a template is often a directed graph, or more generally a category.

Diagram

This is an instantiation of a particular template ∇ in a category $\boldsymbol{C}$. Each node of ∇ is instantiated with an object of $\boldsymbol{C}$, and each edge is instantiated with an arrow of $\boldsymbol{C}$. Of course, there are some obvious source and target restrictions that must be met, and the diagram may require that certain cells commute. This is why we sometimes use a category as a template.

Posed problem

Each diagram in a category $\boldsymbol{C}$ poses two problems, the left (blunt end) problem and the right (sharp end) problem. We never actually say what the problem is (which is perhaps the reason why it is rarely mentioned) but we do say what a solution is. The idea is to find a 'best' solution.

Solution

A solution for a diagram in $\boldsymbol{C}$ is a nominated object X of $\boldsymbol{C}$ together with a collection of arrows. For a left solution all arrows start from X, and such a gadget is often called a cone. For a right solution all arrows finish at X, and such a gadget is often called a co-cone. For both kinds of solutions the arrows must make various triangles commute.

Universal solution

A universal solution for a diagram is a particular solution through which each solution (of that handedness) must pass via a unique mediating arrow. A limit is a universal left solution. A colimit is a universal right solution.

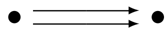
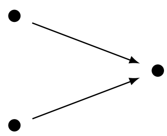
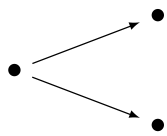
	Limit	Template	Colimit
(1)	final object		initial object
(2)	binary product		binary coproduct
(3)	equalizer		co-equalizer
(4)	pullback		
(5)			pushout

Table 4.1: Some simple limits and colimits – a repeat of Table 2.1

We now begin to look at each these notions in more detail.

4.1 Template and diagram – a first pass

Roughly speaking a template is a collection of nodes, each drawn as a •, and a collection of edges, each drawn as an arrow. Each edge passes from a particular node (its source) to a particular node (its target). In other words, a template is a directed graph. There may also be some commuting conditions on the edges, in which case the template is a category. It is customary to draw the edges as pointing from left to right.

Eventually we instantiate the template in a category $\boldsymbol{C}$ to produce a diagram in $\boldsymbol{C}$. We replace each node by an object of $\boldsymbol{C}$ and we replace each edge by an arrow of $\boldsymbol{C}$. Of course, we respect the source and target conditions and any commuting conditions that the template requires. We then look for the left solutions (blunt end solutions) or the right solutions (sharp end solutions). In particular, we look for a universal solution on the appropriate side, to obtain a limit (universal left solution) or a colimit (universal right solution).

Table 4.1 gives a few small templates. Let's look at a few more examples that are not so simple.

4.1 EXAMPLES. (1) Suppose we have a collection of nodes with no edges. It is convenient of think of this collection of nodes

$$\begin{array}{c}\vdots\\ \bullet\\ \bullet\\ \bullet\\ \bullet\\ \vdots\end{array}$$

arranged vertically. There may be infinitely many of these, finitely many, or none at all.

A limit for a corresponding diagram is a product and a colimit is a coproduct (sometimes called a sum). We have already seen the case where there are zero, one, or just two nodes.

(2) Suppose we have a collection of nodes arranged in a line with an edge between adjacent nodes.

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

If there are only finitely many nodes then the posed problem isn't interesting (since any associated diagram has a left-most object and a right-most object). Thus we may as well suppose there are infinitely many nodes. We use the integers as nodes. This gives us three (or perhaps four) different templates.

The template may have a left-most node and go off to the right

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

in which case it is the colimit (right universal solution) that is interesting.

The template may have a right-most node and go off to the left

$$\cdots \longrightarrow 4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$$

in which case it is the limit (left universal solution) that is interesting. Notice that we have again used he natural numbers to label the nodes.

The template may go off to the left and the right

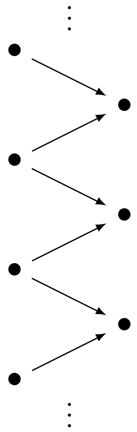
$$\cdots \longrightarrow -3 \longrightarrow -2 \longrightarrow -1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

$$\cdots \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow -2 \longrightarrow -3 \longrightarrow \cdots$$

in which case we may use the integers to label the nodes in one of two ways. Both these are useful in different circumstances.

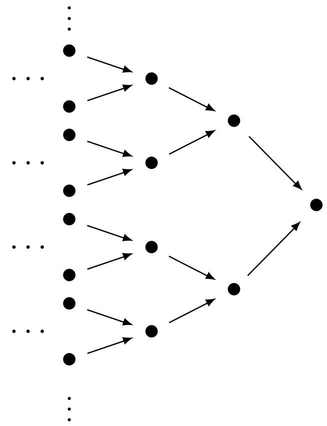
For a diagram of this shape both the limit and the colimit will be interesting.

(3) The template may be a collection of zig-zags



and may be finite or infinite. Even the finite case (with at least four nodes) leads to interesting solutions.

(4) There are more complicated examples.



This is a tree growing to the left and keeps growing for ever, but with the edges pointing to the right. A limit is something that is put out on the far left. What can that be?

(5) Finally, consider the template in Table 4.2. The nodes are arranged in vertical layers each of which is infinite both ways, and there are infinitely many layers progressing leftwards. Each edge passes between one layer and the next.

Think of passing through this graph from right to left, moving backwards along the edges. In some intuitive sense something is being approached out on the far left. The notion of a limit makes this precise.

In this example the template stops at the vertical layer on the right. We could also continue the same kind of pattern moving off to the right. This would not change the limit (to the left) but could have a dramatic impact on the colimit (to the right). ■

Let's now try to make the idea of these examples precise.

4.2 DEFINITION. A template (of the first kind)

$$\nabla = (\mathbb{I},\ \mathbb{E})$$

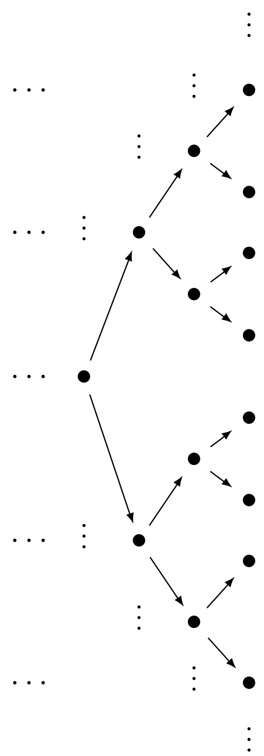


Table 4.2: A more exotic template

is a directed graph consisting of

$$\text{nodes } i, j, k, \ldots \text{ in } \mathbb{I} \qquad \text{edges } e, f, g, \ldots \text{ in } \mathbb{E}$$

where each edge

$$i \xrightarrow{\ e\ } j$$

has a nominated source and target, each of which is a node. ■

Notice that this version says nothing about any commuting conditions in the template. We are being a bit coy here, but eventually we will look at that condition. However, notice that all the templates of Table 4.1 and Examples 4.1 match this definition.

4.3 DEFINITION. Let $\nabla = (\mathbb{I},\ \mathbb{E})$ be a directed graph viewed as a template of the first kind. Let $\boldsymbol{C}$ be a category. A ∇-diagram in $\boldsymbol{C}$ is

- An $\mathbb{I}$-indexed family of objects of $\boldsymbol{C}$ $\qquad \mathsf{A} = \big(A(i) \mid i \in \mathbb{I}\big)$
- An $\mathbb{E}$-indexed family of arrows of $\boldsymbol{C}$ $\qquad \mathcal{A} = \big(A(e) \mid e \in \mathbb{E}\big)$

where each edge

$$i \xrightarrow{\ e\ } j$$

produces an arrow

$$A(i) \xrightarrow{A(e)} A(j)$$

with indicated source and target restrictions. ■

In other words, this is just a 'functor' from $\nabla = (\mathbb{I}, \mathbb{E})$ but with any commuting conditions ignored.

There aren't many exercises concerned solely with templates, but the following construction should be looked at. It and the associated exercises could go elsewhere, but here it is clearly relevant.

Consider the notion of a directed graph as given in Definition 4.2. Such a gadget consists of nodes and edges with two source and target assignments. This looks a bit like the notion of a category. Each category is a directed graph, but not conversely. A directed graph has no notion of composition of edges, and no notion of identity edges. However, there is a construction that converts each directed graph into a category.

4.4 DEFINITION. Let ∇ be a directed graph.

For each $l \in \mathbb{N}$ a path through ∇ of length l is a list of l edges

$$i(0) \xrightarrow{e(1)} i(1) \xrightarrow{e(2)} i(2) \longrightarrow \cdots\cdots \xrightarrow{e(l)} i(l)$$

where the target of each edge is the source of the next one. A path of length 1 is just an edge. A path of length 0 is just a node.

We create a category $\boldsymbol{Pth}(\nabla)$, the category of paths through ∇.

The objects of $\boldsymbol{Pth}(\nabla)$ are the nodes of ∇.

The arrows of $\boldsymbol{Pth}(\nabla)$ are the paths through ∇.

Given two paths

$$i(0) \longrightarrow i(1) \longrightarrow \cdots \longrightarrow i(l) \qquad j(0) \longrightarrow j(1) \longrightarrow \cdots \longrightarrow j(m)$$

with $i(l) = j(0)$ the composite path is

$$i(0) \longrightarrow i(1) \longrightarrow \cdots \longrightarrow i(l) = j(0) \longrightarrow j(1) \longrightarrow \cdots \longrightarrow j(m)$$

formed by sticking one path after the other. ■

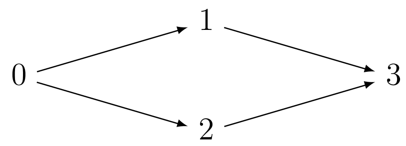
Of course, there is something to be checked here, and in this case it is not entirely trivial.

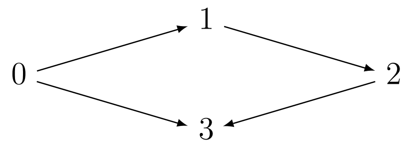
Exercises

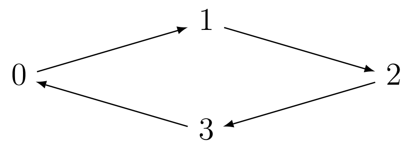
4.1.1 Show that the construction $\boldsymbol{Pth}(\cdot)$ does produce a category.

Make sure you verify all the required properties. This is one of the few examples where the required identity properties are not immediately obvious.

4.1.2 Consider the following four different graphs.

(a) 

(b) 

(c) 

In each case describe the generated category of paths.

4.1.3 Suppose the graph you start from is already a category. Do you merely reconstruct the category?

4.1.4 Let ∇ be a diected graph viewed as a template. Let A be a ∇-diagram in some category $\boldsymbol{C}$.

Show that A extends uniquely to a functor $\boldsymbol{Pth}(\nabla) \longrightarrow \boldsymbol{C}$.

Show that each functor $\boldsymbol{Pth}(\nabla) \longrightarrow \boldsymbol{C}$ is the unique extension of some ∇-diagram in $\boldsymbol{C}$.

4.2 Functor categories

In this section we replace the directed graph $(\mathbb{I}, \mathbb{E})$ by a category ∇. We think of ∇ as an indexing gadget, and to emphasize this we refer to the objects $i, j, k, \ldots$ of ∇ as nodes and its arrows $e, f, g, \ldots$ as edges.

4.5 DEFINITION. Let ∇ be an arbitrary category viewed as a template. Let $\boldsymbol{C}$ be an arbitrary category. These combine to produce the category

$$\boldsymbol{C}^{\nabla}$$

of ∇-diagrams in $\boldsymbol{C}$.

Each object of $\boldsymbol{C}^{\nabla}$ is a functor

$$\nabla \xrightarrow{\ A\ } \boldsymbol{C}$$

from the template to $\boldsymbol{C}$.

Given two such functors

$$\nabla \underset{B}{\overset{A}{\rightrightarrows}} \boldsymbol{C}$$

an arrow

$$A \xrightarrow{\ \sigma\ } B$$

is a natural transformation between the functors.

Given three such functors and two such natural transformations

$$\begin{array}{ccccc} \nabla & & \nabla & & \nabla \\ A\downarrow & \xrightarrow{\ \sigma\ } & B\downarrow & \xrightarrow{\ \tau\ } & \downarrow C \\ \boldsymbol{C} & & \boldsymbol{C} & & \boldsymbol{C} \end{array}$$

the composite

$$\begin{array}{ccc} \nabla & & \nabla \\ A\downarrow & \xrightarrow{\ \tau\circ\sigma\ } & \downarrow C \\ \boldsymbol{C} & & \boldsymbol{C} \end{array}$$

is given by

$$A(i) \xrightarrow{\ (\tau\circ\sigma)_i = \tau_i \circ \sigma_i\ } C(i)$$

for each index i. ■

Of course, there is something to prove here. We must show that the definition of the composite $\tau \circ \sigma$ of two natural transformations is a natural transformation. We also need to show that this composition is associative. However, the proofs are straight forward.

As the terminology of Definition 4.2 suggests, for each category ∇ viewed as a template, and each category $\boldsymbol{C}$, a ∇-diagrams in $\boldsymbol{C}$ is merely a functor A from ∇ to $\boldsymbol{C}$. In other words, such a diagram is

a family of objects $A(i)$ of $\boldsymbol{C}$ a family of arrows $A(e)$ of $\boldsymbol{C}$

indexed by the

nodes edges

of ∇, respectively. As with a diagram over a directed graph we require that each edge

$$i \xrightarrow{\ e\ } j$$

of ∇ produces an arrow

$$A(i) \xrightarrow{\ A(e)\ } A(j)$$

of $\boldsymbol{C}$. We now also require that for each pair

$$i \xrightarrow{\ e\ } j \xrightarrow{\ f\ } k$$

of composible edges of ∇, the induced triangle in $\boldsymbol{C}$

$$\begin{array}{ccc} A(i) & \xrightarrow{\quad A(f\circ e)\quad} & A(k) \\ & {\scriptstyle A(e)}\searrow \qquad \nearrow{\scriptstyle A(f)} & \\ & A(j) & \end{array}$$

commutes. Finally, we now also require that

$$A(\boldsymbol{id}_i) = \boldsymbol{id}_{A(i)}$$

for each index i.[1]

In many cases of interest we don't use an arbitrary category as a template. We use a partially ordered set, or occasonally a pre-ordered set.

Let $\mathbb{I}$ be a pre-ordered set, let

$$i, j, k, \dots$$

range over $\mathbb{I}$ and think of these as nodes. We may view $\mathbb{I}$ as a category in one of two ways. For each pair of nodes i, j there is at most one edge

$$i \xrightarrow{\quad (j,i)\quad} j$$

from i to j. There is such an edge precisely when there is a comparison between i and j. We orientate these edges in one of two ways.

$$\begin{array}{ccc} \text{Upwards} & & \text{Downwards} \\ i \le j & \quad i \xrightarrow{\quad (j,i)\quad} j \quad & j \le i \end{array}$$

Depending on which view we take such an arrow always points upwards or always points downwards in the pre-ordered set.

Why might we want to do that?

4.6 DEFINITION. A pre-ordered set is directed or upwards directed if for each pair i, j of nodes there is at least one node k with $i \le k$ and $j \le k$. ■

Sometimes we want a pre-ordered diagram that is directed to the right (directed to the sharp end). In that case we index the diagram by a directed pre-order with its edges pointing upwards.

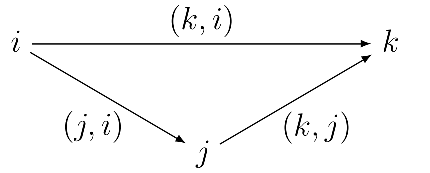
Sometimes we want a pre-ordered diagram that is directed to the left (directed to the blunt end). In that case we index the diagram by a directed pre-order with its edges pointing downwards. We could achieve the same effect using a *downwards* directed pre-order, but that rarely seems to be used.

To conclude this section let's take a closer look at the way edges in a preset are labelled. Since there is at most one edge between two indexes we can use these to label the edge.

$$i \xrightarrow{\quad (j,i)\quad} j$$

[1] I have seen an example – in the Topology book by Dugundji, page 427 – where this last condition is *not* required. I don't know if this is a useful idea or merely an aberration on the part of the author.

At first sight the two components of the edge seem to be the wrong way round. But consider what happens when we compose two such edges. The common node should dissapear.



$$(k, j) \circ (j, i) = (k, i)$$

This convention is neater.

Exercises

Throughout these exercises ∇ and $\boldsymbol{C}$ are an arbitrary pair of categories.

4.2.1 Show that the construction of $\boldsymbol{C}^{\nabla}$ does produce a category.

(This construction is also dealt with elsewhere in the Notes. The composition of natural transformations is sometimes called vertical composition.)

4.2.2 For each $\boldsymbol{C}$-object X let ΔX be the ∇-diagram with X at each node and $\boldsymbol{id}_X$ at each edge.

Show that $X \longmapsto \Delta X$ is the object assignment of a functor

$$\boldsymbol{C}^{\nabla} \longleftarrow \boldsymbol{C}$$

from $\boldsymbol{C}$ to $\boldsymbol{C}^{\nabla}$. This is sometimes called the diagonal functor.

4.2.3 Describe typical arrows

$$\Delta X \longrightarrow A \qquad\qquad A \longrightarrow \Delta X$$

where X is a $\boldsymbol{C}$-object and A is a ∇-diagram.

4.3 Problem and solution

Let ∇ be a template, let $\boldsymbol{C}$ be a category, and let A be a ∇-diagram in $\boldsymbol{C}$. This diagram A is a collection of objects and arrows given by the shape ∇. This diagram poses two problems in $\boldsymbol{C}$, the blunt end problem and the sharp end problem. When we draw the diagram we usually let the arrows point from left to right, so it is more common to speak of the left problem and the right problem.

The following definition is two definitions in one given in parallel.

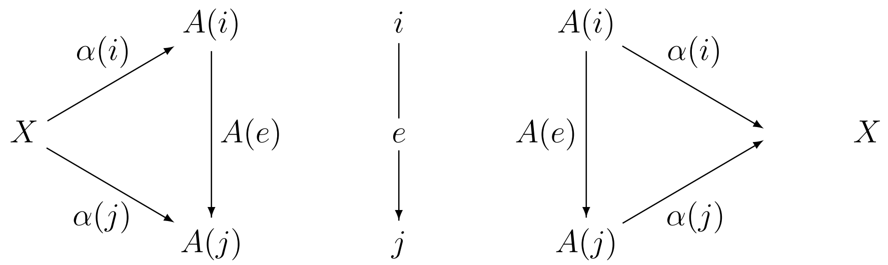
4.7 DEFINITION. Let ∇ be a template, and let A be a ∇-diagram in a category $\boldsymbol{C}$. A

left   right

solution is a $\boldsymbol{C}$-object X together with a family of arrows

$$X \xrightarrow{\alpha(i)} A(i) \qquad\qquad A(i) \xrightarrow{\alpha(i)} X$$

indexed by the nodes of ∇, such that for each edge e of ∇ the induced $\boldsymbol{C}$-triangle



commutes. ■

Any given diagram may have many different left solutions or it may have none. It may have many different right solutions or it may have none. In general there is little or no relationship between left solutions and right solutions.

In due course, the next section, we make precise the notion of a 'best' left solution or a 'best' right solution, but we don't need to worry about that just yet.

4.8 EXAMPLE. Suppose the category $\boldsymbol{C}$ is a poset with its arrows pointing upwards. One kind of diagram in $\boldsymbol{C}$ is simply a subset S with the induced comparisons. Then a

left　　　　　　right

solution for S is merely a

lower　　　　　　upper

bound of S in the poset.

Of course, the orientations here have got a bit twisted, but that is just an historical accident. ■

You may think there is something missing in Definition 4.7. Suppose the template imposes some commuting conditions on the diagram. Shouldn't those conditions be observed in the corresponding solutions? No!

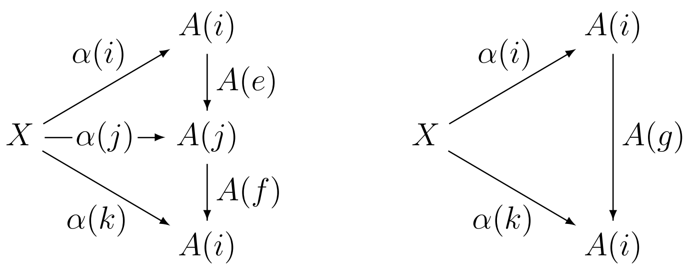
Suppose the template ∇ has composible edges

$$i \xrightarrow{\ e\ } j \xrightarrow{\ f\ } k$$

with

$$i \xrightarrow{g = f \circ e} k$$

as the composite edge. For a left solution we certainly require that the three triangles



commute. But the notion of a diagram requires

$$A(g) = A(f) \circ A(e)$$

so that the fact that the two triangles on the left commute ensures that the triangle on the right also commutes. For this reason sometimes only a 'generating part' of a template is used.

4.9 EXAMPLE. Suppose we use the integers $\mathbb{Z}$

$$\cdots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \cdots$$

as a poset template with its edges pointing upwards. Thus we have an edge

$$n \xrightarrow{(m,n)} m$$

for all $n \leq m$ in $\mathbb{Z}$. In fact

$$(m,n) = (m, m-1) \circ (m-1, m-2) \circ \cdots \circ (n+2, n+1) \circ (n+1, n)$$

using the 1-step edges. Any $\mathbb{Z}$-diagram requires

$$A(m,n) = A(m, m-1) \circ A(m-1, m-2) \circ \cdots \circ A(n+2, n+1) \circ A(n+1, n)$$

so, in practice, we often describe just the 1-step arrows

$$A(n+1, n)$$

for $n \in \mathbb{Z}$. ■

On the whole we need not worry too much about the difference between a diagram indexed by a directed graph and one indexed by a category. In fact, as we have seen each graph ∇ has an associated path category $\boldsymbol{Pth}(\nabla)$ and these produce 'equivalent' diagrams in any category. These two diagrams have the same solutions.

Exercises

4.3.1 Let ∇ be a directed graph and let $\boldsymbol{C}$ be an arbitrary category. Recall that each ∇-diagram in $\boldsymbol{C}$ extends uniquely to a $\boldsymbol{Pth}(\nabla)$-diagram, and each $\boldsymbol{Pth}(\nabla)$-diagram arises from a to a ∇-diagram

Show that such a corresponding pair of diagrams have exactly the same left solutions and exactly the same right solutions.

4.3.2 Given a ∇-diagram A in a category $\boldsymbol{C}$, describe the notions of a left solution and a right solution using the diagonal functor.

4.4 Universal solution

Each diagram in a category poses two problems, the left problem and the right problem. Each of these problems may have many solutions, and in general there is no relationship between the left solutions and the right solutions. We now look for a 'best' solution on either side. Of course, on any particular side such a solution need not exists, but if there is one, then all the 'best' solutions are canonically isomorphic.

A universal solution of a diagram is a particular solution that is as economical as possible, in the sense that it is as 'near' to the diagram as possible. Here is the formal definition. Of course, it is two definitions in one.

4.10 DEFINITION. Let ∇ be a template and let A be a ∇-diagram in a category $\boldsymbol{C}$. A

left universal solution   right universal solution

or

limit   colimit

for A is a particular solution

$$S \xrightarrow{\sigma(i)} A(i) \qquad\qquad A(i) \xrightarrow{\sigma(i)} S$$

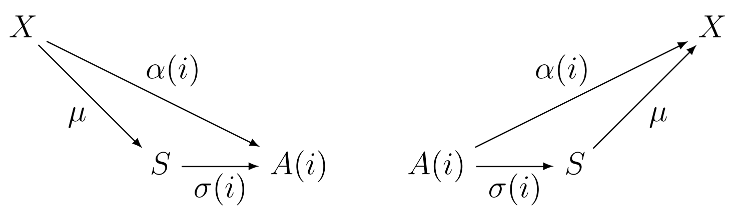
such that for each solution

$$x \xrightarrow{\alpha(i)} A(i) \qquad\qquad A(i) \xrightarrow{\alpha(i)} X$$

there is a *unique* arrow

$$X \xrightarrow{\mu} S \qquad\qquad S \xrightarrow{\mu} X$$

such that for each node i the triangle



commutes.

We call μ the mediator. ■

Of course, a limit or a co-limit of a diagram need not exists. Also, a diagram can have one without the other. However, once we have a universal solution (left or right) we can obtain all other solutions of the same handedness. We simply weaken the universal solution by an arrow to (for left) or from (for right) the carrying object, the apex of the universal solution.

Limits and colimits of diagrams (when they exist) are essentially unique. To prove that for limits we make a preliminary observation.

4.11 LEMMA. *Let ∇ be a template, let A be a diagram in a category $\boldsymbol{C}$, and suppose*

$$S \xrightarrow{\sigma(i)} A(i)$$

is a limit of that diagram indexed by the nodes i of ∇. Then this family of arrows is collection-wise monic. That is, if

$$X \overset{\theta}{\underset{\psi}{\rightrightarrows}} S$$

is a parallel pair of arrows with

$$\sigma(i) \circ \theta = \sigma(i) \circ \psi$$

for each node i, then $\theta = \psi$.

Proof. Consider any such parallel pair θ, ψ of arrows. Let

$$X \xrightarrow{\alpha(i)} A(i)$$

be the arrow given by

$$\alpha(i) = \sigma(i) \circ \theta = \sigma(i) \circ \psi$$

for node i.

Consider any edge

$$i \xrightarrow{e} j$$

of ∇. By passing through S we see that the triangle

$$\begin{array}{ccc} & & A(i) \\ & \overset{\alpha(i)}{\nearrow} & \\ X & & \Big\downarrow A(e) \\ & \underset{\alpha(j)}{\searrow} & \\ & & A(j) \end{array}$$

commutes, and hence we have a left solution of the diagram.

Since S is the apex of a universal left solution there is a unique arrow

$$X \xrightarrow{\mu} S$$

such that

$$\alpha(i) = \sigma(i) \circ \mu$$

for each node i. Since both θ and ψ do this job, we have $\theta = \psi$, as required. ■

This result has a simple consequence.

4.12 COROLLARY. *Let ∇ be a template, let A be a diagram in a category $\boldsymbol{C}$, and suppose*

$$S \xrightarrow{\sigma(i)} A(i)$$

is a limit of that diagram indexed by the nodes i of ∇. Suppose also that

$$S \xrightarrow{\epsilon} S$$

is an endo-arrow of S such that

$$\sigma(i) \circ \epsilon = \sigma(i)$$

for each node i. Then $\epsilon = \boldsymbol{id}_S$.

Proof. We apply Lemma 4.11 to the pair $\theta = \epsilon$ and $\psi = \boldsymbol{id}_S$. ■

So far we have been careful to speak of *a* limit of a diagram. We can now show that we needn't be so cautious.

4.13 THEOREM. *Let ∇ be a template, let A be a diagram in a category $\boldsymbol{C}$, and suppose each of*

$$S \xrightarrow{\sigma(i)} A(i) \qquad\qquad T \xrightarrow{\tau(i)} A(i)$$

is a limit of that diagram indexed by the nodes i of ∇. Then there is a unique arrow

$$T \xrightarrow{\tau} S$$

such that

$$\tau(i) = \sigma(i) \circ \tau$$

for each node i. Furthermore, τ is an isomorphism.

Proof. Since the arrows $\sigma(\cdot)$ form a limit, and the arrows $\tau(\cdot)$ form a left solution, there is a unique arrow τ, with indicated type, such that

$$\tau(i) = \sigma(i) \circ \tau$$

for each node i. This τ is just the mediator.

By symmetry, there is a unique arrow

$$S \xrightarrow{\sigma} T$$

such that

$$\sigma(i) = \tau(i) \circ \sigma$$

for each node i.

Now consider the endo-arrow

$$\epsilon = \tau \circ \sigma$$

of S. For each node i we have

$$\sigma(i) \circ \epsilon = \sigma(i) \circ \tau \circ \sigma = \tau(i) \circ \sigma = \sigma(i)$$

and hence $\epsilon = \boldsymbol{id}_S$ by Corollary 4.12. By symmetry we have

$$\tau \circ \sigma = \boldsymbol{id}_S \qquad \sigma \circ \tau = \boldsymbol{id}_T$$

and hence these mediators σ and τ form an inverse pair of isomorphisms. ■

This shows that if a diagram has a limit then that limit is essentially unique. Thus we may speak of *the* limit of a diagram (provided it does exist).

Of course, there is a similar result for colimits with the same proof but where the arrows point the other way.

In the remaining three sections of this chapter we gather together a random collection of examples to show how limits and colimits can be calculated in appropriate circumstances.

Exercises

4.4.1 State and prove the right hand version of each of the three results of this section.

4.4.2 *Find some more*

4.5 A geometric limit and colimit

In this block we first look at a geometric example involving the circle group and topological spaces. After that there is an exercise which is similar in nature, but discrete and simpler.

For the template we use the integers $\mathbb{Z}$ as a poset. There are two ways to do this, upwards or downwards. Here is is convenient to use the downward version.

$$\cdots \longrightarrow 2 \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow -2 \longrightarrow \cdots$$

We do this because it helps with some of the calculations. Notice that the limit will occur at the positive end of $\mathbb{Z}$, and the colimit will occur at the negative end.

We look at a rather simple diagram in ***Top***, the caegory of topological spaces. We put the same space at each node, and the same map at each edge.

Let $\mathbb{O}$ be the circle group. We think of $\mathbb{O}$ as the circle in the cartesian plane with radious 1 and centre at the origin. Thus each point on $\mathbb{O}$ is determined by its unique polar co-ordinate θ with $0 \leq \alpha < 2\pi$. Here addition (mod 2π) is important.

Put a copy of $\mathbb{O}$ at each index. For each edge

$$\mathbb{O} \xrightarrow{\ \delta\ } \mathbb{O}$$

we take the doubling map

$$\delta(\alpha) = 2\alpha \pmod{2\pi}$$

for co-ordinate α. This function wraps the source circle twice round the target circle.

What can a left solution be?

The limit

The limit is some kind of topological space A furnished with a family of functions

$$A \xrightarrow{\phi_m} \mathbb{O} \qquad m \in \mathbb{N}$$

such that

$$\delta \circ \phi_{m+1} = \phi_m$$

for each $m \in \mathbb{N}$. Thus, for each $a \in A$ we have

$$\phi_m(a) = 2\phi_{m+1}(a) \pmod{2\pi}$$

and hence

$$\phi_m(a) = 2^r\phi_{m+r}(a) \pmod{2\pi}$$

for each $m, r \in \mathbb{N}$.

Since we can divide any real number by 2, we seem to have

$$\phi_r(a) = \frac{1}{2^r}\phi_0(a)$$

for all $a \in A$ and $r \in \mathbb{N}$. This is not quite right, for we have forgotten the 2π aspect.

Suppose we have

$$\phi_0(a) = \alpha$$

for some $0 \le \alpha < 2\pi$. Then one of

$$\phi_1(a) = \frac{\alpha}{2} \qquad \phi_1(a) = \frac{\alpha}{2} + \pi = \frac{\alpha + 2\pi}{2}$$

must hold, and these can arise from

$$\phi_2(a) = \frac{\alpha}{4} \quad \phi_2(a) = \frac{\alpha + 4\pi}{4} \qquad \phi_2(a) = \frac{\alpha + 2\pi}{4} \quad \phi_2(a) = \frac{\alpha + 6\pi}{4}$$

and so on. These possibilities are conveniently displayed as a tree

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{\alpha}{8} \quad \frac{\alpha+8\pi}{8} \qquad \frac{\alpha+4\pi}{8} \quad \frac{\alpha+12\pi}{8} \qquad \frac{\alpha+2\pi}{8} \quad \frac{\alpha+10\pi}{8} \qquad \frac{\alpha+6\pi}{8} \quad \frac{\alpha+14\pi}{8}$$

$$\frac{\alpha}{4} \qquad \frac{\alpha+4\pi}{4} \qquad\qquad \frac{\alpha+2\pi}{4} \qquad \frac{\alpha+6\pi}{4}$$

$$\frac{\alpha}{2} \qquad\qquad \frac{\alpha+2\pi}{2}$$

$$\alpha$$

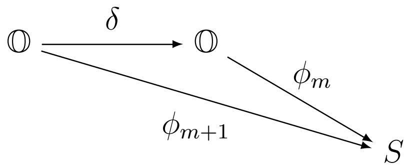
and we see that each element $a \in A$ with $\phi_0(a) = \alpha$ generates a branch of the tree.

Each point of A corresponds to a certain region of $\mathbb{O}$. At some point in your life you should find out more about this.

The colimit

The colimit is easier to understand, in the sense that we can give a full description of it.

We first look at a couple of properties of a general right solution.



Thus we consider a space S furnished with a $\mathbb{Z}$ indexed family of functions ϕ_m such that each indicated triangle commutes. In other words

$$2\alpha \equiv \beta \Longrightarrow \phi_{m+1}(\alpha) = \phi_m(\beta)$$

for all coordinates α, β and all $m \in \mathbb{Z}$. Here and below

$$\lambda \equiv \rho \quad \text{means} \quad \lambda = \rho \pmod{2\pi}$$

with all the usual properties.

Using a trivial induction the equivalence gives

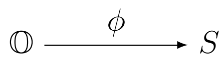
$$2^r\alpha \equiv \beta \Longrightarrow \phi_{k+r}(\alpha) = \phi_k(\beta)$$

for all coordinates α, β, all $k \in \mathbb{Z}$, and all $r \in \mathbb{N}$.

We need a variant of this last result, namely

$$(\star) \quad 2^m\alpha \equiv \beta \Longrightarrow \phi_m(\alpha) = \phi_0(\beta)$$

for all coordinates α, β and all $m \in \mathbb{Z}$. Notice how this shows that the map ϕ_0 determines the whole of the structure of the solution. In particular, *any* function



to a space can be used to generate a solution.

To prove this implication suppose that

$$2^m\alpha \equiv \beta$$

holds.

If $m \leq 0$ then, with $r = m$ the previous result gives

$$\phi_{k+m}(\alpha) = \phi_k(\beta)$$

for all $k \in \mathbb{Z}$. Thus we take $k = 0$.

Suppose $m \leq 0$, say $m = -r$ for $r \in \mathbb{N}$. Then

$$0 \leq 2^m\alpha = 2^{-r}\alpha \leq \alpha$$

so that

$$2^{-r}\alpha = \beta$$

and hence

$$\alpha = 2^r\beta$$

holds. A version of the previous result now gives

$$\phi_{k+r}(\beta) = \phi_k(\alpha)$$

for all $k \in \mathbb{Z}$. Thus taking $k = m$ gives the required result.

Consider the maps

$$\mathbb{O} \xrightarrow{\rho_m} \mathbb{O}$$

given by

$$\rho_m(\alpha) \equiv 2^m\alpha$$

for all $m \in \mathbb{Z}$ and coordinates α. In particular, note that ρ_0 is the identity function on $\mathbb{O}$. We check that these maps furnish $\mathbb{O}$ as a right solution.

We have

$$\rho_{m+1}(\alpha) \equiv 2^{m+1}\alpha$$

for all $m \in \mathbb{Z}$ and coordinates α. Let

$$\beta \equiv 2\alpha$$

so that

$$2^m\beta \equiv 2^{m+1}\alpha$$

and hence

$$(\rho_m \circ \delta)(\alpha) = \rho_m(\beta) \equiv 2^m\beta \equiv \rho_{m+1}(\alpha)$$

so that

$$\rho_m \circ \delta = \rho_{m+1}$$

as required.

Finally, we show that $\mathbb{O}$ with these furnishings is the colimit of the diagram.

To do that consider an arbitrary right solution, as above. We require a unique map

$$\mathbb{O} \xrightarrow{\mu} S$$

such that

$$\phi_m = \mu \circ \rho_m$$

for each $m \in \mathbb{Z}$.

By considering the case $m = 0$ we see that $\mu = \phi_0$ is the only possible map. We check that this does mediate.

Consider any $m \in \mathbb{Z}$ and any coordinate α. With

$$\beta \equiv 2^m\alpha$$

we have

$$(\mu \circ \rho_m)(\alpha) = \mu(\beta) = \phi_0(\beta)$$

so that

$$\phi_m(\alpha) = \phi_0(\beta)$$

is the requirement. This is precisely the result $(\star)$.

After reading this you might think this colimit example is a bit of a cheat. You could be right.

Exercises

4.5.1 For the template use $\mathbb{Z}$ as a poset. It doesn't matter which way you order $\mathbb{Z}$, but you might find it easier to use the upwards version, positives to the right and negatives to the left.

Consider the following diagram in ***Set***.

At each node place $\mathbb{Z}$ (so now $\mathbb{Z}$ is playing two different roles). At each edge place the doubling function.

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{d} & \mathbb{Z} \\ z & \longmapsto & 2z \end{array}$$

Show that the arrows of any left solution have a rather simple behaviour.

Describe the limit of the diagram. You should verify your claims.

Investigate the behaviour of an arbitrary right solution.

Show that when suitably furnished the dyadic rationals are the colimit of the diagram.

4.5.2 ***Find another one.***

4.6 How to calculate certain limits

For many categories the objects are structured sets, and the arrows are structure preserving functions. In other words, each object is a single set, its carrier, furnished with some predetermined structure, perhaps restricted by certain required properties (axioms). Each arrow is a function between the carriers which respects the structure in an appropriate fashion. For instance

$$\textbf{\textit{Pos}} \qquad \textbf{\textit{Mon}} \qquad \textbf{\textit{Top}}$$

are three categories of this kind.

For each such category $\boldsymbol{C}$ there is a forgetful functor

$$\boldsymbol{C} \longrightarrow \textbf{\textit{Set}}$$

which sends each object to its carrier, and views each arrow as the mere function. Often this functor is named U, for underlying.

In this section we show how to compute a limit in such a category $\boldsymbol{C}$. To do that we first compute the corresponding limit in ***Set***, and then lift back up to $\boldsymbol{C}$. Thus we must first show how to calculate limits in ***Set***. The construction we give works for many categories of structured sets, but not necessarily all.

It is rather strange that it is often easier to calculate limits than colimits. However, we do look at certain colimits later.

It is convenient to fix a bit of notation, or rather to recall the notation we have used throughout this chapter.

We assume we have a diagram in the category under investigation given by a template ∇. We let $i, j, k \ldots$ range over the family $\mathbb{I}$ of nodes of ∇, and we let $e, f, g, \ldots$ range over the family $\mathbb{E}$ of edges of ∇. The construction works when ∇ is graph and when ∇ is a category.

4.6.1 Limits in ***Set***

In this block we show how to compute the limit of a diagram in ***Set***. This then forms the basis for limits in the other three and similar categories.

We begin with a review of products in ***Set***.

Binary products are easy, we take the cartesian product – the set of ordered pairs – of the two component sets. Products of finitely many components are just as easy, again we take the cartesian product of the components.

What about the product of an arbitrary indexed family

$$\mathsf{A} = \big(A(i) \mid i \in \mathbb{I}\big)$$

of sets?

Let

$$\bigcup \mathsf{A}$$

be the union of the family A. (In practice it is often useful to tag each component $A(i)$ so we have a disjoint union. But that is not needed here.) We look at certain functions

$$\mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

from nodes to elements.

4.14 DEFINITION. Let A be an $\mathbb{I}$-indexed family of sets, as above. A choice function for A is a function

$$a(\cdot) : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

such that

$$a(i) \in A(i)$$

for each node $i \in \mathbb{I}$. ■

A choice function has the wit to select one member from each component set $A(i)$. When $\mathbb{I}$ is finite such a choice function can be coded as a tuple. When $\mathbb{I}$ is not finite we need this more general idea.

While we are here, one version of the Axiom of Choice says that there always is at least one choice function unless, of course, one component $A(i)$ is empty. Some people seem to have doubts about the validity of the property. But then, that's their choice.

4.15 DEFINITION. Let A be an $\mathbb{I}$-indexed family of sets, let $\prod \mathsf{A}$ be the set of all choice functions for A, and for each node $i \in \mathbb{I}$ let

$$\begin{array}{rcl} \prod \mathsf{A} & \xrightarrow{\alpha(i)} & A(i) \\ a & \longmapsto & a(i) \end{array} \qquad \alpha(i)(a) = a(i)$$

be the 'evaluation at i' function. ■

Before you continue reading you might try to show that

$$\big(\prod \mathsf{A} \xrightarrow{\alpha(i)} A(i) \mid i \in \mathbb{I}\big)$$

is a product wedge in ***Set***. We are going to prove something more general.

We have a template ∇ with nodes $\mathbb{I}$ and edges $\mathbb{E}$. Suppose also we have a diagram in ***Set***

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

which instantiates the template. We want to produce a limit of this diagram.

4.16 DEFINITION. Given a ∇-diagram $(\mathsf{A}, \mathcal{A})$, as above, a thread is a choice function

$$a(\cdot) : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

such that

$$A(e)\big(a(i)\big) = a(j)$$

for each edge

$$i \xrightarrow{\;e\;} j$$

of ∇. ■

A choice function merely selects a member from each component set $A(i)$. A thread has the decency to ensure that these selections are compatible. If we pass from one component $A(i)$ to another $A(j)$ using an edge $A(e)$, then we can take the selected element with us knowing that we will arrive at the selected element at the end. This, of course, is a sophisticated idea. Something that bus, train, and airline companies find hard to grasp.

If $\mathbb{E} = \emptyset$ then every choice function is a thread.

4.17 DEFINITION. Given a ∇-diagram $(\mathsf{A}, \mathcal{A})$, as above, let A be the set of all threads, and for each node $i \in \mathbb{I}$ let

$$\begin{array}{ccc} A & \xrightarrow{\;\alpha(i)\;} & A(i) \\ a & \longmapsto & a(i) \end{array} \qquad \alpha(i)(a) = a(i)$$

be the 'evaluation at i' function. ■

This notion, of course, does not conflict with that of Definition 4.15. It merely extends the idea to a more general context.

4.18 LEMMA. *Given a ∇-diagram $(\mathsf{A}, \mathcal{A})$, as above, the evaluation functions furnish the set of threads as a left solution of the diagram.*

Proof. We must show that for each edge e the induced triangle

$$\begin{array}{ccccc} & & A(i) & & i \\ & \overset{\alpha(i)}{\nearrow} & & & \\ A & & \Big\downarrow A(e) & & \Big\downarrow e \\ & \underset{\alpha(j)}{\searrow} & & & \\ & & A(j) & & j \end{array}$$

commutes. In other words we require

$$\big(A(e) \circ \alpha(i)\big)(a) = \alpha(j)(a)$$

for each edge e and thread $a \in A$. But since a is a thread we have

$$\big(A(e) \circ \alpha(i)\big)(a) = (A(e)\big(\alpha(i)(a)\big) = (A(e)\big(a(i)\big) = a(j) = \alpha(j)(a)$$

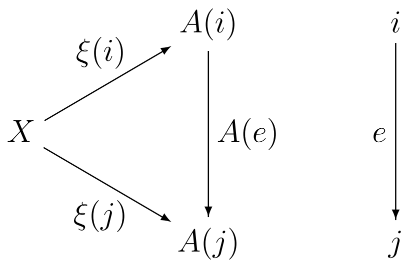
as required. ■

With this we have the result we want.

4.19 THEOREM. *Given a ∇-diagram $(\mathsf{A}, \mathcal{A})$, as above, the evaluation functions furnish the set of threads as a limit of the diagram.*

Proof. By Lemma 4.18 we already know that we have a left solution of the diagram. Thus it suffices to show that this left solution is universal.

To this end let



be a typical part of an arbitrary left solution of the diagram. We require a unique function

$$X \xrightarrow{\ \mu\ } A$$

such that

$$\xi(i) = \alpha(i) \circ \mu$$

for each $i \in \mathbb{I}$.

To obtain one such function set

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. Since

$$\xi(i)(x) \in A(i)$$

we see that $\mu(x)$ is a choice function. For each edge e, as indicated, we have

$$A(e)\big(\mu(x)(i)\big) = A(e)\big(\xi(i)(x)\big) = \big(A(e) \circ \xi(i)\big)(x) = \xi(j)(x) = \mu(x)(j)$$

to show that $\mu(x)$ is a thread. Thus we do have a function μ of the required type. Finally, for each node i we have

$$\big(\alpha(i) \circ \mu\big)(x) = \alpha(i)\big(\mu(x)\big) = \mu(x)(i) = \xi(i)(x)$$

to show that

$$\alpha(i) \circ \mu = \xi(i)$$

as required.

This deals with the existence of a mediating arrow. Now we must deal with the uniqueness. To this end suppose we have a function ν with

$$\alpha(i) \circ \nu = \xi(i)$$

for each node i. Consider any $x \in X$ and the corresponding thread $a = \nu(x)$. For each node i we have

$$\nu(x)(i) = a(i) = \alpha(i)(a) = \alpha(i)\big(\nu(x)\big) = \big(\alpha(i) \circ \nu\big)(x) = \xi(i)(x) = \mu(x)(i)$$

to show $\nu = \mu$, as required. ■

In the next three blocks we show how to calculate limits in certain categories $\boldsymbol{C}$ of structured sets. The process is the same for these categories. In each case there is a forgetful functor

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

to the category of sets. It merely forgets the structure. Given a diagram in $\boldsymbol{C}$ this functor converts it into a diagram in $\boldsymbol{Set}$. We calculate the limit of that $\boldsymbol{Set}$-diagram using the method of this block. The problem then is to furnish that set and collection of arrows so that they form a limit in $\boldsymbol{C}$. This last part need some special properties of $\boldsymbol{C}$.

Exercises

4.6.1 Let $\boldsymbol{Set}(D)$ be the category of sets with a distinguished subset, as used in Exercise 1.2.2. Thus each object is a pair (A, R) where A is a set and R is a subset $R \subseteq A$. The arrows are those functions which preserve the selected subset.

Let $\nabla = (\mathbb{I}, \mathbb{E})$ be a template, and consider a ∇-diagram

$$\mathsf{A}(D) = \big((A(i), R(i)) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A}(D) = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in $\boldsymbol{Set}(D)$. By forgetting the distinguished subsets we obtain a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in $\boldsymbol{Set}$.

Consider the limit of the diagram in $\boldsymbol{Set}$, that is the set of threads with the attached functions. Show that this can be furnished to produce a limit of the diagram in $\boldsymbol{Set}(D)$.

4.6.2 Limits in *Pos*

In this block we consider the category $\boldsymbol{Pos}$ of posets and monotone maps.

We continue with the notation of the previous block. Thus we have a template ∇ with a collection $\mathbb{I}$ of nodes and a collection $\mathbb{E}$ of edges. These index objects and arrows in $\boldsymbol{Pos}$. In particular, we assume we have an instantiation of ∇

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

to form a diagram in ***Pos***. Thus each $A(i)$ is a poset, and for each edge

$$i \xrightarrow{\ e\ } j$$

the arrow

$$A(i) \xrightarrow{\ A(e)\ } A(j)$$

is a monotone map. Eventually we put these extra facilities to good use.

For the first step we forget the extra facilities and drop down to ***Set***. We look at the limit of this ***Set***-diagram. Thus we look at the *set* of all threads

$$a : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

together with the $\mathbb{I}$-indexed family

$$\begin{array}{rcl} A & \xrightarrow{\ \sigma(i)\ } & A(i) \\ a & \longmapsto & a(i) \end{array}$$

of evaluation functions. Recall that a thread satisfies

$$A(e)\big(a(i)\big) = a(j)$$

for each edge e, as above. This family of equalities rephrases as

$$A(e) \circ \alpha(i) = \alpha(j)$$

using the evaluation function α.

Our job is to furnish A as a poset, and check that each evaluation function is monotone. In this way we produce a left solution of the ***Pos***-diagram. We then have to show that this solution is universal, a limit in ***Pos***.

How can we partially order A? In other words, how can we compare threads? In this kind of situation there is one trick that should always be tried. We use the pointwise comparison.

We let

$$a \le b \iff (\forall i \in \mathbb{I})[a(i) \le b(i)]$$

for threads a and b. Notice how this works. For threads a and b we pass to each poset $A(i)$ in turn and carry out a comparison there. All of these must give a positive answer.

Why does this give a partial order on A? Verifying the three properties is routine. Let's look at the antisymmetry.

Consider threads a and b with $a \le b \le a$. Then, for each $i \in \mathbb{I}$, we have

$$a(i) \le b(i) \le a(i)$$

and hence $a(i) = b(i)$, since $A(i)$ is a poset. Thus $a = b$.

Next we must show that each evaluation function $\alpha(i)$ is monotone, that is

$$a \le b \Longrightarrow \alpha(i)(a) \le \alpha(i)(b)$$

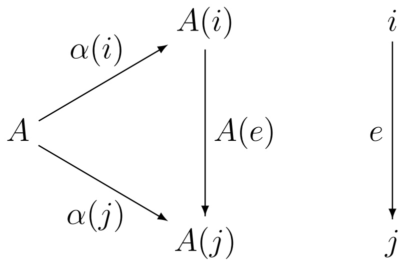
for threads a and b. Since

$$\alpha(i)(a) = a(i) \qquad \alpha(i)(b) = b(i)$$

this is immediate.

This produces the furnishings. Why does it give us a left solution of the diagram in ***Pos***?

We require each cell

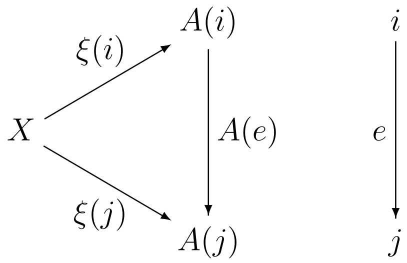


to be a commuting triangle in ***Pos***, for each edge e, as indicated. This is certainly a triangle in ***Pos***. We are given that each $A(e)$ is monotone, and we have ensured that $\alpha(i)$ and $\alpha(j)$ are monotone. This ***Pos***-triangle commutes because it commutes down in ***Set***.

Our next job is to show that this left solution is universal in ***Pos***. To do that we compare it with an arbitrary left solution of the ***Pos***-diagram. Thus we assume given a poset X together with an $\mathbb{I}$-indexed family of monotone maps

$$X \xrightarrow{\ \xi(i)\ } A(i)$$

such that the ***Pos***-triangle



commutes for each edge e, as indicated.

We require a unique mediator

$$X \xrightarrow{\ \mu\ } A$$

which, of course, must be monotone.

Think about this. If there is such a mediator μ then, by passing down to ***Set***, it can only be that function that works for the ***Set***-diagram. If that function turns out to be monotone, it will certainly make all the required ***Pos***-triangles commute, for they commute down in ***Set***. Thus we don't have much choice. The ***Set***-mediator is given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. We have to show that this function is monotone.

Why does

$$x \le y \Longrightarrow \mu(x) \le \mu(y)$$

hold for all $x, y \in X$?

Consider such $x, y \in X$ with $x \le y$. We are given that each $\xi(i)$ is monotone, so that

$$\xi(i)(x) \le \xi(i)(y)$$

and hence

$$\mu(x)(i) \le \mu(y)(i)$$

for each $i \in \mathbb{I}$, by definition of μ. Finally, remember that each of $\mu(x)$ and $\mu(y)$ is a thread, so that this last universally quantified comparison gives

$$\mu(x) \le \mu(y)$$

as required.

The construction of his block is fairly typical. We will use it again with minor variations to produce limits in two more categories.

Exercises

4.6.2 (a) Let ***Eqv*** be the category of equivalence relations. Each object is a pair $(A, \sim)$ where A is a set and $\sim$ is an equivalence relation on A.

Make sure you know what the arrows are. To do that a look at Exercises 1.2.3 will be useful.

(b) Let $\nabla = (\mathbb{I}, \mathbb{E})$ be a template, and consider a ∇-diagram

$$\mathsf{A}(\sim) = \big((A(i), \sim_i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A}(D) = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in ***Eqv***. By forgetting the distinguished subsets we obtain a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in ***Set***.

Consider the limit of the diagram in ***Set***, that is the set of threads with the attached functions. Show that this can be furnished to produce a limit of the diagram in ***Eqv***.

4.6.3 Limits in *Mon*

In this block we show how the construction of Block 4.6.1 also produces limits in ***Mon***, the category of monoids. The general procedure is the same as outlined at the beginning of Block 4.6.2. Starting from a diagram in ***Mon***, we pass to ***Set*** and take the limit of that ***Set***-diagram. The main problem is to furnish that ***Set***-limit to become a limit of the original ***Mon***-diagram.

Recall that a monoid is a furnished set

$$(A,\ \bullet,\ 1)$$

where '$\bullet$' is a binary operation on A, and 1 is a distinguished element. These attributes must satisfy

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \qquad 1 \bullet a = a = a \bullet 1$$

for all $a, b, c \in A$. In general, we write

$$ab \quad \text{for} \quad a \bullet b$$

but there is one case when we will explicitly show that operation symbol.

Monoids are the objects of ***Mon***. The arrows are the monoid morphism. Recall that a monoid morphism

$$(A, \bullet, 1) \xrightarrow{f} (B, \bullet, 1)$$

is a function

$$f : A \longrightarrow B$$

between the carriers such that

$$f(ab) = f(a)f(b) \qquad f(1) = 1$$

for all $a, b \in A$.

As usual we have a template ∇ with a collection $\mathbb{I}$ of nodes and a collection $\mathbb{E}$ of edges. We also have an instantiation

$$\mathsf{A} = \big(A(i) \mid i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \mid e \in \mathbb{E}\big)$$

to form a ***Mon***-diagram. Thus each $A(i)$ is a monoid and each $A(e)$ is a monoid morphism.

By passing down to ***Set*** we obtain the set A of all threads

$$a : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

together with the $\mathbb{I}$-indexed family

$$A \xrightarrow{\alpha(i)} A(i)$$

of evaluation functions. Our first job is to furnish A as a monoid, and check that each evaluation function is a monoid morphism.

How can we combine a pair of threads a and b to form

$$a \star b$$

a third thread? This is the case where we will explicitly indicate the operation.

Let

$$(a \star b)(i) = a(i)b(i)$$

for each $i \in \mathbb{I}$. Notice how we use the operation on $A(i)$. This construction certainly gives a function

$$a \star b : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

and it is a choice function since $a(i)b(i)$ lives in $A(i)$.

Why is $a \star b$ a thread? Consider an arbitrary edge e from i to j. We remember that $A(e)$ is a monoid morphism. Thus

$$A(e)\big((a \star b)(i)\big) = A(e)\big(a(i)b(i)\big) = \Big(A(e)\big(a(i)\big)\Big)\Big(A(e)\big(b(i)\big)\Big) = a(j)b(j) = (a \star b)(j)$$

as required.

Why is this operation associative?

Because

$$\begin{aligned}\big((a\star b)\star c\big)(i) &= \big((a\star b)(i)\big)c(i)\\ &= \big(a(i)b(i)\big)c(i)\\ &= a(i)\big(b(i)c(i)\big)\\ &= a(i)\big((b\star c)(i)\big) \quad = \big(a\star(b\star c)\big)(i)\end{aligned}$$

for all $a, b, c \in A$ and $i \in \mathbb{I}$.

We need a distinguished element of A. We set $1(i) = 1_i$ for each $i \in \mathbb{I}$. Here 1_i is the distinguished element of $A(i)$. It is easy to check that this function 1 is a thread, and almost trivially, we have

$$1 \star a = a = a \star 1$$

for each thread a. Thus we have furnished A as a monoid.

Why is each evaluation function

$$A \xrightarrow{\ \alpha(i)\ } A(i)$$

a monoid morphism?

Because

$$\alpha(i)\big(a\star b\big) = \big(a\star b\big)(i) = a(i)b(i) = \big(\alpha(i)(a)\big)\big(\alpha(i)(b)\big)$$

for each $a, b \in A$.

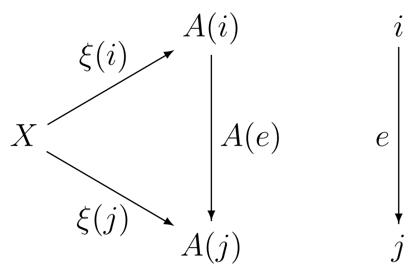
This set up the furnishings. Why does it give a left solution of the ***Mon***-diagram?

To be a left solution we require that certain triangles commute in ***Mon***. These triangles do commutes down in ***Set***, and they are triangles in ***Mon***, so they commute in ***Mon***.

Our main job is to show that this left solution is universal in ***Mon***. To do that we compare it with an arbitrary left solution of the ***Mon***-diagram. Thus we assume given a monoid X together with an $\mathbb{I}$-indexed family of monoid morphisms

$$x \xrightarrow{\ \xi(i)\ } A(i)$$

such that the ***Mon***-triangle



commutes for each edge e, as indicated.

We require a unique mediator

$$X \xrightarrow{\ \mu\ } A$$

which, of course, must be a monoid morphism.

By passing down to ***Set*** we see there is only one possible function μ we can use. That given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. Thus it suffices to show that this function μ is a monoid morphism.

For each $x, y \in x$ and $i \in \mathbb{I}$, remembering that $\xi(i)$ is a monoid morphism, we have

$$\mu(xy)(i) = \xi(i)(xy) = \big(\xi(i)(x)\big)\big(\xi(i)(y)\big) = \big(\mu(x)(i)\big)\big(\mu(y)(i)\big) = \big(\mu(x) \star \mu(y)\big)(i)$$

so that

$$\mu(xy) = \mu(x) \star \mu(y)$$

to show that μ passes across the operation. The other requirement

$$\mu(1) = 1$$

is even easier.

This kind of construction works in many categories of an algebraic nature.

Exercises

4.6.3 (a) Let ***CMon*** be the category of commutative monoids. Show that the construction of this block produces limits in ***CMon***.

(b) Let ***Grp*** be the category of groups. Show that the construction of this block produces limits in ***Grp***.

(c) Let ***Rng*** be the category of unital rings. Show that ***Rng*** has limits for $\nabla = (\mathbb{I}, \mathbb{E})$ diagrams.

4.6.4 A partially ordered monoid (a pom) is a structure

$$(A, \leq, \cdot, 1)$$

where $(A, \leq)$ is a poset and $(A, \cdot, 1)$ is a monoid, and

$$\left.\begin{matrix} x \leq a \\ y \leq b \end{matrix}\right\} \Longrightarrow xy \leq ab$$

for all $a, b, x, y \in A$. These are the objects of the category ***Pom***. A arrow of ***Pom*** is a function between two poms which is both a monotone map and a monoid morphism.

For an arbitrary template $\nabla = (\mathbb{I}, \mathbb{E})$ show that each ∇-diagram in ***Pom*** has a limit.

4.6.4 Limits in *Top*

In this block we show how the construction of Block 4.6.1 also produces limits in ***Top***, the category of topological spaces and continuous maps.

Before we get into the details of the construction, it is worth making a few remarks about the notion as it appears in the literature.

In topological circles a left limit is usually called an inverse limit. More often than not the template is a partial order or even a pre-order, and the indexing is contravariant. We won't deal with that aspect here.

As usual we have a template ∇ of nodes $\mathbb{I}$ and edges

$$\mathsf{A} = \big(A(i) \mid i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \mid e \in \mathbb{E}\big)$$

to form a ***Top***-diagram. Thus each $A(i)$ is a topological and each $A(e)$ is a continuous map.

By passing down to ***Set*** we obtain the set A of all threads

$$a : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

together with the $\mathbb{I}$-indexed family

$$A \xrightarrow{\ \alpha(i)\ } S(i)$$

of evaluation functions.

Our first job is to convert A into a topological space in such a way that each $\alpha(i)$ is continuous. We do this in a minimalist fashion.

Consider any node $i \in \mathbb{I}$ and any open $U \in \mathcal{O}A(i)$ of that component. We certainly require the inverse image set

$$i(U) = \alpha(i)^{\leftarrow}(U) = \{a \in A \mid a(i) \in U\}$$

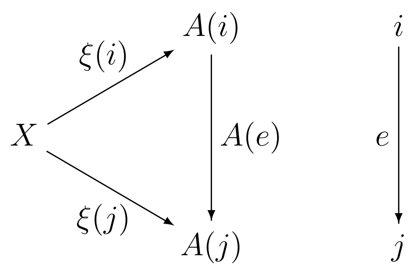
to be open. Thus we take the family of all these subsets of A as a subbase of a topology on A. This ensures that each $\alpha(i)$ is continuous without being too energetic.

In the usual way this gives us a left solution of the diagram. This has nothing much to do with the topological aspects. It's merely that certain triangles of function do commute.

Our main job is to show that this left solution is universal in ***Top***. To do that we compare it with an arbitrary left solution of the ***Top***-diagram. Thus we assume given a topological space X together with an $\mathbb{I}$-indexed family of continuous maps

$$X \xrightarrow{\ \xi(i)\ } A(i)$$

such that the ***Top***-triangle



commutes for each edge e, as indicated.

We require a unique mediator

$$X \xrightarrow{\ \mu\ } A$$

which, of course, must be a continuous map.

By passing down to ***Set*** we see that the only possible mediating function is given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. Thus it suffices to show that this function μ is continuous.

To show that μ is continuous it suffices to show that for each subbasic open set of A the inverse image across μ is open. Thus we require

$$\mu^{\leftarrow}(i(U)) \in \mathcal{O}R$$

for each node $i \in \mathbb{I}$ and open $U \in \mathcal{O}A(i)$. For each $x \in X$ we have

$$\begin{aligned} x \in \mu^{\leftarrow}(i(U)) &\iff \mu(x) \in i(U) = \alpha(i)^{\leftarrow}(U) \\ &\iff \alpha(i)(\mu(x)) \in U \\ &\iff \mu(x)(i) \in U \\ &\iff \xi(i)(r) = \mu(x)(i) = U \qquad \iff x \in \xi(i)^{\leftarrow}(U) \end{aligned}$$

to show that

$$\mu^{\leftarrow}(i(U)) = \xi(i)^{\leftarrow}(U)$$

for each pair i and U. Since each $\xi(i)$ is continuous this shows that each $\mu^{\leftarrow}(i(U))$ is open, for the required result.

Notice how this construction and proof works. To ensure that each $\alpha(i)$ is continuous we need at least all the sets $i(U)$ in the topology on S. To show that a mediator μ is continuous we can't deal with more than the sets $i(U)$.

Exercises

[`Need to sort some out`]

4.7 Confluent colimits in *Set*

So far we have looked at particular examples of limits. In this section we describe how to calculate a certain kind of colimit. We work in the category ***Set*** of sets, but the same method works for many other categories of structured sets.

For this example we assume the template is a poset.

Thus let $\mathbb{I}$ be a poset with nodes

$$i, j, k, \ldots$$

and for each comparison $i \leq j$ let

$$i \xrightarrow{(j,i)} j$$

be the corresponding edge.

We assume $\mathbb{I}$ satisfies a certain restriction

4.20 DEFINITION. The poset $\mathbb{I}$ is directed if for each $i, j \in \mathbb{I}$ there is some $k \in \mathbb{I}$ with $i, j \leq k$.

The poset $\mathbb{I}$ is confluent if for each $i, j, l \in \mathbb{I}$ with $l \leq i, j$, there is some $k \in \mathbb{I}$ with $i, j \leq k$. ■

Trivially, each directed poset is confluent, but there are confluent posets that are not directed. For example each discrete set is confluent but not directed (if it has more than one node).

Notice that if a poset is directed then (by repeated use of this property) each finite subset has at least one upper bound.

Similarly, if a poset is confluent, then (by repeated use of this property) each finite subset which has a lower bound also has an upper bound.

We assume the template poset $\mathbb{I}$ is confluent.

Let A be an $\mathbb{I}$-diagram in ***Set***. Thus A is an $\mathbb{I}$-indexed family of sets

$$A(i)$$

together with connecting functions

$$A(i) \xrightarrow{A(j,i)} A(j)$$

one for each comparison $i \leq j$ in $\mathbb{I}$. These functions must compose in the usual way, that is

$$A(i,i) = \boldsymbol{id}_{A(i)}$$

and

$$A(k,j) \circ A(j,i) = A(k,i)$$

for $i \leq j \leq k$.

To help with the later calculations it is convenient to write

$$\begin{array}{rcl} A(i) & \longrightarrow & A(j) \\ x & \longmapsto & x|j \end{array}$$

for the function $A(j,i)$. Thus

$$x|j = A(j,i)(x)$$

and we may think of this as the 'restriction' of $x \in A(i)$ to j. Note that

$$x|i = x \qquad x|j|k = x|k$$

for $i \leq j \leq k$ with $x \in A(i)$.

To obtain the co-limit of A we first produce what turns out to be the coproduct of A.

Let

$$\amalg A$$

be the disjoint union of the sets $A(i)$. We need to set this up with some care. Thus we let

$$\amalg A = \bigcup \{A(i) \times \{i\} \mid i \in \mathbb{I}\}$$

that is we take the set of all pairs

$$(x,i)$$

for $i \in \mathbb{I}$ and $x \in A(i)$. In this way an element x may occur many times, but each occurrence is tagged with the parent index.

For each $i \in \mathbb{I}$ there is a function

$$\begin{array}{rcl} A(i) & \longrightarrow & \amalg A \\ x & \longmapsto & (x,i) \end{array}$$

and it turns out that these functions structure $\amalg A$ as the coproduct of A in ***Set***. The proof of that is the discrete case of the more general proof to follow.

For the more general situation we need to take a quotient of $\amalg A$.

Consider the relation $\sim$ on $\amalg A$ given by

$$(x,i) \sim (y,j) \Longleftrightarrow (\exists i,j \le k)[x|k = y|k]$$

for (x,i) and $y,j)$ from $\amalg A$. Here, of course, the quantifier looks for a node k which is an upper bound of the two given nodes. Trivially, this relation is reflexive and symmetric. We prove that it is also transitive, and hence is an equivalence relation on $\amalg A$. This is where we use the confluence of $\mathbb{I}$.

Consider any situation

$$(x,i) \sim (y,j) \sim (z,k)$$

so that $(x,i) \sim (z,k)$ is required.

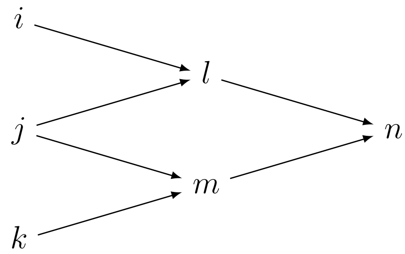
We are given nodes

$$i,j \le l \qquad j,k \le m$$

such that

$$x|l = y|l \qquad y|m = z|m$$

hold. Now look how the nodes sit in $\mathbb{I}$.



The nodes l,m have a lower bound and hence, by confluence, they have some upper bound n. But now

$$x|n = x|l|n = y|l|n = y|n = y|m|n = z|m|n = z|n$$

to show that

$$(x,i) \sim (z,k)$$

as required.

Since $\sim$ is an equivalence relation on $\amalg A$ we may let

$$L = \amalg A/{\sim}$$

the family of blocks (equivalence classes) of $\sim$ in $\amalg A$. Let

$$\begin{array}{ccc} \amalg A & \longrightarrow & L \\ (x,i) & \longmapsto & [x,i] \end{array}$$

be the associated quotient function.

For later observe that

$$[x,i] = [x|j,j]$$

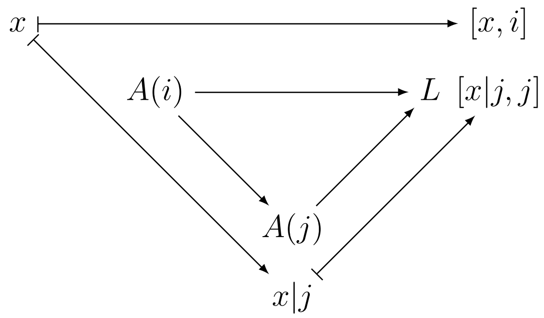
for all nodes $i \le j$.

We show that the family of composite functions

$$\begin{array}{ccccc} A(i) & \longrightarrow & \amalg A & \longrightarrow & L \\ x & \longmapsto & (x,i) & \longmapsto & [x,i] \end{array}$$

structure L as the co-limit of the $\mathbb{I}$-diagram A.

Our first job is to show that we do have a right solution of the diagram. Consider any pair of nodes $i \le j$. We require that the inner triangle commutes.



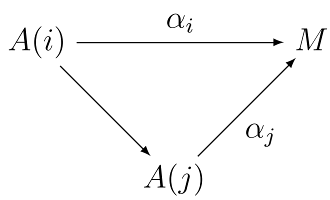
To prove that we track an element from the top left hand corner to the top right hand corner by the short trip and the long trip. Thus we require

$$[x,i] = [x|j,j]$$

and this is nothing more than the observation made above.

Our second job is to show that we have a universal right solution.

Consider any right solution M where for nodes $i \le j$ the commuting triangle



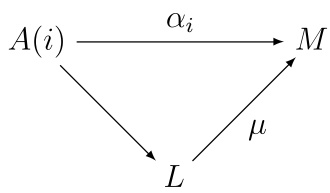
is a typical part of the structure. Thus we have

$$\alpha_j(x|j) = \alpha_i(x)$$

for each $x \in A(i)$. We require a unique function

$$L \xrightarrow{\ \mu\ } M$$

such that for each node i the triangle



commutes.

If there is such a function μ, then

$$\mu\big([x,i]\big) = \alpha_i(x)$$

for each $i \in \mathbb{I}$ and $x \in A(i)$. This shows that there is at most one such μ. To show there is at least one, it suffices to show that this assignment is well-defined, that is

$$[x,i] = [y,j] \Longrightarrow \alpha_i(x) = \alpha_j(y)$$

for $i, j \in \mathbb{I}$ and $x \in A(i), y \in A(j)$.

Assuming

$$[x,i] = [y,j]$$

we have

$$(x,i) \sim ([y,j)$$

to give

$$x|k = y|k$$

for some node k with $i, j \leq k$. But now

$$\alpha_i(x) = \alpha_k(x|k) = \alpha_k(y|k) = \alpha_j(y)$$

for the required result.

At the moment there are 15 exercises in this chapter, with more to come.

5
Adjunctions

The isolation of the notion of an adjunction is one of the most important contributions of category theory. In a sense adjoints form the first 'non-trivial' part of category theory, at least it can seem that way now that all the basic stuff has been sorted out. Adjunctions abound in mathematics, and many examples were known before the categorical notion was formalized. We have already met several examples, and later I will point you to them.

In this chapter we go through the various aspects of adjunctions quite slowly. We look at each facet in some detail but, I hope, not in so much detail that you get bogged down in the mire, and lose the big picture.

There is a lot going on with adjunctions, and you will probably get confused more than once. You might get things mixed up, forget which way an arrow is suppose to go, not be able to spell contafurious, and so on. Don't worry. I've been at it for over 40 years and I still can't remember some of the details. In fact, I don't try to, and neither should you.

You should get yourself to the position where you can recognize that perhaps there is an adjunction lurking around, but you may not be quite sure where. You can then look up the details. That's exactly what I do. If you ever have to use adjunctions everyday, then the details will become second nature to you.

It all a bit like learning to ride a bike. Except that sometimes you have to do it backwards and stand on your head, in a different notation. The good thing is that you are unlikely to get hurt, unless you stab yourself with your pencil.

5.1 Adjunctions defined

When first seen in their full categorical glory adjunctions can seem a bit daunting. There is a lot going on; many balls to keep in the air. Don't worry. In many particular examples much of this complexity can simply disappear.

In this section we look at all the various components and eventually arrive at the formal definition. Then in the later section we analyse the content of the notion. Thus for most of this section we merely run through the various bits of gadgetry that make up an adjunction. We don't look at the details of the restrictions imposed on these gadgets.

An adjunction is an interaction between two categories

$$\boldsymbol{Src} \qquad\qquad \boldsymbol{Trg}$$

which we may think of as the

$$\text{source} \qquad\qquad \text{target}$$

category. This is an entirely conventional distinction, but it can help. The two categories play similar roles, but some of their attributes are in mirror image.

We have two covariant functors

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{Trg}$$

going between the two categories, but in opposite directions. These functors are related in a certain way. We get to these appropriate details in due course. By convention we call

$$F \qquad\qquad G$$

the

$$\text{left} \qquad\qquad \text{right}$$

adjoint of the pair, and we write

$$F \dashv G$$

to indicate this relationship. We also write any of

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}}_{\dashv} \boldsymbol{Trg} \qquad \boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}}_{\perp} \boldsymbol{Trg} \qquad \boldsymbol{Src} \underset{F \dashv G}{\rightleftarrows} \boldsymbol{Trg}$$

to indicate the relationship. Usually the left functor is placed above the right functor.

There are other bit of notation used with an adjunction. Sometimes we write

$$F^* \quad \text{for} \quad F \qquad\qquad F_* \quad \text{for} \quad G$$

where the position of the decoration indicates which is the left and which is the right component. This can be useful when there is more than one adjunction around and we don't want to waste the alphabet. In those circumstances we sometimes write F for the pair (F^*, F_*). For the time being we will stick with $F \dashv G$, but eventually we will use some of this other notation.

By convention we think of an adjunction as passing in the direction of its left adjoint. We write any of

$$\boldsymbol{Src} \rightharpoondown \boldsymbol{Trg} \qquad \boldsymbol{Src} \xrightharpoondown{F \dashv G} \boldsymbol{Trg} \qquad \boldsymbol{Src} \xrightharpoondown{F^* \dashv F_*} \boldsymbol{Trg}$$

for the adjunction making use of the harpoon arrow to alert us.

This is merely conventional terminology and notation. Some of the older literature was written before these conventions were established. You might find that left and right are called something else, such as right and left. Also, some of the current ignoscenti are never quite sure what day it is, although they think they do.

The two functors can be composed to give endofunctors

$$G \circ F \qquad\qquad F \circ G$$

on

$$\boldsymbol{Src} \qquad\qquad \boldsymbol{Trg}$$

respectively. These composites are related to the corresponding identity functors by natural transformations

$$\boldsymbol{Id}_{\boldsymbol{Src}} \xrightarrow{\eta} G \circ F \qquad\qquad F \circ G \xrightarrow{\epsilon} \boldsymbol{Id}_{\boldsymbol{Trg}}$$

called the

unit co-unit

of the adjunction. Notice the left/right antisymmetry of these gadgets.

Consider any pair

$$A \qquad S$$

of objects from

$$\boldsymbol{Src} \qquad \boldsymbol{Trg}$$

respectively. An adjunction tries to compare these objects. Of course, there isn't a direct comparison, for they live in different worlds. To compare them we transport one of the objects to the other category, and do the comparison there. Thus we use one of the two arrow sets

$$\boldsymbol{Src}[A, GS] \qquad \boldsymbol{Trg}[FA, S]$$

of the indicated category.

Notice that the

left functor F right functor G

occurs in the

left position right position

of the appropriate arrow set. This helps us to remember which functor is doing which job.

Each of the two arrow sets provides a place where we might compare the two objects. But which place should we use? It doesn't matter, for parts of the gadgetry of an adjunction is an inverse pair of bijections

$$\begin{array}{ccc} f & \longmapsto & f^{\sharp} \\ \boldsymbol{Src}[A, GS] & & \boldsymbol{Trg}[FA, S] \\ g_{\flat} & \longleftarrow\!\!\dashv & g \end{array}$$

between the two arrows sets.

Furthermore, and this is what makes an adjunction what it is, these two assignment

$$(\cdot)^{\sharp} \qquad (\cdot)_{\flat}$$

must be natural for variation of both A and S.

There is quite a lot going on here. In the next section we work through this data again to see exactly what it means. For now let's give the formal definition of an adjunction.

5.1 DEFINITION. An adjunction

$$\big(F,\ G,\ (\cdot)^{\sharp},\ (\cdot)_{\flat}\big)$$

consists of a pair of covariant functors

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{Trg} \quad (F \dashv G)$$

where, for each

$\boldsymbol{Src}$-object A $\boldsymbol{Trg}$-object S

the two transposition assignments

$$\boldsymbol{Src}[A, GS] \underset{(\cdot)_\flat}{\overset{(\cdot)^\sharp}{\rightleftarrows}} \boldsymbol{Trg}[FA, S]$$

form an inverse pair of bijections, where each is natural in A and S. ■

Observe that the data for an adjunction is not just a pair (F, G) of functors. We also need the nominated transposition assignments $((\cdot)^\sharp, (\cdot)_\flat)$. It is possible for two functors to be adjoint in two different ways. [1] Of course, we only need one of $(\cdot)^\sharp$ and $(\cdot)_\flat$, for each is the inverse of the other.

Notice that I seem to have forgotten the unit and co-unit. That is because we can show that

$$\eta_A = (\boldsymbol{id}_{FA})_\flat \qquad\qquad \epsilon_S = (\boldsymbol{id}_{GS})^\sharp$$

do those jobs. We look at this in Section 5.3.

It turns out that various selections of the data

$$F \quad G \quad (\cdot)^\sharp \quad (\cdot)_\flat \quad \eta \quad \epsilon$$

can be put together in different ways to form an adjunction. We look at these various combinations in the following sections.

Exercises 5.1.1, 5.1.2, and 5.1.3 are rather straight forwrad. You have seen most of the gadgetry before. Exercises 5.1.4, 5.1.5, and 5.1.6 are more complicated, and you may have to return to them more than once.

Exercises

5.1.1 Each poset can be viewed as a category. Show that a poset adjunction is nothing more than a categorical adjunction.

5.1.2 Each set S can be converted into a preset in two extreme ways. The discrete version uses equality as the comparison. For the indiscrete version any two elements are comparable.

Show that the forgetful functor

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Pre}$$

has both a left and a right adjoint, and these are different.

5.1.3 Show that the forgetful functor

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Top}$$

has both a left and a right adjoint, and these are different.

[1] I can't think of an example where this is necessary.

5.1.4 For a poset S let $\mathcal{L}S$ be the poset of lower sections of S (under inclusion).

(a) Show that each monotone function

$$S \xrightarrow{\ f\ } T$$

between posets induces a monotone functions

$$\mathcal{L}S \xleftarrow{\ f^{\leftarrow}\ } \mathcal{L}T$$

via inverse image.

(b) Show that the monotone function $f^{\leftarrow}$ has both a left adjoint and a right adjoint, and in general these are different. One of these is given by direct image, and the other isn't.

5.1.5 Let ∇ be a category viewed as a template, let $\boldsymbol{C}$ be an arbitrary category, and consider the diagonal functor.

$$\boldsymbol{C}^{\nabla} \xleftarrow{\ \Delta\ } \boldsymbol{C}$$

Show that $\boldsymbol{C}$ has a limit for each ∇-diagram precisely when Δ has a right adjoint, and sort out the corresponding result for colimits.

5.1.6 Let

$$\boldsymbol{Src} \xleftarrow[\ G\ ]{} \boldsymbol{Trg}$$

be a functor which does have a left adjoint.

Show that each limit cone in $\boldsymbol{Trg}$ is transported by G into a limit cone in $\boldsymbol{Src}$.

5.2 Adjunctions illustrated

Definition 5.1 says exactly what an adjunction is, but doesn't tell us very much. There are several hidden consequences of the definition and these are difficult to see. In the following sections we take the definition apart, and look at the various components of an adjunction in some detail. To do that it will help if there are some examples we can get our teeth into. There are some simple examples. Exercises 5.1.2 and 5.1.3 give a couple of these, and we will see a few more. In this section we look at two (or perhaps it's three) examples with a bit more content. The idea is that as you read the following sections you can use these examples to illustrate what is going on. Thus you shouldn't expect to understand these examples immediately. Keep coming back to them as you are getting to grips with the various aspects of adjunctions.

The two examples, an algebraic example and a topological example, are miniature versions of more involved, and quite important, adjunctions. I will say what these larger versions are, but at this stage we don't go into any details.

There is also a rather simple set-theoretic example (as a preliminary for the topological example). This is, perhaps, the best example to start with. However, you should be careful with it. In some ways it is too simple to bring out all of the different facets that an adjunction can have.

5.2.1 An algebraic example

An involution algebra is a structure

$$(A,\ (\cdot)^\bullet)$$

carried by a set A where $(\cdot)^\bullet$ is an involution on this carrier. In other words $(\cdot)^\bullet$ is a 1-placed operation on A with

$$a^{\bullet\bullet} = a$$

for each $a \in A$. An involution morphism

$$A \xrightarrow{\ \phi\ } B$$

between involution algebras is a function, as indicated, such that

$$\phi(a^\bullet) = \phi(a)^\bullet$$

for each $a \in A$. This gives us the category

$$\boldsymbol{Inv}$$

of involution algebra and involution morphisms.

Let

$$\boldsymbol{Set} \xleftarrow{\ U\ } \boldsymbol{Inv}$$

be the underlying functor, the forgetful functor that loses the involution. We show that U has adjoints on both sides

$$\boldsymbol{Set} \underset{\Pi}{\overset{\Sigma}{\rightrightarrows}} \boldsymbol{Inv} \qquad \boldsymbol{Set} \xleftarrow{\ U\ } \boldsymbol{Inv}$$

and these are quite different.

Before we begin to construct these adjoints let's see where this example comes from. There are three levels of generalization each of which is worth analysing in its own right.

For the first level recall the notion of a (right) R-sets for a monoid R. An involution algebra is nothing more than an R-set for an appropriate monoid R. (Can you see which one?) The forgetful functor

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Set}\text{-}R$$

has both a left adjoint and a right adjoint.

For the next level consider an arbitrary morphism

$$S \longrightarrow R$$

between monoids. This induces a functor

$$\boldsymbol{Set}\text{-}S \longleftarrow \boldsymbol{Set}\text{-}R$$

called restriction of scalars. This functor has both a left adjoint and a right adjoint. When S is the trivial monoid this example reduces to the previous one.

Finally let

$$S \longrightarrow R$$

be a ring morphism. This induces a restriction of scalars functor

$$\boldsymbol{Mod}\text{-}S \longleftarrow \boldsymbol{Mod}\text{-}R$$

between the module categories. Again this has both a left adjoint and a right adjoint. In this case the left adjoint is given by a tensor product, and the right adjoint is an enriched hom-functor.

All of these adjunctions are worth looking at sometime.

Here we concentrate on the involutary case. From now on in this block the only algebras and morphisms we meet are involutary, so we drop the qualifier 'involution'.

To produce the two adjoints it will help if we let

Set		***Inv***
$X, Y, \ldots$	Objects	$A, B, \ldots$
$f, g, \ldots, k, \ldots$	Arrows	$\phi, \psi, \ldots, \lambda, \ldots$

range over the indicated gadgets.

We describe the initial parts of the two constructions in parallel.

For each set X let

$$\Sigma X = X + X \qquad \Pi X = X \times X$$

that is

$$\Sigma X = \{(x, i) \mid x \in X,\ i = 0, 1\} \qquad \Pi X = \{(x, y) \mid x, y \in X\}$$

the set of $\{0, 1\}$-tagged elements of X and the set of ordered pairs from X. It is not hard to furnish each of these with an involution. For ΣX we flip the tag, and for ΠX we swap the components.

We require both Σ and Π to be a functor to ***Inv***. Given a function

$$Y \xrightarrow{\ k\ } X$$

what should

$$\Sigma Y \xrightarrow{\ \Sigma(k)\ } \Sigma X \qquad \Pi Y \xrightarrow{\ \Pi(k)\ } \Pi X$$

be? By fiddling around we soon find the answer, but verifying that it does produce morphisms takes a little bit of work. Think about this before you continue.

This gives us two of the four components of each of the two adjunctions. Next we require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, UA] & & \boldsymbol{Inv}[\Sigma X, A] \\ \psi_\flat & \longleftarrow\!\shortmid & \psi \end{array} \qquad \begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Inv}[A, \Pi X] & & \boldsymbol{Set}[UA, X] \\ g_\flat & \longleftarrow\!\shortmid & g \end{array}$$

for each set X and each algebra A. Producing these requires a little bit of thought.

For the left hand bijection it helps if we set

$$a^{(i)} = \begin{cases} a^{\bullet} & \text{if } i = 1 \\ a & \text{if } i = 0 \end{cases}$$

for each $a \in A$ and tag i. In particular we have

$$a^{(i)\bullet} = a^{(1-i)} = a^{\bullet(i)} \qquad \phi(a^{(i)}) = \phi(a)^{(i)}$$

for each $a \in A$, tag i, and morphism ϕ.

For the right hand bijection remember that ϕ is given as a morphism.

You should now think about these constructions for a while, and do Exercises 5.2.1 and 5.2.2.

Of course, this doesn't quite prove that each of

$$\big(\Sigma, U, (\cdot)^{\sharp}, (\cdot)_{\flat}\big) \qquad \big(U, \pi, (\cdot)^{\sharp}, (\cdot)_{\flat}\big)$$

is an adjunction. We need to show that each $(\cdot)^{\sharp}$ and each $(\cdot)_{\flat}$ is natural. However, let's leave that until we have a better idea of what that means.

Exercises

5.2.1 Describe the action of Σ and Π on functions, and verify that each result is a morphism.

5.2.2 Set up the two inverse pair of bijections $(\cdot)^{\sharp}$ and $(\cdot)_{\flat}$. At this stage don't worry about the required naturality.

5.2.2 A set-theoretic example

In this block we describe a rather easy example of an adjunctions where the category ***Set*** is both the source and target. We have met most of the components before. The reason for doing this example here is that in the next block we produce an enriched version with ***Set*** replaced by ***Top***. That has far more content, but much of the gadgetry is the same as this ***Set*** example.

Let I be a fixed set. We know that

$$- \times I$$

is an endo-functor on ***Set***. We show that this functor has a right adjoint.

This right adjoint is another functor that we already know. It is the hom-functor $\boldsymbol{Set}[I, -]$. Thus, we attach to each set Y the set of all functions

$$I \longrightarrow Y$$

from I to Y. To do that we use what at first may seem an odd notation for this function set. However, in the end you will find that it is better than the more usual notations.

For each set Y let

$$I \Rightarrow Y$$

be the set of all functions from I to Y.

This gives us two endo-functors

$$-\times I \qquad I \Rightarrow -$$

on ***Set***. Recall that each function

$$X_2 \xrightarrow{k} X_1 \qquad Y_1 \xrightarrow{l} Y_2$$

is sent to

$$\begin{array}{ccc} X_2 \times I & \longrightarrow & X_1 \times I \\ (x,i) & \longmapsto & (k(x),i) \end{array} \qquad \begin{array}{ccc} I \Rightarrow Y_1 & \longrightarrow & I \Rightarrow Y_2 \\ p & \longmapsto & l \circ p \end{array}$$

respectively.

To show that

$$-\times I \dashv I \Rightarrow -$$

we at least require an inverse pair of bijections.

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, I \Rightarrow Y] & & \boldsymbol{Set}[X \times I, Y] \\ g_\flat & \longleftarrow & g \end{array}$$

for each pair X, Y of sets. If you think about it this is an almost trivial exercise. In many mathematical situations we wouldn't even distinguish between f and $f^\sharp$, nor between g and $g_\flat$.

We should also prove that each of $(\cdot)^\sharp$ and $(\cdot)_\flat$ is natural. For this example, that is not difficult, but let's leave it until we have more of an understanding of what it entails.

For the record let us state the result we are aiming at.

5.2 THEOREM. *For each set I, we have*

$$-\times I \dashv I \Rightarrow -$$

an adjunction of endofunctors on ***Set***.

As with any adjunction, this one has a unit and a counit

$$X \xrightarrow{\eta_X} I \Rightarrow (X \times I) \qquad (I \Rightarrow Y) \times I \xrightarrow{\epsilon_Y} Y$$

natural in X and Y, respectively. In this case these are more or less obvious.

Exercises

5.2.3 Write down the definitions of $(\cdot)^\sharp$ and $(\cdot)_\flat$. These two assignments are little more than inserting or omitting brackets.

5.2.4 Write down the unit and counit, and show that each is natural.

5.2.3 A topological example

In this block we re-do the adjunction of Block 5.2.2 with ***Set*** replaced by ***Top***. As we will see this is not entirely straight forward. We need to impose appropriate conditions on the pivotal object I.

With this kind of result there are two interacting themes, a general and a particular. The general theme is that of the categorical constructions and calculations. Here we find that most of these have been done in Block 5.2.2. That is why we did that simple example before this one. The particular theme is that of handling the topological restrictions.

It is useful whenever possible to separate the two themes. Category theory is good at handling the generalities. By separating these from the particularities we are able to see more clearly the specific content of a result.

As in the ***Set*** example, in this ***Top*** case there are three players, the central fixed object I and the two varying objects X and Y. Here all three of X, I, Y are topological spaces, and any functor we produce must return topological spaces and continuous maps.

So let

$$X \qquad I \qquad Y$$

be topological spaces with carried families

$$\mathcal{O}X \qquad \mathcal{O}I \qquad \mathcal{O}Y$$

of open sets, the topologies.

We know that the cartesian product $X \times I$ carries the product topology. This is the smallest topology for which the two projections

$$X \times I \longrightarrow X \qquad\qquad X \times I \longrightarrow I$$

are continuous. Thus for each $U \in \mathcal{O}X$ and $W \in \mathcal{O}I$ the product $U \times W$ is open in $X \times I$, and this set of products forms a subbase of the whole topology. This space $X \times I$ with these two continous projections form a product wedge in ***Top***. (That is why the product topology is defined in the way it is.) Thus we have an endo-functor

$$- \times I$$

on ***Top***.

Our aim is to find a right adjoint to this functor. And we want this to be an enriched hom-functor.

For spaces I and Y let

$$I \Rightarrow Y$$

be the set of all *continuous* maps from I to Y. This is smaller than the set of all functions from I to Y, so perhaps we should use a different notation. But we won't. However, if you do become confused then try something like

$$I \stackrel{\cdot}{\Rightarrow} Y$$

for the set of continuous maps.

With I fixed this certainly gives us a functor

$$\boldsymbol{Top} \xrightarrow{\ I \Rightarrow -\ } \boldsymbol{Set}$$

but this isn't good enough. The functor must output to ***Top***, not just ***Set***. This means we have to find a way of topologizing $I \Rightarrow Y$.

Let

$$\mathcal{K}I$$

be the family of compact subsets K of I. Before we continue you should make sure you know what a compact subset is.

5.3 DEFINITION. For $K \in \mathcal{K}I$ and $V \in \mathcal{O}Y$ let

$$\langle K, V \rangle$$

be the set of continuous maps

$$I \xrightarrow{\ \theta\ } Y$$

such that

$$\theta[K] \subseteq V$$

that is $\theta(i) \in V$ for all $i \in K$.

The compact open topology on $I \Rightarrow Y$ has the family of all $\langle K, V \rangle$ as a subbase. ■

This certainly topologizes $I \Rightarrow Y$, but we want $I \Rightarrow -$ to be an endofunctor on ***Top***, so we also need an action on arrows.

Consider any map

$$Y_1 \xrightarrow{\ \psi\ } Y_2$$

between two spaces. This induces a function

$$\begin{array}{ccc} (I \Rightarrow Y_1) & \xrightarrow{\ \Psi\ } & (I \Rightarrow Y_2) \\ \theta & \longmapsto & \psi \circ \theta \end{array}$$

between the two functions spaces. We check that Ψ is continuous.

Consider any $K \in \mathcal{K}I$ and $V \in \mathcal{O}Y_2$. We need

$$\Psi^{\leftarrow}\big(\langle K, V \rangle\big)$$

to be open in $I \Rightarrow Y_1$. But a simple calculation gives

$$\Psi^{\leftarrow}\big(\langle K, V \rangle\big) = \langle K, \psi^{\leftarrow}(V) \rangle$$

which, since ψ is continuous, is open in $I \Rightarrow Y_1$. (You should do that calculation.)

This gives the arrow assignment for $I \Rightarrow -$, and the functorial properties are almost immediate.

We now have a pair of endo-functors on ***Top***, but these don't necessarily form an adjoint pair. For that we need I to be a particular kind of space.

Read the following definition carefully. Many texts get it wrong.

5.4 DEFINITION. A topological space I is locally compact if for each situation

$$r \in V \in \mathcal{O}I$$

we have

$$r \in W \subseteq K \subseteq V$$

for some $K \in \mathcal{K}I$ and $W \in \mathcal{O}I$. ■

With these preliminaries we are ready to prove the following.

5.5 THEOREM. *For each locally compact space I, we have*

$$- \times I \;\dashv\; I \Rightarrow -$$

*an adjunction of endofunctors on **Top**.*

Theorem 5.5 is important in its own right, but it also has an important refinement. We consider the case where I is the real interval $[0,1]$, and modify the two spaces

$$X \times I \qquad\qquad I \Rightarrow Y$$

to produce the

suspension space of X loop space of Y

respectively.

A loop in the space Y is a continuous map

$$\ell : I \longrightarrow Y$$

with $\ell(0) = \ell(1)$. We consider the set of all such loops as a subspace of $I \Rightarrow Y$.

We modify the product space $X \times I$ by pincing together all points

$$(x, 0) \quad \text{for } x \in X$$

and all points

$$(x, 1) \quad \text{for } x \in X$$

to obtain two pinch points. Technically we take a certain quotient space of $X \times I$.

It can be shown that

$$\text{Suspension} \dashv \text{Loop}$$

by adding to the proof of Theorem 5.2. (There are a couple of technicalities that I have omitted, but this description is not too far from the truth.)

The proof of Theorem 5.5 will take some time. We will do it in little bits as we get to know the general categorical notions. The overall method of proof is the same as for Theorem 5.2 with an extra layer of complexity. We must check that various functions are continuous.

Throughout we fix I, an arbitrary locally compact space.

The first thing we must do is set up an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Top}[X, I \Rightarrow Y] & & \boldsymbol{Top}[X \times I, Y] \\ \psi_\flat & \longleftarrow\!\shortmid & \psi \end{array}$$

for arbitrary spaces X and Y. In fact we use the same trick as for the ***Set*** case. Thus we set

$$\psi_\flat(x)(i) = \psi(x,i) \qquad \phi^\sharp(x,i) = \phi(x)(i)$$

for each $x \in X$ and $i \in I$. Of course, we must show that $\phi^\sharp$ and $\psi_\flat$ are continuous, but once we have done that the rest of the proof is trivial (as for the ***Set*** case).

We look first at the construction

$$\psi_\flat \longleftarrow\!\!| \ \psi$$

since this doesn't make use of the local compactness of I.

We are given a continuous function

$$X \times I \xrightarrow{\ \psi\ } Y$$

where $X \times I$ carries the product topology. Since ψ is continuous we see that for each $x \in X$ the function

$$\psi(x,\cdot) : I \longrightarrow Y$$

is continuous. Thus we may define a function

$$\psi_\flat : X \longrightarrow (I \Rightarrow Y)$$

by

$$\psi_\flat(x)(i) = \psi(x,i)$$

for each $x \in X$ and $i \in I$. Our job is to show that $\psi_\flat$ is continuous, where $(I \Rightarrow Y)$ carries the compact open topology.

5.6 LEMMA. *For each pair X and Y of topological spaces, and each continuous map*

$$X \times I \xrightarrow{\ \psi\ } Y$$

the induced function

$$X \xrightarrow{\ \psi_\flat\ } (I \Rightarrow Y)$$

as defined above is continuous.

Proof. Consider any subbasic open set

$$\langle K, V\rangle$$

of $I \Rightarrow Y$, where

$$K \in \mathcal{K}I \qquad V \in \mathcal{O}Y$$

are the two components. We require

$$\psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

to be open in X. To show that consider any member s of this set. We require

$$s \in U \subseteq \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

for some $U \in \mathcal{O}X$.

We have

$$\psi_\flat(s) \in \langle K, V \rangle$$

that is

$$\psi(s,i) = \psi_\flat(s)(i) \in V$$

for each $i \in K$. We now use the continuity of ψ and the compactness of K. ■

Next we look at the construction

$$\phi \longmapsto \phi^\sharp$$

and this does make use of the local compactness of I.

5.7 LEMMA. *Let I be a locally compact topological space. For each pair X and Y of topological spaces, and for each continuous map*

$$X \xrightarrow{\ \phi\ } (I \Rightarrow Y)$$

where $I \Rightarrow Y$ carries the compact open topology, there is a continuous map

$$X \times I \xrightarrow{\ \phi^\sharp\ } Y$$

given by

$$\phi^\sharp(x,i) = \phi(x)(i)$$

for each $x \in X$ and $i \in I$.

Proof. Consider any $V \in \mathcal{O}Y$ and any member

$$(s,r) \in \phi^{\sharp\leftarrow}(V)$$

of the inverse image of V across $\phi^\sharp$. Remembering how $X \times I$ is topologized, it suffices to produce open neighbourhoods

$$s \in U \in \mathcal{O}X \qquad r \in W \in \mathcal{O}I$$

such that

$$U \times W \subseteq \phi^{\sharp\leftarrow}(V)$$

holds. We satisfy the conditions one by one.

We know that

$$\phi(s) : I \longrightarrow Y$$

is continuous, and hence

$$\phi(s)^{\leftarrow}(V)$$

is open on I. But

$$\phi(s)(r) = \phi^\sharp(s,r) \in V$$

so that

$$r \in \phi(s)^{\leftarrow}(V)$$

and hence the local compactness of I gives

$$r \in W \subseteq K \subseteq \phi(s)^{\leftarrow}(V)$$

for some $K \in \mathcal{K}I$ and $W \in \mathcal{O}I$. This W is one of the open sets that we need.

The pair K and V give us a subbasic open set

$$\langle K, V \rangle$$

of $I \Rightarrow Y$. Thus, since ϕ is continuous, we see that

$$U = \phi^{\leftarrow}\big(\langle K, V \rangle\big)$$

is open in X. We may check that this is the other open set that we need. ■

This sets up the inverse pair of bijections for a given pair X, Y. However, we need these bijections to be natural for variation of X and Y. We can postpone a proof of that for a while until we have analysed the general notion in more detail. However, I can tell you that the proof is exactly the same as for the ***Set*** case. There is no more work to be done.

Exercises

5.2.5 Fill in the details required to show that $I \Rightarrow -$ is an endo-functor on ***Top***. Observe that many of the calculations are the same as for the ***Set*** case.

5.2.6 Complete the proof of Lemma 5.6.

As a hint for a fixed s let i range through K to obtain an open covering of K.

5.2.7 Complete the proof of Lemma 5.7.

As a hint observe that for $i \in I$ we have

$$i \in K \Longrightarrow i \in \phi(s)^{\leftarrow}(V) \Longrightarrow \phi(s)(i) \in V$$

and hence $s \in U$.

5.2.8 For the ***Top*** adjunction write down the unit and the counit, and show that each is natural.

5.3 Adjunctions uncouple

In this and the next two sections we look at various aspects of Definition 5.1. As we do this you should keep going back to the examples of Section 5.2. This will help you understand the general notion. The set-theoretic example is always a good place to start, but you should also investigate at least one of the other two examples.

To form an adjunction the data

$$(F,\ G,\ (\cdot)^{\sharp},\ (\cdot)_{\flat})$$

must satisfy two requirements; the bijection requirement, and the naturality requirement. The bijection requirement is easy to understand. The naturality requirement needs a good going over.

(Bij) For all

$$\begin{array}{ccc} \boldsymbol{Src} & & \boldsymbol{Trg} \\ A & \text{objects} & S \\ A \xrightarrow{f} GS & \text{arrows} & FA \xrightarrow{g} S \end{array}$$

from the indicated categories, both

$$(f^\sharp)_\flat = f \qquad (g_\flat)^\sharp = g$$

hold.

(Nat) For all

$$\begin{array}{ccc} \boldsymbol{Src} & & \boldsymbol{Trg} \\ B \quad A & \text{objects} & S \quad T \\ B \xrightarrow{k} A & & S \xrightarrow{l} T \\ & \text{arrows} & \\ A \xrightarrow[f]{} GS & & FA \xrightarrow[g]{} S \end{array}$$

from the indicated categories, both

$$(\sharp) \quad (G(l) \circ f \circ k)^\sharp = l \circ f^\sharp \circ F(k) \qquad G(l) \circ g_\flat \circ k = (l \circ g \circ F(k))_\flat \quad (\flat)$$

hold.

Table 5.1: The various requirements for an adjunction

It turns out that the naturality property can be split into several smaller parts. Furthermore, these can be put together in different ways, sometimes in conjunction with unit or co-unit properties, to determine an adjunction. We begin to look at these combinations in this section.

The bijection requirement, (Bij), is given in Table 5.1. It merely says that for each pair of objects $A \in \boldsymbol{Src}$ and $S \in \boldsymbol{Trg}$, the two assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Src}[A, GS] & & \boldsymbol{Trg}[FA, S] \\ g_\flat & \longleftarrow\!\!| & g \end{array}$$

form an inverse pair of bijections. Of course, each of $(\cdot)^\sharp$ and $(\cdot)_\flat$ determines the other. Thus in any particular example it suffices to mention just one of them, and say that it is a bijection. Its inverse is the other one.

The naturality requirement is more complicated. It says that each of the transposition assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ is natural. But what does that mean?

Recall that we may form a product category

$$\boldsymbol{Src} \times \boldsymbol{Trg}$$

whose objects are pairs

$$(A, S)$$

of objects $A \in \boldsymbol{Src}$ and $S \in \boldsymbol{Trg}$. We won't say what the arrows are just yet, and you will see why not in just a moment. The two functors F and G give functors $\mathfrak{F}$ and $\mathfrak{G}$[2]

$$\begin{array}{ccc} (A,S) & \overset{\mathfrak{F}}{\longmapsto} & \boldsymbol{Trg}[FA,S] \\ \boldsymbol{Src} \times \boldsymbol{Trg} & \longrightarrow & \boldsymbol{Set} \\ (A,S) & \underset{\mathfrak{G}}{\longmapsto} & \boldsymbol{Trg}[A,GS] \end{array}$$

to the category of sets. The naturality requirement says that $(\cdot)^\sharp$ and $(\cdot)_\flat$ provide natural isomorphisms between $\mathfrak{F}$ and $\mathfrak{G}$.

Now you can see that we have to be a bit careful. Arrows sets have different variance, namely

$$[\text{Contra} \ , \text{Co}]$$

in the two positions. Thus technically we are dealing with a pair of functors

$$\boldsymbol{Src}^{\mathrm{op}} \times \boldsymbol{Trg} \underset{\mathfrak{G}}{\overset{\mathfrak{F}}{\rightrightarrows}} \boldsymbol{Set}$$

where we use the opposite of $\boldsymbol{Src}$.

That's a slightly flashy way of describing the requirements. Let's get down to basics.

Consider a pair of arrows

$$B \xrightarrow{\ k\ } A \qquad\qquad S \xrightarrow{\ l\ } T$$

from the two categories. Notice how we have anticipated the contravariance on the $\boldsymbol{Src}$ component. The pair (k, l) form an arrow in the product category

$$\boldsymbol{Src}^{\mathrm{op}} \times \boldsymbol{Trg}$$

which is the source of both $\mathfrak{F}$ and $\mathfrak{G}$. Using (k, l) we obtain the pair of commuting diagrams in $\boldsymbol{Set}$ given in Table 5.2. In other words, each arrow

$$A \xrightarrow{\ f\ } GS \qquad\qquad FA \xrightarrow{\ g\ } S$$

of

$$\boldsymbol{Src} \qquad\qquad \boldsymbol{Trg}$$

is sent to

$$B \xrightarrow{\ G(l) \circ f \circ k\ } GT \qquad\qquad FB \xrightarrow{\ l \circ g \circ F(k)\ } T$$

[2]Set this as an exercise earlier

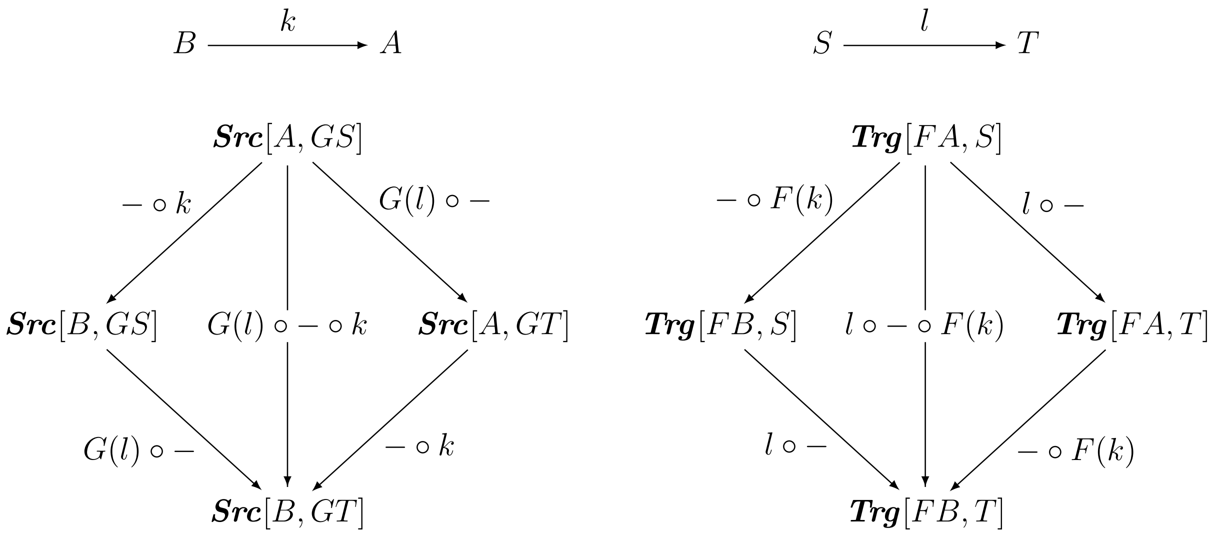
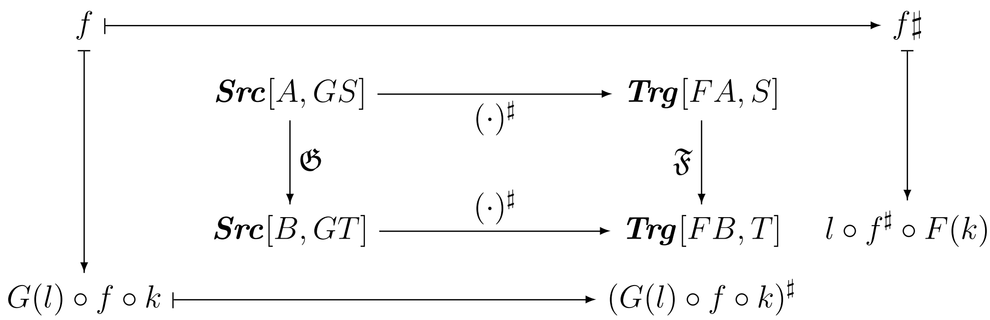


Table 5.2: Two commuting diagrams in ***Set***

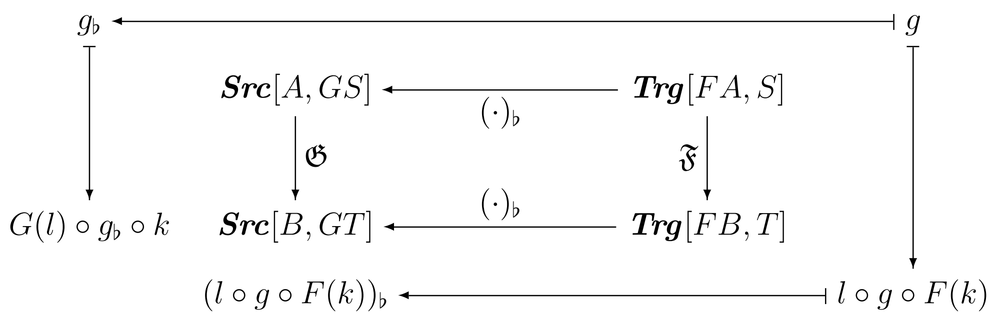
respectively

Now, for given k and l, consider the two paths



from $\mathfrak{G}$ to $\mathfrak{F}$. The assignment $(\cdot)^\sharp$ being natural is simply that these two paths agree for all k, l, f.

In the same way, $(\cdot)_\flat$ being natural is that the two paths



agree for all k, l, g.

The naturality requirement, (Nat), is given in Table 5.1. It is stated as two identities

$$(\sharp) \qquad (\flat)$$

in the arrows k, l, f, g. You should note the type and naming of these arrows. We invoke these identities many times, and we don't (can't) always use matching letters. And sometimes we need only some of the arrows.

In this account we try to stick to the notation used in (Nat). However, sometimes we have to use different letters for the arrows, and sometimes these seem to appear out of position.

We know that if each component of a natural transformation is a bijection, then the inverse is also natural. This leads to a simplification in (Nat).

5.8 LEMMA. *In the presence of* (Bij), *each of the two identities* ($\sharp$), ($\flat$) *of* (Nat) *implies the other.*

Proof. Assuming ($\sharp$) let's check ($\flat$). For this we need arrows k, l, g of the indicated type. Let $f = g_\flat$ to obtain the fourth arrow. From (Bij) we have $g = f^\sharp$.

Using ($\sharp$) at the second step we have

$$(G(l) \circ g_\flat \circ k)^\sharp = (G(l) \circ f \circ k)^\sharp = l \circ f^\sharp \circ F(k) = l \circ g \circ F(k)$$

so that

$$G(l) \circ g_\flat \circ k = (l \circ g \circ F(k))_\flat$$

by a second use of (Bij). ■

This shows that in (Nat) only one of the two identities ($\sharp$) and ($\flat$) is needed. So why did we bother to state both? Because some particular examples are best handled using one condition rather than the other. In fact, we can go even further. We can decompose each of ($\sharp$) and ($\flat$) into two bits, and use various combinations of these bits. By taking either k or l to be an identity arrow we obtain four instances of ($\sharp$) and ($\flat$).

$$\begin{array}{llll} (\sharp\uparrow) & (G(l) \circ f)^\sharp = l \circ f^\sharp & (\flat\uparrow) & g_\flat \circ k = (g \circ F(k))_\flat \\ (\sharp\downarrow) & (f \circ k)^\sharp = f^\sharp \circ F(k) & (\flat\downarrow) & G(l) \circ g_\flat = (l \circ g)_\flat \end{array}$$

In these each occurring arrow has the type given in (Nat). Also, each condition is quantified. For instance, ($\sharp\uparrow$) says

For each pair of arrows

$$A \xrightarrow{\ f\ } GS \qquad\qquad S \xrightarrow{\ l\ } T$$

we have ...

and requires only two arrows, as indicated.

The following result gathers together many of the combinations you may meet.

5.9 LEMMA. *Consider a pair of functors*

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{Trg}$$

and a pair of assignments $(\cdot)^\sharp$ *and* $(\cdot)_\flat$ *satisfying* (Bij). *Then*

$$(\sharp\uparrow) \Longleftrightarrow (\flat\downarrow) \qquad (\sharp\downarrow) \Longleftrightarrow (\flat\uparrow)$$

and the data forms an adjunction precisely when one of the four pairs of conditions

$$(\sharp\uparrow),(\flat\uparrow)$$
$$(\sharp\uparrow),(\sharp\downarrow) \qquad\qquad (\flat\uparrow),(\flat\downarrow)$$
$$(\sharp\downarrow),(\flat\downarrow)$$

hold.

Proof. The proof of the top two equivalences is more or less the same as that of Lemma 5.8. We take

$$k = \boldsymbol{id}_A \qquad l = \boldsymbol{id}_S$$

as appropriate. Similarly, each adjunction satisfies the four conditions $(\sharp\uparrow,\sharp\downarrow,\flat\uparrow,\flat\downarrow)$.

It remains to verify that any one of the four pairs ensures that we have an adjunction. Let's look at the pair $(\sharp\uparrow,\sharp\downarrow)$. It suffices to show

$$(\sharp\uparrow,\sharp\downarrow) \Longrightarrow (\sharp)$$

and then invoke Lemma 5.8.

Consider arrows k, l, f as in (Nat). Using $(\sharp\uparrow)$ and then $(\sharp\downarrow)$ we have

$$(G(l)\circ f\circ k)^\sharp = l\circ(f\circ k)^\sharp = l\circ f^\sharp\circ F(k)$$

as required. Notice that $(\sharp\uparrow)$ is applied to the pair

$$B \xrightarrow{\ f\circ k\ } GS \qquad\qquad S \xrightarrow{\ l\ } T$$

which is allowed by the quantified condition. ■

As you can see, there are several combinations of the structured data

$$(F,\ G,\ (\cdot)^\sharp,\ (\cdot)_\flat)$$

which lead to an adjunction. It is best not to try to remember the details, but merely that there are variants.

Exercises

5.3.1 Definition 5.1 requires that the bijections $(\cdot)^\sharp, (\cdot)_\flat$ are natural for variation of both A and S. In Lemma 5.8 both A and S vary at the same time.

What happens if we fix one of A, S and let the other vary? What conditions arise out of that kind of naturality?

5.3.2 Consider the two algebraic constructions of Subsection 5.2.1.

For both case show that each of the two assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ is natural. In each case draw the square that must commute.

5.3.3 Consider the set-theoretic construction of Subsection 5.2.2.

Show that each of the two assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ is natural. In each case draw the square that must commute.

5.3.4 Consider the topological construction of Subsection 5.2.3.

Show that each of the two assignments $(\cdot)^\sharp$ and $(\cdot)_\flat$ is natural. In each case draw the square that must commute.

Did you notice something about the proof?

5.4 The unit and the co-unit

In Section 5.1 the unit and the co-unit of an adjunction made a brief appearance, but were not part of the official gadgetry discussed in Section 5.2. In fact, the unit and co-unit are important attributes of an adjunction, and can be more important than the transposition assignments. When appropriately restricted they can determine the adjunction.

5.10 DEFINITION. Given an adjunction, as in Definition 5.1, we set

$$\eta_A = (\boldsymbol{id}_{FA})_\flat \qquad\qquad \epsilon_S = (\boldsymbol{id}_{GS})^\sharp$$

for each

$$\boldsymbol{Src}\text{-object } A \qquad\qquad \boldsymbol{Trg}\text{-object } S$$

to obtain arrows

$$A \xrightarrow{\ \eta_A\ } (G\circ F)A \qquad (F\circ G)S \xrightarrow{\ \epsilon_S\ } S$$

the

$$\textsf{unit} \qquad\qquad \textsf{co-unit}$$

of the adjunction. ■

This section is devoted to an analysis of these gadgets. Naturally, we begin with their most important property.

5.11 LEMMA. *The unit and the co-unit of an adjunction are both natural.*

Proof. We deal with the unit. The co-unit is dealt with in a symmetric way.
Consider any arrow

$$B \xrightarrow{\ k\ } A$$

of ***Src***. We must show that the square

$$\begin{array}{ccccc}
B & \xrightarrow{\ \eta_B\ } & (G\circ F)B & \qquad & \eta_B = (\boldsymbol{id}_{FB})_\flat \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle (G\circ F)(k)} & & \\
A & \xrightarrow[\ \eta_A\ ]{} & (G\circ F)A & \qquad & \eta_A = (\boldsymbol{id}_{FA})_\flat
\end{array}$$

commutes, that is we must check that

$$(G\circ F)(k)\circ\eta_B = \eta_A\circ k$$

holds. To do this we use the identity ($\flat$) of (Nat) twice, but in different instantiations.
Thus with

$$\begin{array}{cc|cc}
B \xrightarrow{\ \boldsymbol{id}_B\ } B & FB \xrightarrow{\ F(k)\ } FA & B \xrightarrow{\ k\ } A & FA \xrightarrow{\ \boldsymbol{id}_{FA}\ } FA \\
 & FB \xrightarrow[\ \boldsymbol{id}_{FB}\ ]{} FB & & FA \xrightarrow[\ \boldsymbol{id}_{FA}\ ]{} FA
\end{array}$$

we have

$$G(F(k)) \circ (\boldsymbol{id}_{FB})_\flat = (F(k) \circ \boldsymbol{id}_{FB})_\flat = (\boldsymbol{id}_{FA} \circ F(k))_\flat = (\boldsymbol{id}_{FA})_\flat \circ k$$

as required. The central equality is a trivial property of the identity arrows. ∎

Once we know the unit or co-unit of an adjunction, we can retrieve the transposition assignments.

5.12 LEMMA. *Given an adjunction, as in Definition 5.1, we have*

$$f^\sharp = \epsilon_S \circ F(f) \qquad g_\flat = G(g) \circ \eta_A$$

for all arrows f and g as in (Nat).

Proof. We verify the left hand equality.
Given an arrow

$$A \xrightarrow{\;f\;} GS$$

we apply $(\sharp\downarrow)$ with

$$(k) \qquad A \xrightarrow{\;f\;} GS$$
$$(f) \qquad GS \xrightarrow[\boldsymbol{id}_{GS}]{} GS$$

as the component arrows. Thus

$$f^\sharp = (\boldsymbol{id}_{GS} \circ f)^\sharp = (\boldsymbol{id}_{GS})^\sharp \circ F(f) = \epsilon_S \circ F(f)$$

as required. ∎

By setting

$$f = \eta_A \qquad g = \epsilon_S$$

and remembering Definition 5.10 we obtain an important particular case of this result.

5.13 COROLLARY. *Given an adjunction, as in Definition 5.1, we have*

$$\epsilon_{FA} \circ F(\eta_A) = \boldsymbol{id}_{FA} \qquad G(\epsilon_S) \circ \eta_{GS} = \boldsymbol{id}_{GS}$$

*for each **Src**-object A and **Trg**-object S.*

These are important identities, for in the appropriate circumstances they determine the adjunction.

5.14 THEOREM. *Let*

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{Trg}$$

be a pair of functors, and let

$$\boldsymbol{Id}_{\boldsymbol{Src}} \xrightarrow{\;\eta\;} G \circ F \qquad F \circ G \xrightarrow{\;\epsilon\;} \boldsymbol{Id}_{\boldsymbol{Trg}}$$

be a pair of natural transformations satisfying the identities of Corollary 5.13. Then η and ϵ are the unit and co-unit of a unique adjunction $F \dashv G$.

Proof. By Lemma 5.12 we know that if this data does arise from an adjunction then

$$f^\sharp = \epsilon_S \circ F(f) \qquad g_\flat = G(g) \circ \eta_A$$

for all arrows

$$A \xrightarrow{\ f\ } GS \qquad\qquad FA \xrightarrow{\ g\ } S$$

of ***Src*** and ***Trg***, respectively.

Thus it is sufficient to show that these two assignments $(\cdot)^\sharp, (\cdot)_\flat$ form an inverse pair of bijections which satisfy (Nat).

Remember that the given naturality of η ensures that

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & (G \circ F)A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(F(f))} \\ GS & \xrightarrow[\eta_{GS}]{} & (G \circ F \circ G)S \end{array}$$

commutes. With this and one of the given conditions we have

$$\begin{aligned} (f^\sharp)_\flat &= (\epsilon_S \circ F(f))_\flat && \text{by the definition of } (\cdot)^\sharp \\ &= G(\epsilon_S \circ F(f)) \circ \eta_A && \text{by the definition of } (\cdot)_\flat \\ &= G(\epsilon_S) \circ G(F(f)) \circ \eta_A && \text{by the functorality of } G \\ &= G(\epsilon_S) \circ \eta_{GS} \circ f && \text{by the above naturality} \\ &= id_{GS} \circ f && \text{by the given right hand identity} \\ &= f && \text{by the neutral property of } id_{GS} \end{aligned}$$

to give one half of the bijection property. The other half $(g_\flat)^\sharp = g$ is proved in the same way.

The required naturality can be verified in several ways, but let's go straight for ($\sharp$). To this end consider arrows k, l, f as in (Nat). The given naturality of ϵ ensures that

$$\begin{array}{ccc} (F \circ G)S & \xrightarrow{\epsilon_S} & S \\ {\scriptstyle F(G(l))}\downarrow & & \downarrow{\scriptstyle l} \\ (F \circ G)T & \xrightarrow[\epsilon_T]{} & T \end{array}$$

commutes. Using this we have

$$\begin{aligned} l \circ f^\sharp \circ F(k) &= l \circ \epsilon_S \circ F(f) \circ F(k) && \text{by the definition of } (\cdot)^\sharp \\ &= l \circ \epsilon_S \circ F(f \circ k) && \text{by the functorality of } F \\ &= \epsilon_T \circ F(G(l)) \circ F(f \circ k) && \text{by the naturality given above} \\ &= \epsilon_T \circ F(G(l) \circ f \circ k) && \text{by the functorality of } F \\ &= (G(l) \circ f \circ k)^\sharp && \text{by the definition of } (\cdot)^\sharp \end{aligned}$$

to give ($\sharp$). The identity ($\flat$) can be verified in the same way. ■

For any particular adjunction only some of the data

$$(F,\ G,\ (\cdot)^\sharp,\ (\cdot)_\flat,\ \eta,\ \epsilon)$$

is needed. However, given an adjunction you should get into the habit of working out which each component is. Sometimes you find a simpler way of looking at the whole set up.

Exercises

5.4.1 Show that the counit of an adjunction is natural.

5.4.2 Show how to retrieve the transpostion $(\cdot)_\flat$ from the unit η of an adjunction.

5.4.3 Do the other half of the proof of Theorem 5.14.

5.4.4 Consider the two algebraic constructions of Subsection 5.2.1.
Show that for each set X and algebra A there are assignments

$$X \xrightarrow{\eta_X} (U \circ \Sigma)X \qquad (\Sigma \circ U)A \xrightarrow{\delta_A} A$$

$$(U \circ \Pi)X \xrightarrow{\epsilon_X} X \qquad A \xrightarrow{\zeta_A} (\Pi \circ U)A$$

where δ_A and ζ_A are morphisms.
Show that each of the families $\eta, \epsilon, \delta, \zeta$ is a natural transformation.
(These assignments have to do a particular job, so you might not hit on the correct ones at first go. The required naturality and the following exercise should help.)

5.4.5 For the two algebraic constructions of Subsection 5.2.1, verify directly the identities of Lemma 5.12 and Corollary 5.13. In each case draw the composite arrow that is dealt with.

5.4.6 For the set-theoretic and topological constructions of Subsection 5.2.2 and 5.2.3, verify directly the identities of Lemma 5.12 and Corollary 5.13. In each case draw the composite arrow that is dealt with.
Did you notice anything about the calculations?

5.5 Free and co-free constructions

Adjunctions can arise in several different guises, and some of these don't look at all like the official notion. These differences are probably the reason why the full blown notion wasn't recognized earlier.

In this section we look at two such guises (almost disguises). You will recognize the notions from earlier. They are the idea of the universal

free co-free

solution across a functor which now need not be forgetful.

I am going to describe the two notions in parallel. I suggest that you read just one side first, and perhaps the free side is easier. Once you are almost happy with that, go through the other side. You should note the symmetry between the two sides.

At the first reading do not try to connect the ideas with that of an adjunction, even though some of the notation is the same. We will sort that out later.

For both notions we again have a pair

$$\boldsymbol{Src} \qquad\qquad \boldsymbol{Trg}$$

of categories. But now we have just one functor

free	co-free
$\boldsymbol{Src} \xleftarrow[G]{} \boldsymbol{Trg}$	$\boldsymbol{Src} \xrightarrow{F} \boldsymbol{Trg}$

depending on which side we are considering, as indicated. Often in particular examples this is a forgetful functor, but need not be.

The idea is that we want to convert each

$$\boldsymbol{Src}\text{-object } A \qquad\qquad \boldsymbol{Trg}\text{-object } S$$

into an object of the other category. Furthermore, we want to do this in the most economical fashion, whatever that means. Thus we pose two problems the

free problem co-free problem

respectively.

For the problem a solution is an arrow

$$A \xrightarrow{f} GS \qquad\qquad FA \longrightarrow gS$$

comparing a

$$\boldsymbol{Src}\text{-object } A \qquad\qquad \boldsymbol{Trg}\text{-object } S$$

with a transposed

$$\boldsymbol{Trg}\text{-object } S \qquad\qquad \boldsymbol{Src}\text{-object } A$$

respectively. Note the direction of the comparison. It is from ***Src*** to ***Trg*** in both problems. The difference between the two problems is the object that is transported.

We look for a universal solution to the problem which applies to each

$$\boldsymbol{Src}\text{-object } A \qquad\qquad \boldsymbol{Trg}\text{-object } S$$

respectively.

Thus we look for an object assignment

$\boldsymbol{Src}$	$\boldsymbol{Trg}$	$\boldsymbol{Src}$	$\boldsymbol{Trg}$
$A \longmapsto FA$		$GS \longleftarrow\!\shortmid\; S$	

together with a selected arrow

$$A \xrightarrow{\eta_A} (G \circ F)A \qquad\qquad (F \circ G)S \xrightarrow{\epsilon_S} S$$

for each object. Observe that, as yet, part of the solution is just an object assignment, not a functor.

So far these gadgets merely select a solution to the problem. We want a *universal* solution, a solution through which each other solution must factor.

Free

Let

$$\textbf{\textit{Src}} \xleftarrow{\;G\;} \textbf{\textit{Trg}}$$

be a functor and let

$$A \longmapsto FA$$

be an object assignment in the opposite direction. A ***Src***-indexed family

$$A \xrightarrow{\eta_A} (G \circ F)A$$

of ***Src***-arrows form a

G-free

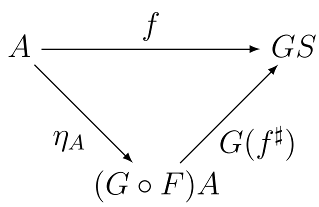
solution if for each ***Src***-arrow

$$A \xrightarrow{f} GS$$

with $S \in \textbf{\textit{Trg}}$, there is a *unique* ***Trg***-arrow

$$FA \xrightarrow{f^\sharp} S$$

such that



commutes.

Cofree

Let

$$\textbf{\textit{Src}} \xrightarrow{\;F\;} \textbf{\textit{Trg}}$$

be a functor and let

$$GS \longleftarrow\!\shortmid S$$

be an object assignment in the opposite direction. A ***Trg***-indexed family

$$(F \circ G)S \xrightarrow{\epsilon_S} S$$

of ***Trg***-arrows form a

F-co-free

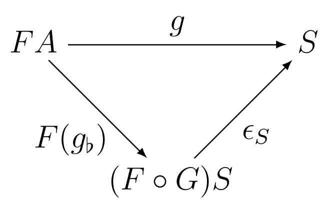
solution if for each ***Trg***-arrow

$$FA \xrightarrow{g} S$$

with $A \in \textbf{\textit{Src}}$, there is a *unique* ***Src***-arrow

$$A \xrightarrow{g_\flat} GS$$

such that



commutes.

Table 5.3: Free and Cofree solutions

5.15 DEFINITION. The notions of a

free co-free

solution are defined in unison as in Table 5.3. You should read each column separately, perhaps starting with the free (left hand) column. ■

Where might we find such universal solutions? If you can't guess the answer, then you should give up now.

5.16 THEOREM. *Let*

$$\boldsymbol{Src} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{Trg} \quad (\dashv)$$

be an adjunction with associated gadgets in standard notation.

The object assignment F, the unit η, and the transposition $(\cdot)^\sharp$ provide the data for a G-free solution.

The object assignment G, the co-unit ϵ, and the transposition $(\cdot)_\flat$ provide the data for a F-co-free solution.

Proof. We look at the free result.

Consider any arrow

$$A \xrightarrow{\ f\ } GS$$

of $\boldsymbol{Src}$.

We first check that

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & GS \\ & {\scriptstyle \eta_A}\searrow \quad \nearrow{\scriptstyle G(f^\sharp)} & \\ & (G\circ F)A & \end{array}$$

does commute (and then consider the required uniqueness).

We use the selection of arrows

$$A \xrightarrow{\ \boldsymbol{id}_A\ } A \qquad \begin{array}{c} FA \xrightarrow{\ f^\sharp\ } S \\ FA \xrightarrow[\ \boldsymbol{id}_{FA}\ ]{} FA \end{array}$$

and then apply ($\flat$) of (Nat). Thus

$$G(f^\sharp)\circ\eta_A = G(f^\sharp)\circ(\boldsymbol{id}_{FA})_\flat\circ\boldsymbol{id}_A = (f^\sharp\circ\boldsymbol{id}_{FA}\circ F(\boldsymbol{id}_A))_\flat = (f^\sharp)_\flat = f$$

as required.

For the uniqueness we consider any arrow

$$FA \xrightarrow{\ g\ } S$$

for which

$$f = G(g)\circ\eta_A$$

and show that, in fact, $g = f^\sharp$.

We use the selection of arrows

$$A \xrightarrow{\ \boldsymbol{id}_A\ } A \qquad \begin{array}{c} FA \xrightarrow{\ g\ } S \\ FA \xrightarrow[\ \boldsymbol{id}_{FA}\ ]{} FA \end{array}$$

and then apply ($\flat$) of (Nat). Thus

$$f = G(g) \circ \eta_A = G(g) \circ (\boldsymbol{id}_{FA})_\flat = G(g) \circ (\boldsymbol{id}_{FA})_\flat \circ \boldsymbol{id}_A = (g \circ \boldsymbol{id}_{FA} \circ F(\boldsymbol{id}_A))_\flat = g_\flat$$

and hence

$$g = (g_\flat)^\sharp = f^\sharp$$

by a use of (Bij). ■

This shows that each adjunction gives us universal solutions of both parities. The more important result is that every universal solution must arise from an adjunction.

5.17 THEOREM. *Let*

$$\boldsymbol{Src} \xleftarrow[G]{} \boldsymbol{Trg}$$

be a functor, and suppose

$$F \quad \eta \quad (\cdot)^\sharp$$

is the data that provides a G-free solution. Then the object assignment F fills out to a functor for which

$$F \dashv G$$

with $(\cdot)^\sharp$ *as the transposition assignment and* η *as the unit.*

Proof. The proof is quite long, but not very deep. There are many small parts each of which is straight forward.

Remember that the G-free solution says

For each arrow f (of a certain kind), there is
a *unique* arrow $f^\sharp$ (to do a certain job).

and it is this uniqueness that is important. We use it some 8 or 9 times.

Our first job is to produce an arrow assignment to create the functor F.

Consider any arrow

$$B \xrightarrow{k} A$$

of $\boldsymbol{Src}$. Let f be the composite

$$B \xrightarrow{k} A \xrightarrow{\eta_A} (G \circ F)A$$

and consider the commuting square

$$\begin{array}{ccc} B & \xrightarrow{k} & A \\ \eta_B \downarrow & (\square) & \downarrow \eta_A \\ (G \circ F)B & \xrightarrow[G(f^\sharp)]{} & (G \circ F)A \end{array}$$

provided by the G-free solution. Here we have

$$FB \xrightarrow{f^\sharp} FA$$

and we take this to be $F(k)$. Thus, we set

$$F(k) = (\eta \circ k)^\sharp$$

for each ***Src***-arrow k, as above.

This gives us an arrow assignment, but we need to show that F is a functor. We must check that F passes across composition, and preserves identity arrows.

Consider a composite

$$C \xrightarrow{\;l\;} B \xrightarrow{\;k\;} A$$

with

$$m = k \circ l$$

in ***Src***. We require

$$F(m) = F(k) \circ F(l)$$

in ***Trg***.

Remember what $F(m)$ is. It is the *unique* arrow

$$FC \longrightarrow FA$$

such that

$$\begin{array}{ccc} C & \xrightarrow{\quad m \quad} & A \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ (G \circ F)C & \xrightarrow[G(F(m))]{} & G(\circ F)A \end{array}$$

commutes. But, by construction of $F(k)$ and $F(l)$, both of the squares

$$\begin{array}{ccccc} C & \xrightarrow{\quad l \quad} & B & \xrightarrow{\quad k \quad} & A \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_A} \\ (G \circ F)C & \xrightarrow[G(F(l))]{} & (G \circ F)B & \xrightarrow[G(F(k))]{} & (G \circ F)A \end{array}$$

commute, and hence

$$\begin{array}{ccc} C & \xrightarrow{\qquad\qquad m \qquad\qquad} & A \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ (G \circ F)C & \xrightarrow[G(F(k) \circ F(l)) = G(F(k)) \circ G(F(l))]{} & G(\circ F)A \end{array}$$

commutes (for we know that G is a functor). The *uniqueness* now gives

$$F(k \circ l) = F(m) = F(k) \circ F(l)$$

which is what we want.

We mustn't forget to check that F preserves identity arrows, that is

$$F(\boldsymbol{id}_A) = \boldsymbol{id}_{FA}$$

for each object A of $\boldsymbol{Src}$. But $F(\boldsymbol{id}_A)$ is the *unique* arrow

$$FA \longrightarrow g \qquad FA$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\boldsymbol{id}_A} & A \\ \eta_A \downarrow & & \downarrow \eta_A \\ (G\circ F)A & \xrightarrow[G(g)]{} & (G\circ F)A \end{array}$$

commutes. Since

$$G(\boldsymbol{id}_{FA}) = \boldsymbol{id}_{(G\circ F)A}$$

we see that

$$g = \boldsymbol{id}_{FA}$$

does make this latest square commute, and hence

$$F(\boldsymbol{id}_A) = g = \boldsymbol{id}_{FA}$$

by yet another appeal to the *uniqueness*.

This gives us a functor F, and the commuting square (□) shops that η is natural. We now check that

$$F \dashv G$$

using the assignment $(\cdot)^\sharp$.

We look first at (Bij).

Fix $A \in \boldsymbol{Src}$ and $s \in \boldsymbol{Trg}$. We certainly have an assignment

$$\begin{array}{ccc} \boldsymbol{Src}[A, GS] & \longrightarrow & \boldsymbol{Trg}[FA, S] \\ f & \longmapsto & f^\sharp \end{array}$$

between the indicated arrow sets.We show that this is a bijection (and then $(\cdot)_\flat$ is its inverse).

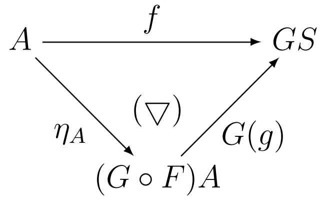
By definition of G-free, for each arrow

$$A \xrightarrow{f} GS$$

the arrow

$$FA \xrightarrow{g = f^\sharp} S$$

must ensure that



commutes, and be the *unique* arrow to do this.

To show that $(\cdot)^\sharp$ is injective, suppose

$$f_1^\sharp = f_2^\sharp$$

for two arrows taken from $\boldsymbol{Src}[A, GS]$. Then

$$f_1 = G(f_1^\sharp) = G(f_1^\sharp) = f_2$$

as required.

To show that $(\cdot)^\sharp$ is surjective, consider any arrow g taken from $\boldsymbol{Trg}[FA, S]$. Let

$$f = G(g) \circ \eta_A$$

so that $(\triangledown)$ commutes. But now, by the *uniqueness* we have

$$f^\sharp = g$$

to give the required result.

Next we verify (Nat). Because we already have (Bij), it suffices to check ($\sharp$).

To this end consider arrows

$$B \xrightarrow{\ k\ } A \qquad\qquad S \xrightarrow{\ l\ } T$$

$$A \xrightarrow[\ f\ ]{} GS$$

and let

$$h = (G(l) \circ f \circ k)^\sharp$$

so that

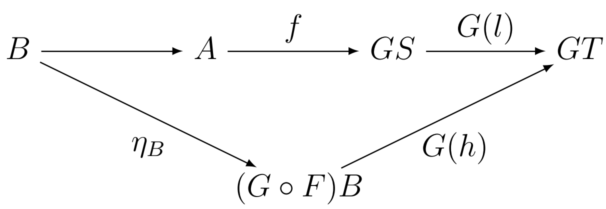
$$h = l \circ f^\sharp \circ F(k)$$

is our problem.

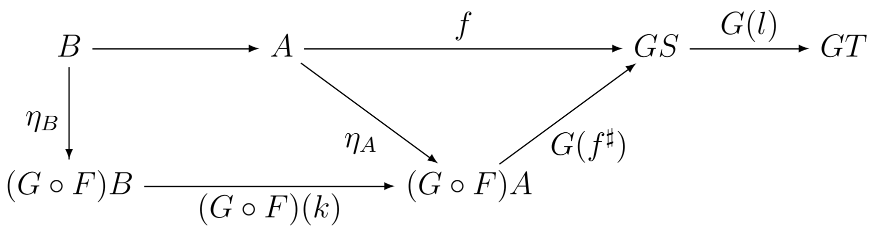
By construction, h is the *unique* arrow

$$FB \xrightarrow{\ h\ } T$$

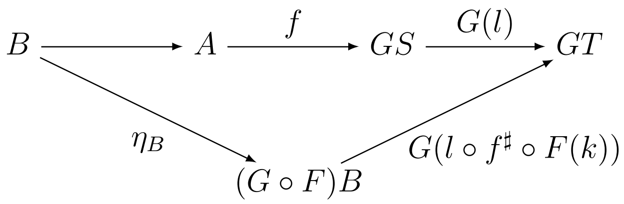
such that



commutes. Now consider the diagram



and observe that the left hand square and central triangle do commute. Since G is a functor, this gives a commuting triangle



and hence once again the *uniqueness* is the answer to our problem.

Don't go away just yet, for we haven't quite finished. Read the statement of the result again. This says that under the given circumstances we have

$$F \dashv G$$

with $(\cdot)^\sharp$ as the transposition assignment *and* with η as the unit. We still have this last clause to verify.

We require

$$\eta_A = (\boldsymbol{id}_{FA})_\flat$$

for each object A of ***Src***.

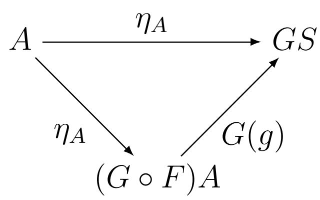
With $S = FA$, consider the arrow

$$A \xrightarrow{\ \eta_A\ } GS$$

of ***Src***. What is the job done by $\eta^\sharp$. It is the *unique* arrow

$$FA \xrightarrow{\ g\ } S$$

such that



commute. But clearly, since

$$G(\boldsymbol{id}_{FA}) = \boldsymbol{id}_{GS}$$

we see that

$$g = \boldsymbol{id}_{FA}$$

does this job, and hence

$$\eta_A^\sharp = \boldsymbol{id}_{FA}$$

by a final appeal to *uniqueness*. Since $(\cdot)_\flat$ is the inverse of $(\cdot)^\sharp$, we have

$$\eta_A = (\eta_A^\sharp)_\flat = (\boldsymbol{id}_{FA})_\flat$$

as required. ■

Now you can have a drink. But mind you, not too many. You still have to come back and do the co-free proof.

Exercises

5.5.1 Complete the proof of Theorem 5.16. That is, deal with the co-free part.

5.5.2 Do the co-free analogue of Theorem 5.17.

5.5.3 In this exercise the forgetful functor has been omitted. You should insert it where necessary.

Consider the algebraic construction Σ of Subsection 5.2.1.

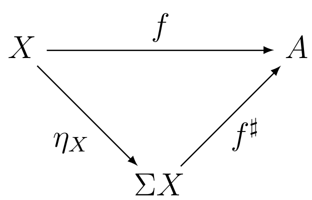
Show that for each function

$$X \xrightarrow{\ f\ } A$$

from a set to an algebra, there is a unique morphism

$$\Sigma X \xrightarrow{\ f^\sharp\ } A$$

such that



commutes in ***Set***. (This shows that ΣX is the free algebra generated by X, via η_X.)

5.5.4 In this exercise the forgetful functor has been omitted. You should insert it where necessary.

Consider the algebraic construction Π of Subsection 5.2.1.

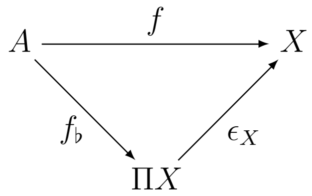
Show that for each function

$$A \xrightarrow{\ f\ } X$$

from an algebra to a set, there is a unique morphism

$$A \xrightarrow{\ f_\flat\ } \Pi X$$

such that



commutes in ***Set***. (This shows that ΠX is the co-free algebra co-generated by X, via ϵ_X. However, this is not a notion that occurs often in most parts of mathematics.)

5.5.5 Verify the free and co-free properties for the constructions of Blocks 5.2.2 and 5.2.3.

5.6 Contravariant adjunctions

So far we have been looking at adjoint pairs of covariant functors. There is a similar notion for contravariant functors. In some ways this is easier because the notions are completely symmetric between the two component functors.

5.18 DEFINITION. Let

$$\boldsymbol{Alg} \xrightarrow{\ \mathfrak{S}\ } \boldsymbol{Spc} \qquad\qquad \boldsymbol{Alg} \xleftarrow{\ \mathfrak{A}\ } \boldsymbol{Spc}$$

be a pair of contravariant functors between a pair of categories These form a contravariant adjunction if for each

$$\boldsymbol{Alg}\text{-object } A \qquad \boldsymbol{Spc}\text{-object } S$$

there is a bijective correspondence

$$\begin{array}{ccc} \boldsymbol{Alg}[A, \mathfrak{A}S] & & \boldsymbol{Spc}[S, \mathfrak{S}A] \\ f & \longmapsto & f^{\sigma} \\ \phi^{\alpha} & \longleftarrow\!\!\dashv & \phi \end{array}$$

between the two arrow sets. Furthermore, this correspondence must be natural for variation of A and S. ■

The notation here is chosen to be suggestive.

The two categories

$$\boldsymbol{Alg} \qquad \boldsymbol{Spc}$$

are often of an

algebraic spatial

nature. Each of the two functors

$$\mathfrak{A} \qquad \mathfrak{S}$$

is named after its target.

As with a covariant adjunction, each identity arrow

$$\mathfrak{A}S \xrightarrow{\ \boldsymbol{id}_{\mathfrak{A}S}\ } \mathfrak{A}S \qquad\qquad \mathfrak{S}A \xrightarrow{\ \boldsymbol{id}_{\mathfrak{S}A}\ } \mathfrak{S}A$$

can be transferred to the other side to produce arrows

$$A \xrightarrow{\ h\ } (\mathfrak{A} \circ \mathfrak{S})A \qquad\qquad S \xrightarrow{\ \eta\ } (\mathfrak{S} \circ \mathfrak{A})S$$

the analogues of the unit. Often one or both of these form a representation of the parent object in terms of a gadget of the other kind. At the heart of many representation result there is a contravariant adjunction. We may look at an example of this later.

The required naturality is worth looking at.

Consider any pair of arrows

$$B \xrightarrow{\ l\ } A \qquad\qquad T \xrightarrow{\ \lambda\ } S$$

from the two categories. These induce square

$$\begin{array}{ccccc}
f & \boldsymbol{Alg}[A, \mathfrak{A}S] & \longleftrightarrow & \boldsymbol{Spc}[S, \mathfrak{S}A] & \phi \\
\Big\downarrow & \Big\downarrow & & \Big\downarrow & \Big\downarrow \\
f^{\bullet} & \boldsymbol{Alg}[B, \mathfrak{A}S] & \longleftrightarrow & \boldsymbol{Spc}[T, \mathfrak{S}B] & \psi^{\bullet}
\end{array}$$

between the arrow sets, which must commute. Here, for arrows f and ϕ as indicated, we have

$$f^{\bullet} = \mathfrak{A}(\lambda) \circ f \circ l \qquad \phi^{\bullet} = \mathfrak{S}(l) \circ \phi \circ \lambda$$

and we require

$$f^{\bullet\alpha} = f^{\sigma\bullet} \qquad \phi^{\alpha\bullet} = \phi^{\bullet\sigma}$$

to hold.

Exercises

5.6.1 This exercise is about a contravariant adjunction. At a first go it might get you kiated, but when looked at in the right way it a fairly straight forward.

We set out two contravariant functors

$$\boldsymbol{Pos} \xrightarrow{\ \Upsilon\ } \boldsymbol{Top} \qquad\qquad \boldsymbol{Pos} \xleftarrow{\ \mathcal{O}\ } \boldsymbol{Top}$$

between the category of posets and monotone functions and the category of topological spaces and continuous maps.

For each space S its topology $\mathcal{O}S$ is a poset under inclusion. The action of $\mathcal{O}$ on maps is via inverse images. You should check that this is a functor.

For each poset A we convert its family ΥA of upper sections into a space. Note that we view each $p \in \Upsilon A$ as a point of the constructed space. We topologize ΥA by setting down a base. For each finite subset a from A we use

$$p \in \langle \mathsf{a} \rangle \iff \mathsf{a} \subseteq p$$

to extract a subset $\langle \mathsf{a} \rangle$ of ΥA. Since

$$\langle \mathsf{a} \rangle \cap \langle \mathsf{b} \rangle = \langle \mathsf{a} \cup \mathsf{b} \rangle$$

for finite subsets a, b of A, we see that these extracted subsets of ΥA do form a base for a topology.

(a) Show that for each monotone function

$$B \xrightarrow{\ f\ } A$$

between poset, the inverse image map

$$\Upsilon a \xrightarrow{\ f^{\leftarrow}\ } \Upsilon B$$

is continuous (relative to the carried topologies), and hence Υ is a contravariant functor, as claimed.

(b) Show that the two functors form a contravariant adjunction.

(c) Can you see a neater way of setting up this adjunction?

Need to sort out several more

At the moment there are 30 exercises in this chapter

6

There may be another chapter
mainly of examples of various adjunction
and this will alter the page numbers
of the following solutions

Part II

Solutions

A
Categories

A.1 Categories defined

1.1.1 Not needed? ■

1.1.2 These examples are dealt with in Section 1.5. ■

A.2 Categories of structured sets

1.2.1 (c) Consider the function

$$f(r) = \alpha^r(a)$$

which sends each $r \in \mathbb{N}$ to the r^{th} iterate of α applied to a. A routine calculation show that this is a ***Pno***-arrow. A simple proof by induction shows that it is the only possible arrow. ■

1.2.2 Consider a pair

$$(A, X) \xrightarrow{\quad f \quad} (B, Y) \xrightarrow{\quad g \quad} (C, Z)$$

of such morphisms. We show that the function composite $g \circ f$ is also a morphism, that is

$$x \in X \Longrightarrow g(f(x)) \in Z$$

for each element x of A. But the morphism property of first f and then g gives

$$x \in X \Longrightarrow y = f(x) \in Y \Longrightarrow g(f(x)) = g(y) \in Z$$

for the required result.

This doesn't quite prove that we have a category, but the other requirements – that arrow composition is associative, and there are identity arrows – are just as easy. ■

1.2.3 The approriate notion of arrow

$$(A, R) \xrightarrow{\ f \ } (B, S)$$

is a function between the carrying sets such that

$$(x, y) \in R \Longrightarrow (f(x), f(y) \in S$$

for all $x, y \in A$. This generalizes the idea used in ***Pre*** and ***Pos***. ■

1.2.4 Consider a pair of continuous maps

$$R \xrightarrow{\quad \psi \quad} S \xrightarrow{\quad \phi \quad} T$$

between topological spaces. A simple calculation gives

$$(\phi \circ \psi)^{\leftarrow} = \psi^{\leftarrow} \circ \phi^{\leftarrow}$$

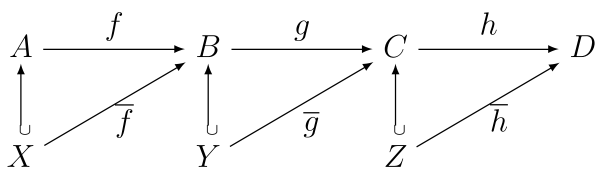
which is the required property. ■

1.2.5 Let $R = \boldsymbol{C}[A, A]$. We have a binary operation $\circ$ on R, namely arrow composition. This operation is associative (by one of the axioms of being a category). We also have a distinguished element $\mathbf{1}_A$ of R, namely the identity arrow on A. This is the required unit to show that R is a monoid.

(Strictly speaking, this do not show that R is a monoid, for we don't know that R is a set. There are some categories for which $\boldsymbol{C}[A, A]$ is so large it is not a set. This is rather weired but it shouldn't frighten us too much (unless you suffer from one of the several forms of a constructive bent). ■

1.2.6 To show that ***Pfn*** is a category we must at least show that composition of arrows is associative.

Consider three composible partial functions

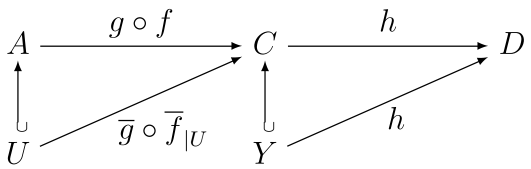


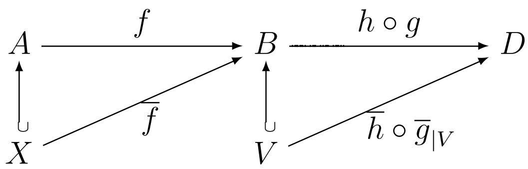
as indicated. We must describe

$$h \circ (g \circ f) \qquad (h \circ g) \circ f$$

and show that they are the same.

We need

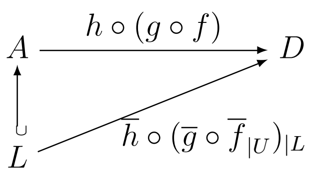


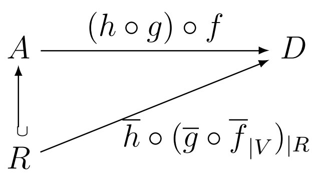


where

$$a \in U \Longleftrightarrow a \in X \text{ and } \overline{f}(a) \in Y \qquad b \in V \Longleftrightarrow b \in Y \text{ and } \overline{g}(b) \in Z$$

for $a \in A$ and $b \in B$. We also need





where

$$a \in L \Longleftrightarrow a \in U \text{ and } (\overline{g} \circ \overline{f}_{|U})(a) \in Z \qquad a \in R \Longleftrightarrow a \in X \text{ and } \overline{f}(a) \in V$$

for $a \in A$.

We show that $L = R$ and that the two function composites are equal.

For $a \in L$ we have $a \in U$, so that $\overline{f}_{|U}(a) = \overline{f}(a)$. Thus, remembering the definition of U we have

$$a \in L \iff a \in X \text{ and } \overline{f}(a) \in Y \text{ and } (\overline{g} \circ \overline{f}_{|U})(a) \in Z$$

for $a \in A$. Remembering the definition of V we have

$$\overline{f}(a) \in V \iff \overline{f}(a) \in Y \text{ and } \overline{g}(\overline{f}(a)) \in Z$$

and hence

$$a \in R \iff a \in X \text{ and } \overline{f}(a) \in Y \text{ and } (\overline{g} \circ \overline{f}_{|U})(a) \in Z$$

for $a \in A$. This shows that $L = R$.

Consider any $a \in L = R$. We have

$$a \in U \qquad \overline{f}(a) \in V$$

so that

$$\big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \big(\overline{g} \circ \overline{f}_{|U}\big)(a) = \overline{g}(\overline{f}(a))$$

to give

$$\overline{h} \circ \big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \overline{h}(\overline{g}(\overline{f}(a)))$$

and

$$\big((\overline{h} \circ \overline{g}_{|V}) \circ \overline{f}_{|R}\big)(a)(\overline{h} \circ \overline{g}_{|V})(\overline{f}(a)) = \overline{h}(\overline{g}(\overline{f}(a)))$$

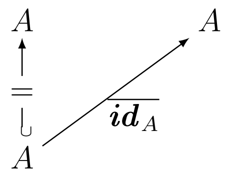
that is

$$\overline{h} \circ \big(\overline{g} \circ \overline{f}_{|U}\big)_{|L}(a) = \overline{h}(\overline{g}(\overline{(}a))) = \big((\overline{h} \circ \overline{g}_{|V}) \circ \overline{f}_{|R}\big)(a)$$

to show that the two function composites are the same.

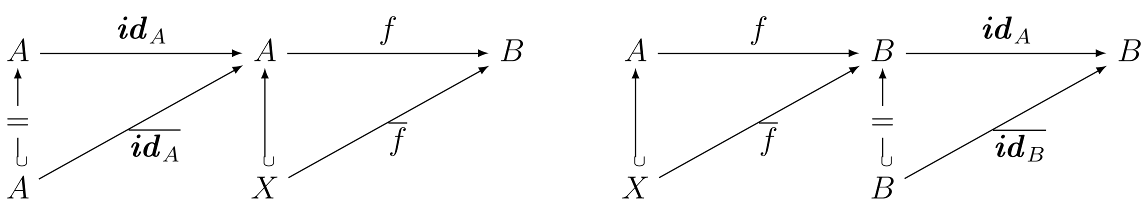
What about identity arrows?

Every total function is also a partial function. Each set A carries an identity function $\boldsymbol{id}_A$ which is



when viewed as a partial function.

Consider the composites



where f is an arbitrary partial function. To compute these composites we first use

$$a \in L \iff a \in A \text{ and } \overline{\boldsymbol{id}_A}(a) \in X \qquad a \in R \iff a \in X \text{ and } \overline{f}(a) \in B$$

(for $a \in A$) to extract $L, R \subseteq A$. Notice that, in fact

$$L = X = R$$

but for different reasons. The arrow composites are

$$\begin{array}{ccc} A & \xrightarrow{f \circ \boldsymbol{id}_A} & B \\ \uparrow & \nearrow_{\overline{f} \circ \overline{\boldsymbol{id}}_{A|X}} & \\ X & & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\boldsymbol{id}_B \circ f} & B \\ \uparrow & \nearrow_{\overline{\boldsymbol{id}}_B \circ \overline{f}_{|X}} & \\ X & & \end{array}$$

and these function composites are

$$\overline{f} \circ \overline{\boldsymbol{id}}_{A|X} = \overline{f} \circ \overline{\boldsymbol{id}}_X = \overline{f} \qquad \overline{\boldsymbol{id}}_X \circ \overline{f}_X = \overline{\boldsymbol{id}}_B \circ \overline{f} = \overline{f}$$

to show

$$f \circ \boldsymbol{id}_A = f = \boldsymbol{id}_B \circ f$$

as required. ■

1.2.7 We set up a pair of translations

$$\boldsymbol{Pfn} \underset{M}{\overset{L}{\rightleftarrows}} \boldsymbol{Set}_\perp$$

between the two categories, and then show that each 2-step trip takes us back to where we started.

We set

$$LA = A \cup \{\perp\} \qquad MS = S - \{\perp\}$$

for each object A of $\boldsymbol{Pfn}$ and each object S of $\boldsymbol{Set}_\perp$. In other words, L adjoints the distinguished point, and M removes the distinguished point. Almost trivially we have

$$(M \circ L)A = A \qquad (L \circ M)S = S$$

for each such A and S.

The way we deal with arrows is more intricate.

For each arrow of $\boldsymbol{Pfn}$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & \nearrow_{\overline{f}} & \\ X & & \end{array}$$

we let

$$\begin{array}{rcll} LA & \xrightarrow{L(f)} & LB & \\ a & \longmapsto & \overline{f}(a) & \text{for } a \in X \\ a & \longmapsto & \perp & \text{for } a \in A - X \\ \perp & \longmapsto & \perp & \end{array}$$

to obtain an arrow of $\boldsymbol{Set}_\perp$. In other words we set

$$L(f)(a) = \begin{cases} \overline{f}(a) & \text{if } a \in X \\ \perp & \text{if } a \notin X \end{cases}$$

$$L(f)(\perp) = \perp$$

for each $a \in A$. By the lower clause, this is an arrow of $\boldsymbol{Set}_\perp$.

Consider any arrow

$$S \xrightarrow{\ \phi\ } T$$

of $\boldsymbol{Set}_\perp$. We extract

$$X \subseteq MS = S - \{\perp\}$$

by

$$X = S - \phi^{\leftarrow}(T)$$

that is

$$s \in X \iff \phi(s) \neq \perp$$

for $s \in S$. In particular, $\perp \notin X$. Thus we have a partial function

$$\begin{array}{ccc} MS & \xrightarrow{M(\phi)} & MT \\ \uparrow & \nearrow_{\phi|_X} & \\ X & & \end{array}$$

controlled by the restriction of ϕ to X.

These constructions give

$$\begin{array}{rcll} S & \xrightarrow{L(M(\phi)f)} & T & \\ s & \longmapsto & \phi(s) & \text{for } s \in X \\ s & \longmapsto & \perp & \text{for } s \in MS - X \\ \perp & \longmapsto & \perp & \end{array}$$

so that

$$L(M(\phi) = \phi$$

which is what we want.

For the other way round each partial function

$$A \xrightarrow{\ f\ } B$$

as above gives a pointed arrow

$$LA \xrightarrow{\ \phi = L(f)\ } LB$$

which we convert back into a partial function. To do that we set

$$W = LA = \phi^{\leftarrow}(\perp)$$

so that

$$a \in W \iff \phi(a) \neq \perp \iff a \in X$$

for each $a \in A$. Thus

$$W = X$$

with

$$\phi|_X = \overline{f}$$

and hence

$$M(L(f)) = f$$

as required.

Do you think that this really shows that $\boldsymbol{Pfn}$ and $\boldsymbol{Set}_\perp$ are 'essentially the same'? ∎

1.2.8 Showing that each of

$$R\text{-}\boldsymbol{Set} \qquad \boldsymbol{Set}\text{-}R$$

is a category is easy.

For both categories an object is a structured set

$$\big(A\ (\alpha_r \,|\, r \in R)\big)$$

a set A furnished with an R-indexed family of 1-placed operations on A. This family must satisfy

$$\alpha_s \circ \alpha_r = \alpha_{rs} \qquad \alpha_s \circ \alpha_r = \alpha_{sr}$$

for $r, s \in R$. Note the

COVARIANCE CONTRAVARIANCE

here.

For both categories an arrow

$$\big(A\ (\alpha_r \,|\, r \in R)\big) \xrightarrow{\ f\ } \big(B\ (\beta_r \,|\, r \in R)\big)$$

is a function f between the carriers such that

$$f \circ \alpha_r = \beta_r \circ f$$

for each $r \in R$. ∎

A.3 An arrow need not be a function

1.3.1 Let $\mathbb{R}^m$ be the vector space of column vectors with m real components. Each $m \times n$ matrix A gives a linear transformation

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ A\ } & \mathbb{R}^m \\ x & \longmapsto & Ax \end{array}$$

and every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ arises in this way from a unique $m \times n$ matrix A. The composite

$$\mathbb{R}^n \xrightarrow{\ B\ } \mathbb{R}^k \xrightarrow{\ A\ } \mathbb{R}^m$$

of two linear transformations is linear, and corresponds to the matrix product AB. ■

1.3.2 The main problem is to define the composition of graph morphisms, and to show that this composition is associative.

Consider a pair of graph morphisms.

$$(N, E) \xrightarrow{\ f\ } (M, F) \xrightarrow{\ g\ } (L.G)$$

(Sorry about the 'backward' notation, but it is always better not to have too many subscripts, dashes, or bits of flapping notation.) In more detail we have

$$\begin{array}{ccccc} E & \xrightarrow{\ f_1\ } & F & \xrightarrow{\ g_1\ } & G \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle \rho_F} & & \downarrow{\scriptstyle \rho_G} \\ N & \xrightarrow[\ f_0\ ]{} & M & \xrightarrow[\ g_0\ ]{} & L \end{array}$$

where all the relevant ***Set***-squares commute, that is

$$f_0 \circ \rho_E = \rho_F \circ f_1 \qquad g_0 \circ \rho_F = \rho_G \circ g_1$$

where ρ is σ or τ throughout.

We observe that by composing both components

$$\begin{array}{ccc} E & \xrightarrow{\ g_1 \circ f_1\ } & G \\ {\scriptstyle \rho_E}\downarrow & & \downarrow{\scriptstyle \rho_G} \\ N & \xrightarrow[\ g_0 \circ f_0\ ]{} & L \end{array}$$

we have

$$g_0 \circ f_0 \circ \rho_E = g_0 \circ \rho_F \circ f_1 = \rho_G \circ g_1 \circ f_1$$

and hence we obtain a graph morphism.

We take this as the composition of two graph morphisms.

A trivial exercise shows that this composition is associative. ■

1.3.3 Consider a composible pair of arrows of this category.

$$(A, R) \xrightarrow{\ (f, \phi)\ } (B, S) \xrightarrow{\ (g, \psi)\ } (C, T)$$

In other words we have two pair of composible functions.

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \qquad\qquad R \xleftarrow{\ \phi\ } S \xrightarrow{\ \psi\ } T$$

The pair

$$(A, R) \xrightarrow{\ (g \circ f, \phi \circ \psi)\ } (C, T)$$

is the composite

$$(A,R) \xrightarrow{(f,\phi)\circ(g,\psi)} (C,T)$$

in the category. Verifying the axioms is more of less trivial.

This is essentially the same as Example 1.9. The category $\boldsymbol{S}$ has been replaced by its opposite $\boldsymbol{S}^{\mathrm{op}}$. See Example 1.16. ■

1.3.4 Consider a pair of arrows in this category

$$\begin{array}{ccccc} A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C \\ R & \xleftarrow[\phi]{} & S & \xleftarrow[\psi]{} & T \end{array}$$

where we have separated the two components. We must show that the pair

$$(A,R) \xrightarrow{(g\circ f,\phi\circ\psi)} (C,T)$$

is an arrow, that is

$$\big(g\circ f\big)\big(a(\phi\circ\psi)(t)\big) = \big(g\circ f\big)(a)t$$

for each $a \in A$ and $t \in T$.

Consider such a pair a, t and let

$$b = f(a) \qquad s = \psi(t)$$

to produce $b \in B$ and $s \in S$. Then

$$\begin{aligned} \big(g\circ f\big)\big(a(\phi\circ\psi)(t)\big) &= g\big(f(a\phi(s)\big) \\ &= g\big(f(a)s\big) \\ &= g\big(b\psi(t)\big) \\ &= g(b)t \qquad\qquad = \big(g\circ f\big)(a)t \end{aligned}$$

for the required result. ■

1.3.5 To show that the composition is associative consider three relations

$$A \xrightarrow{\;F\;} B \xrightarrow{\;G\;} C \xrightarrow{\;H\;} D$$

between sets. For $a \in A$ and $d \in D$ we have

$$\begin{aligned} d\big(H\circ(G\circ F)\big)a &\Longleftrightarrow (\exists c\in C)[dHc(G\circ F)a] \Longleftrightarrow (\exists c\in C, b\in B)[dHcGbFa] \\ d\big((H\circ G)\circ F\big)a &\Longleftrightarrow (\exists b\in B)[d(H\circ G)bFa] \Longleftrightarrow (\exists b\in B, c\in C)[dHcGbFa] \end{aligned}$$

so that a flip of quantifiers gives the required result.

Consider a pair of functions

$$A \xrightarrow[\Gamma(f)]{\;f\;} B \xrightarrow[\Gamma(g)]{\;g\;} C$$

with associated graphs. For $a \in A$ and $c \in C$ we have

$$\begin{aligned} c\big(\Gamma(g)\circ\Gamma(f)\big)a &\iff (\exists b \in B)[c\Gamma(g)b\Gamma(f)a] \\ &\iff (\exists b \in B)[c = g(b) \;\&\; b = f(a)] \\ &\iff c = g(f(a)) \qquad\qquad\qquad\quad \iff c\Gamma(g\circ f)a \end{aligned}$$

for the required result. ■

1.3.6 (a) Suppose first that $f \dashv g$, that is

$$f(a) \le b \iff a \le g(b)$$

for $a \in S$ and $b \in T$. Since

$$f(a) \le f(a) \qquad g(b) \le b(b)$$

a use of this equivalence one way or the other gives

$$a \le (g \circ f)(a) \qquad (f \circ g)(b) \le b$$

for the first required result.

Conversely, suppose the two comparisons hold (for all $a \in S$ and $b \in T$), and suppose

$$f(a) \le bn \qquad a \le g(b)$$

(for some $a \in S$ and $b \in T$). Since both f and g are monotone we have

$$a \le (g \circ f)(a) \le g(b) \qquad f(a) \le (f \circ g)(b) \le b$$

to verify the equivalence.

(b) For each $a \in S$ we have

$$a \le (g \circ f)(a)$$

and hence

$$f(a) \le (f \circ g \circ f)(a)$$

since f is monotone. For each $b \in T$ we have

$$(f \circ g)(b) \le b$$

and hence

$$(f \circ g \circ f)(a) \le f(a)$$

as a particular case. This gives

$$f \circ g \circ f = f$$

and the other equality follows in a similar fashion. ■

1.3.7 Let's use the 'decorated' notation.

We are given a couple of projection pairs

$$A \xrightarrow[f_* \circ f^* = \boldsymbol{id}_A]{f^* \dashv f_*} B \xrightarrow[g_* \circ g^* = \boldsymbol{id}_B]{g^* \dashv g_*} C$$

a couple of adjunctions with the indicated equalities. These certainly gives a composite adjunction

$$A \xrightarrow{h^* \dashv h_*} C$$

where

$$h^* = g^* \circ f^* \qquad h_* = f_* \circ g_*$$

are the two components. But now

$$h_* \circ h^* = f_* \circ g_* \circ g^* \circ f^* = f_* \circ \boldsymbol{id}_B \circ f^* = f_* \circ f^* = \boldsymbol{id}_A$$

to show that $h^* \dashv h_*$ is a projection pair. ■

1.3.8 For each $x \in \mathbb{R}$ let

$$\lambda(x) = \lfloor x \rfloor \qquad \rho(x) = \lceil x \rceil$$

the integer

floor ceiling

of x. ■

1.3.9 For a given monotone map

$$T \xrightarrow{\ \phi\ } S$$

we require monotone maps

$$\mathcal{L}T \underset{f_\flat}{\overset{f^\sharp}{\rightrightarrows}} \mathcal{L}S \qquad \mathcal{L}T \xleftarrow{\ \phi^\leftarrow\ } \mathcal{L}S$$

with

$$f^\sharp(Y) \subseteq X \iff Y \subseteq \phi^\leftarrow(X)$$
$$\phi^\leftarrow(X) \subseteq Y \iff X \subseteq f_\flat(Y)$$

for each $X \in \mathcal{L}S$ and $Y \in LT$.

For $Y \in \mathcal{L}T$ let

$$f^\sharp(Y) = \downarrow\phi[Y]$$

the lower section of S generated by the direct image of Y across ϕ. For $X \in \mathcal{L}S$ and $Y \in \mathcal{L}T$ we have

$$\begin{aligned} f^\sharp(Y) \subseteq X &\iff \phi[Y] \subseteq X \\ &\iff (\forall t \in T)[t \in Y \implies \phi(t) \in X] \\ &\iff (\forall t \in T)[t \in Y \implies t \in \phi^\leftarrow(X)] \iff Y \subseteq \phi^\leftarrow(X) \end{aligned}$$

as required.

For $Y \in \mathcal{L}T$ let

$$f_\flat(Y) = \big(\uparrow\phi[Y']\big)'$$

the complement of the upper section of S generated by the direct image of the complement of Y across ϕ. For $X \in \mathcal{L}S$ and $Y \in \mathcal{L}T$ we have

$$\begin{aligned} X \subseteq f_\flat(Y) &\iff \uparrow\phi[Y'] \subseteq X' \\ &\iff \phi[Y'] \subseteq X' \\ &\iff (\forall t \in T)[t \in Y' \implies \phi(t) \in X'] \\ &\iff (\forall t \in T)[\phi(t) \in X \implies t \in Y] \\ &\iff (\forall t \in T)[t \in \phi^{\leftarrow}(X) \implies t \in Y] \quad \iff \phi^{\leftarrow}(X) \subseteq Y \end{aligned}$$

as required. ■

1.3.10 You will find it instructive to go through the following solution of a more general exercise.

Let

$$\nabla = (N, E)$$

be a graph (in the sense of Exercise 1.3.2). We let

$$i, j, k, \ldots \text{range over } N \qquad e, f, g, \ldots \text{range over } E$$

and think of these as stocks of indexes. As with any graph there is some source and target data, namely

$$\sigma(e) \xrightarrow{\ e\ } \tau(e)$$

for each $e \in E$.

We view ∇ as a template. For an arbitrary category $\boldsymbol{C}$ we produce a new category $\boldsymbol{C}^\nabla$, the category of ∇-diagrams on $\boldsymbol{C}$.

An object of $\boldsymbol{C}^\nabla$ is a pair

$$\mathsf{A} = \big(A(i) \mid i \in N\big) \qquad \mathcal{A} = \big(A(e) \mid e \in E\big)$$

an

$$N\text{-indexed family of objects} \qquad E\text{-indexed family of arrows}$$

of $\boldsymbol{C}$, respectively. These families must satisfy

$$A(\sigma(e)) \xrightarrow{\ A(e)\ } A(\tau(e))$$

for each $e \in E$.

An arrow of $\boldsymbol{C}^\nabla$

$$(\mathsf{A}, \mathcal{A}) \xrightarrow{\ \phi\ } (\mathsf{B}, \mathcal{B})$$

is an N-indexed family of arrows of $\boldsymbol{C}$

$$A(i) \xrightarrow{\ \phi_i\ } B(i)$$

such that the $\boldsymbol{C}$-square

$$\begin{array}{ccc} A(\sigma(e)) & \xrightarrow{\phi_{\sigma(e)}} & B(\sigma(e)) \\ {\scriptstyle A(e)}\big\downarrow & & \big\downarrow{\scriptstyle B(e)} \\ A(\tau(e)) & \xrightarrow[\phi_{\tau(e)}]{} & B(\tau(e)) \end{array}$$

commutes for each $e \in E$.

Composition of arrows is done componentwise.

Given arrows

$$(\mathsf{A}, \mathcal{A}) \xrightarrow{\phi} (\mathsf{B}, \mathcal{B}) \xrightarrow{\psi} (\mathsf{C}, \mathcal{C})$$

in $\boldsymbol{C}^{\nabla}$, we have components

$$A(i) \xrightarrow{\phi_i} B(i) \xrightarrow{\psi_i} C(i)$$

for each $i \in N$. We take this composite as the i^{th} component of $\psi \circ \phi$.

$$A(i) \xrightarrow{(\psi \circ \phi)_i = \psi_i \circ \phi_i} C(i)$$

Of course, we need to show that this does produce an arrow of $\boldsymbol{C}^{\nabla}$, in other words that

$$\begin{array}{ccc} A(\sigma(e)) & \xrightarrow{(\psi \circ \phi)_{\sigma(e)}} & C(\sigma(e)) \\ {\scriptstyle A(e)}\big\downarrow & & \big\downarrow{\scriptstyle C(e)} \\ A(\tau(e)) & \xrightarrow[(\psi \circ \phi)_{\tau(e)}]{} & C(\tau(e)) \end{array}$$

commutes for each $e \in E$. This square can be decomposed as

$$\begin{array}{ccccc} A(\sigma(e)) & \xrightarrow{\phi_{\sigma(e)}} & B(\sigma(e)) & \xrightarrow{\psi_{\sigma(e)}} & C(\sigma(e)) \\ {\scriptstyle A(e)}\big\downarrow & & \big\downarrow{\scriptstyle B(e)} & & \big\downarrow{\scriptstyle C(e)} \\ A(\tau(e)) & \xrightarrow[\phi_{\tau(e)}]{} & B(\tau(e)) & \xrightarrow[\psi_{\tau(e)}]{} & C(\tau(e)) \end{array}$$

and hence the required result is immediate.

The same kind of argument shows that this composition is associative. ■

1.3.11 We have

$(\boldsymbol{Set} \downarrow 1)$ is essentially $\boldsymbol{Set}$

$(1 \downarrow \boldsymbol{Set})$ is essentially $\boldsymbol{Set}_{\perp}$

$(\boldsymbol{Set} \downarrow 2)$ is essentially Sets with a distinguished subset but with a restricted family of arrows

$(2 \downarrow \boldsymbol{Set})$ is essentially Sets with two distinguished points where the third uses the correspondence between subsets and characteristic functions. Let's look at this third example.

Let

$$\mathsf{2} = \{0, 1\}$$

where here it is useful to think of 1 as 'true' and 0 as 'false'. An object of $\boldsymbol{Set} \downarrow \mathsf{2}$

$$A \xrightarrow{\ \alpha\ } \mathsf{2}$$

is a set A with a carried characteristic function α. This function α gives a subset $X \subseteq A$ where

$$a \in X \iff \alpha(a) = 1$$

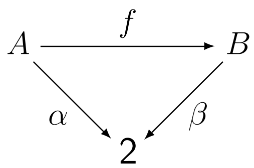
for each $a \in A$. Furthermore this set X determines α since

$$\alpha(a) = \begin{cases} 1 \text{ if } a \in X \\ 0 \text{ if } a \notin X \end{cases}$$

for each $a \in A$. There is a bijective correspondence between characteristic functions carried by A and subsets of A. (If you have never seen this trick before, then take note. This and various generalizations are used throughout mathematics.)

This shows that the objects of $\boldsymbol{Set} \downarrow \mathsf{2}$ are precisely the sets with distinguished subset.

What is an arrow of $\boldsymbol{Set} \downarrow \mathsf{2}$?



$$(A, X) \xrightarrow{\ f\ } (B, Y)$$

On the left we have the official version. It is a function f for which

$$\alpha = \beta \circ f$$

holds. On the right we have the unofficial version. It is a function f such that

$$a \in X \iff \alpha(a) = 1 \iff \beta(f(a)) = 1 \iff f(a) \in Y$$

that is

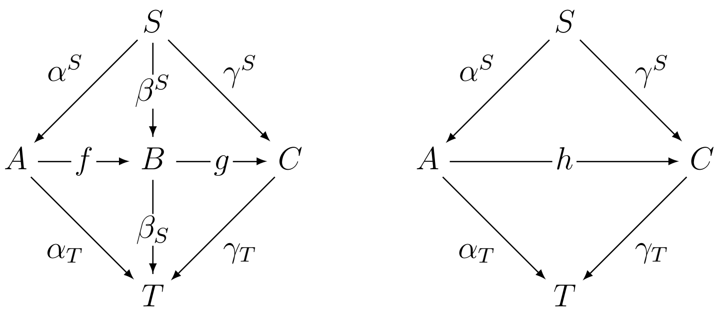
$$a \in X \iff f(a) \in Y$$

for each $a \in A$. An arrow of the category of sets is a function f such that

$$a \in X \implies f(a) \in Y$$

for each $a \in A$. Thus the two categories have the same objects but $\boldsymbol{Set} \downarrow \mathsf{2}$ has a more restrictive kind of arrow. ■

1.3.12 (a) Consider a composible pair of arrows of $(S \downarrow \boldsymbol{C} \downarrow T)$ as on the left. This is a commuting diagram.



Let

$$h = g \circ f$$

be the function composite of f and g. To show that is an arrow of $(S \downarrow \boldsymbol{C} \downarrow T)$ we must check that the diagram on the right commutes. This is a simple exercise in diagram chasing (which we look at in more detail in Section 2.1).

(b) An object I of a category is initial if for each object A there is a unique arrow $I \longrightarrow A$. Not every category has such an object, but many do. (The category ***Set*** has an initial object, and you might worry a bit about what it is.) If I is an initial object of C then

$$(I \downarrow \boldsymbol{C} \downarrow T) \qquad (\boldsymbol{C} \downarrow T)$$

are essentially the same category.

An object F of a category is final if for each object A there is a unique arrow $A \longrightarrow F$. Not every category has such an object, but many do. (The category ***Set*** has a final object, and it is pretty obvious what it is.) If F is a final object of C then

$$(S \downarrow \boldsymbol{C} \downarrow F) \qquad (S \downarrow \boldsymbol{C})$$

are essentially the same category.

We look at initial and final objects in Section 2.4. ■

A.4 More complicated categories

1.4.1 Let's look at the composition of arrows.

Consider a pair of arrows

$$\mathsf{A} \xrightarrow{\ f\ } \mathsf{B} \xrightarrow{\ g\ } \mathsf{C}$$

of $\widehat{S}$. How might we produce the composite arrow

$$\mathsf{A} \xrightarrow{\ h = g \circ f\ } \mathsf{C}$$

in $\widehat{S}$?

For each index $i \in S$ we have a pair of functions

$$A(i) \xrightarrow{\ f_i\ } B(i) \xrightarrow{\ g_i\ } C(i)$$

between sets, and we can certainly form the function composite

$$A(i) \xrightarrow{\ h_i = g_i \circ f_i\ } C(i)$$

at the index. We show this gives an arrow in $\widehat{S}$.

Consider any pair $j \leq i$ of comparable indexes. We have a pair of commuting squares, as on the left

$$\begin{array}{ccccc} A(i) & \xrightarrow{f_i} & B(i) & \xrightarrow{g_i} & C(i) \\ {\scriptstyle A(j,i)}\downarrow & & \downarrow{\scriptstyle B(j,i)} & & \downarrow{\scriptstyle C(j,i)} \\ A(j) & \xrightarrow[f_j]{} & B(j) & \xrightarrow[g_j]{} & C(j) \end{array} \qquad \begin{array}{ccc} A(i) & \xrightarrow{h_i} & C(i) \\ {\scriptstyle A(j,i)}\downarrow & & \downarrow{\scriptstyle C(j,i)} \\ A(j) & \xrightarrow[h_j]{} & C(j) \end{array}$$

and we require a commuting square, as on the right. This is a simple exercise in diagram chasing.

There are several more little bits to be done, but all are just as easy.

This is a nice example of how convenient arrow-theoretic mathods can be. If we always had to expose the inner details of these objects and arrows then some calculations could become a mess. By hiding these innards be begin to get a clearer picture of what is going on. Of course, there are times when we have to get inside a presheaf, but that doesn't mean we should do it all the time. ■

1.4.2 Observe that a chain complex is a special kind of presheaf with $\mathbb{Z}$ as the indexing poset. The conecting arrows are module morphisms with the extra requirement is that if $m + 2 \leq n$ then the conection morphism

$$A_n \longrightarrow A_m$$

is zero.

Category theory can bring out similarities that arec not so obvious when we have to carry around lots of details. ■

A.5 Two simple categories and a bonus

1.5.1 (a) The product as categories is the cartesian product as algebras.

(b) The product as categories is the cartesian product as presets. ■

1.5.2 The category $(S \downarrow s)$ is the principal upper section of S above s.

The category $(s \downarrow S)$ is the principal lower section of S below s.

The category $(s \downarrow S \downarrow t)$ is the convex section of S between s and t. This could is empty if $s \ngeq t$. ■

1.5.3 The category S^{op} is the poset S turned upsidedown.

The category R^{op} is the same set with a new operation $\overline{\star}$ given by

$$r \mathbin{\overline{\star}} s = s \star r$$

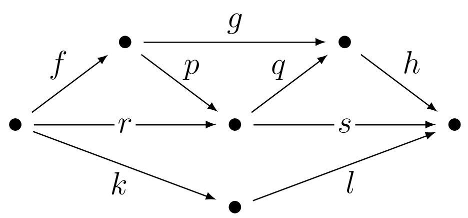
for $r, s \in R$. Here $\star$ is the old operation. ■

1.5.4 It is the category $\boldsymbol{A} \times \boldsymbol{S}^{\mathrm{op}}$. ■

B
Basic gadgetry

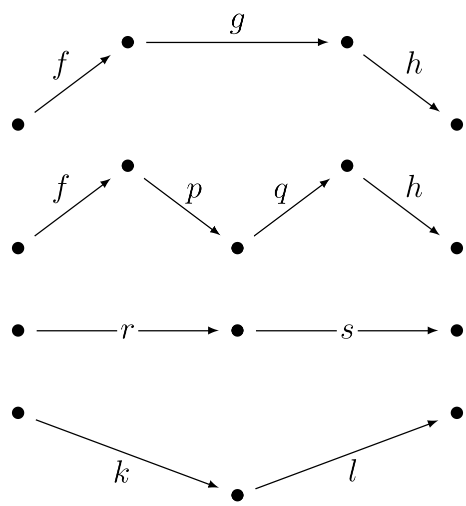
B.1 Diagram chasing

2.1.1 For the equational reasoning we need to label more arrows.



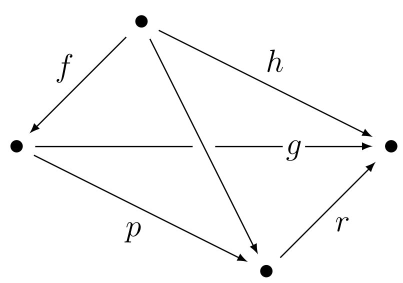
The calculation on the left gives the equational version

$$h \circ g \circ f = h \circ q \circ p \circ f = s \circ r = l \circ k$$



and the diagram chase on the right give the same result. ■

2.1.2 We label the arrows as follows



with q for the unlabelled arrow. Then

$$g \circ f = r \circ p \circ f = r \circ q = h$$

gives the required result. ∎

2.1.3 The sequence of corners

$$1234512345$$

is a trip twice round the pentagram. Because the various triangles we see that

$$\underline{123}45123451$$
$$1\underline{345}123451$$
$$13\underline{512}3451$$
$$135\underline{234}51$$
$$1352\underline{451}$$
$$135241$$

gives the result. At each step the underline indicates the triangle that is collapsed. ∎

B.2 Monics and epics

2.2.1 (a) For instance, consider a section s which is also epic. Since s is a section we have a composite

$$B \xrightarrow{\ s\ } A \xrightarrow{\ r\ } B \qquad r \circ s = \boldsymbol{id}_B$$

which is an identity. This also shows that the parallel pair of arrows

$$B \xrightarrow{\ s\ } A \begin{array}{c} \xrightarrow{\ r\ } B \xrightarrow{\ s\ } \\ \xrightarrow[\boldsymbol{id}_A]{\hspace{6em}} \end{array} A$$

agree, and hence

$$s \circ r = \boldsymbol{id}_A$$

since s is epic. This show that r is the inverse of s. ∎

2.2.2 (a) In a preset there is no more than one arrow

$$i \longrightarrow j$$

between a given pair of elements. Thus for any parallel pair

$$i \rightrightarrows j$$

the two arrows are equal. This shows that every arrow is monic and epic.

(b) A preset is balanced precisely when it is a poset. ∎

2.2.3 An element is monic or epic if it is cancellable on the appropriate side.

An element is a retraction or a section if it has a one sided inverse on the appropriate side.

An element is an isomorphism if it has a two sided inverse

A monoid is balanced precisely when it is embeddable in a group. ■

2.2.4 Consider a pair of arrows

$$X \underset{l}{\overset{k}{\rightrightarrows}} A$$

to A.

Assuming both m, n are monic we have

$$n \circ m \circ k = n \circ m \circ l \Longrightarrow m \circ k = m \circ l \Longrightarrow k = l$$

which more or less shows that $n \circ m$ is monic.

Assuming $n \circ m$ is monic we have

$$m \circ k = m \circ l \Longrightarrow n \circ m \circ k = n \circ m \circ l \Longrightarrow k = l$$

which more or less shows that m is monic. ■

2.2.5 It will help if we get a bit of notation sorted out. Let

$$(A, \cdot, \iota)$$

be an arbitrary monoid written multiplicatively. For once we will display the operation symbol.

Consider any monoid morphism

$$(\mathbb{Z}, +, 0) \xrightarrow{f} (A, \cdot, \iota)$$

from the additively written monoid $\mathbb{Z}$. Thus

$$f(0) = \iota \qquad f(m+n) = f(m) \cdot f(n)$$

for all $m, n \in \mathbb{Z}$.

Consider any situation

$$\mathbb{N} \xhookrightarrow{e} \mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} A$$

where

$$f \circ e = g \circ e$$

that is

$$f(m) = g(m)$$

for all $m \in \mathbb{N}$. We require $f = g$, that is

$$f(-m) = g(-m)$$

for all $m \in \mathbb{N}$. But, taking it slowly, for $m \in \mathbb{N}$ we have

$$\begin{aligned} g(-m) &= g(-m) \cdot \iota \\ &= g(-m) \cdot f(0) \\ &= g(-m) \cdot f(m + (-m)) \\ &= g(-m) \cdot f(m) \cdot f(-m)) \\ &= g(-m) \cdot g(m) \cdot f(-m)) \\ &= g(-m + m) \cdot f(-m)) \\ &= g(0) \cdot f(-m)) \\ &= \iota \cdot f(-m)) \qquad\qquad = f(-m) \end{aligned}$$

as required. Of course, the central equality is the crucial step. ■

2.2.6 The format for this solution is like that of Solution 2.2.5, but now we have a few more algebraic identities we can use, and we don't need to mix the notation.

Consider any situation in ***Rng***

$$\mathbb{Z} \xhookrightarrow{\quad e \quad} \mathbb{Q} \underset{g}{\overset{f}{\rightrightarrows}} A$$

where

$$f \circ e = g \circ e$$

that is

$$f(m) = g(m)$$

for all $m \in \mathbb{Z}$. We require $f = g$, that is

$$f\left(\frac{m}{n}\right) = g\left(\frac{m}{n}\right)$$

for all $m, n \in \mathbb{Z}$ with $n \neq 0$.

Consider any non-zero $n \in \mathbb{Z}$. We have

$$g\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \cdot f(n) \cdot f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right) \cdot g(n) \cdot f\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right)$$

and hence

$$g\left(\frac{m}{n}\right) = g(m) \cdot g\left(\frac{1}{n}\right) = f(m) \cdot f\left(\frac{1}{n}\right) = f\left(\frac{m}{n}\right)$$

as required. ■

2.2.7 Suppose the category $\boldsymbol{C}$ of structured sets has a selector $(S, \star)$. Consider any monic

$$A \xrightarrow{\quad m \quad} B$$

in $\boldsymbol{C}$. Consider $a_1, a_2 \in A$ with $m(a_1) = m(a_2)$. We show $a_1 = a_2$, and hence show that m is injective.

Consider the parallel pair

$$S \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A$$

with $\alpha_1(\star) = a_1$ and $\alpha_2(\star) = a_2$. Each of the two composites

$$S \underset{m \circ \alpha_2}{\overset{m \circ \alpha_1}{\rightrightarrows}} B$$

is uniquely determined by its value at $\star$. But

$$(m \circ \alpha_1)(\star) = m(a_1) = m(a_2) = (m \circ \alpha_2)(\star)$$

so that

$$m \circ \alpha_1 = m \circ \alpha_2$$

to give

$$\alpha_1 = \alpha_2$$

(since m is monic), and hence

$$a_1 = \alpha_1(\star) = \alpha_2(\star) = a_2$$

as required.

(b) It suffices to exhibit a selector for each of the categories.

For ***Set***, ***Pos***, ***Top*** the 1-element structure will do.

For ***Mon*** we use the monoid $(\mathbb{N}, +, 0)$ with $\star = 0$.

For ***Grp*** we use the group $(\mathbb{Z}, +, 0)$ with $\star = 1$.

For ***Rng*** we use the ring of polynomials $\mathbb{Z}[X]$ with $\star = X$.

For ***Set***-R we use R itself with $\star = 1$. ■

2.2.8 An isomorphism in ***Top*** is usually called a homeomorphism.

See Exercise 2.2.7.

Consider any topological space S and let S^d be the set S viewed as a discrete space. The identity function on the set S is a bijective continuous map

$$S^d \longrightarrow S$$

but is not a homeomorphism (unless S is discrete).

(b) Let's go straight to the more general result.

Consider any situation in $\boldsymbol{Top}_2$

$$T \xrightarrow{\ \epsilon\ } S \underset{\psi}{\overset{\phi}{\rightrightarrows}} R$$

where

$$\phi \circ \epsilon = \psi \circ \epsilon$$

and where $\epsilon[T]$ is dense in S. We require $\phi = \psi$.

By way of contradiction suppose $\phi \neq \psi$. We have $\phi(s) \neq \psi(s)$ for some $s \in S$. Since R is T_2, this gives

$$\phi(s) \in U \qquad \psi(s) \in V \qquad U \cap V = \emptyset$$

for some pair U, V of open sets of S. We have

$$s \in \phi^{\leftarrow}(U) \cap \psi^{\leftarrow}(V)$$

and both these sets are open in S. This intersection is non-empty, and so must meet $\epsilon[T]$ (since this set is dense in S). We thus have some $t \in T$ with

$$\epsilon(t) \in \phi^{\leftarrow}(U) \cap \psi^{\leftarrow}(V)$$

that is

$$(\phi \circ \epsilon)(t) \in U \qquad (\psi \circ \epsilon)(t) \in V$$

which is the contradiction since $\phi \circ \epsilon = \psi \circ \epsilon$ and $U \cap V = \emptyset$. ■

2.2.9 (e) Since e is epic an equality

$$j \circ f \circ e = m \circ l \circ e$$

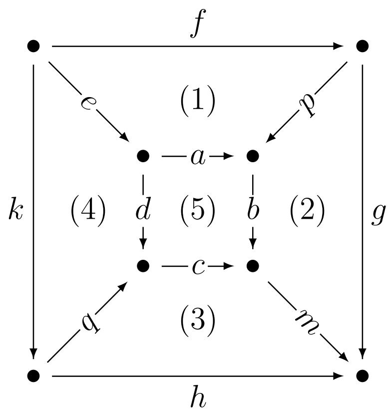
will suffice. But we easily produce a sequence of equalities

$$j \circ f \circ e = j \circ g \circ b = \cdots = m \circ l \circ e$$

by passing across each of the faces in turn.

(m) A dual version of (e). ■

2.2.10 We label the arrows and variois cells



as shown. We are given that cells (1, 2, 3, 4) commute.

(a) Here we are also given that (5) commutes. A use of (2, 1, 5, 4, 3) in that order gives

$$g \circ f = m \circ b \circ p \circ f = m \circ c \circ d \circ f = m \circ c \circ q \circ k = h \circ k$$

to show that the outer cell commutes. You should also look at this in the form of a diagram chase

(b) We have

$$m \circ b \circ a \circ e = m \circ b \circ p \circ f = g \circ f \qquad m \circ c \circ d \circ e = m \circ c \circ q \circ k = h \circ k$$

using (1, 2) on the left hand side and (4, 3) on the right hand side. Assuming the outer square commutes this gives

$$m \circ b \circ a \circ e = g \circ f = h \circ k = m \circ c \circ d \circ e$$

and hence

$$b \circ a = c \circ d$$

by the assumed cancellative properties of m and e. ■

B.3 Simple limits and colimits

2.3.1 For an arbitrary subset X of a poset the

limit colimit

is denoted

$$\bigwedge X \qquad \bigvee X$$

and called

the greatest lower bound or infimum the least upper bound or supremum

of X, provided these exists, of course.

When X is empty we have

$$\bigwedge \emptyset = \top \text{ (top)} \qquad \bigvee \emptyset = \bot \text{ (bottom)}$$

of the poset.

When X is a singleton we have

$$\bigwedge X = \{s\} = \bigvee X$$

where s is the unique member of X.

When $X = \{a, b\}$ we have

$$\bigwedge X = a \wedge b \qquad \bigvee X = a \vee b$$

the

meet join

of the pair.

When S is a preset each of these notions may not determine a unique element, only a family of equivalent elements. ■

B.4 Initial and final objects

2.4.1 Consider any initial object I in a category. Since I is initial there is a unique endo-arrow

$$I \longrightarrow I$$

on I. We already know one example of such an arrow, namely the identity arrow $\boldsymbol{id}_I$. Thus this is the only endo-arrow on I.

Consider any pair I, J of initial objects. There are unique arrows

$$I \xrightarrow{\;f\;} J \qquad\qquad J \xrightarrow{\;g\;} I$$

since

$$I \text{ is initial} \qquad\qquad J \text{ is initial}$$

respectively. The composite

$$g \circ f \qquad\qquad f \circ g$$

is an endo-arrow on

$$I \qquad\qquad J$$

respectively. By the previous observation we have

$$g \circ f = \boldsymbol{id}_I \qquad\qquad f \circ g = \boldsymbol{id}_J$$

and hence f, g are an inverse pair of isomorphisms.

If F, G are two final objects, then there is a unique arrow $I \longrightarrow J$, and this is an isomorphism.

This is proved in exactly the same way, we simply think of the arrows as pinting in the other direction. Equivalently, we apply the 'initial' result to the opposite category. ■

2.4.2 By the uniqueness of mediators the only endo-arrow of I is $\boldsymbol{id}_I$.

Consider any arrow

$$A \xrightarrow[r]{} I$$

and let

$$A \xleftarrow{\;s\;} I$$

be the unique arrow given by the initial property of I. Then $r \circ s$ is an endo-arrow of I, and hence

$$r \circ s = \boldsymbol{id}_I$$

be the above remark.

By duality, each arrow

$$F \xrightarrow{\;s\;} F$$

from a final object is a section.

Consider any arrow

$$F \xrightarrow{\ f\ } I$$

passing from a final object to an initial object. From above we know there are arrows

$$F \underset{s}{\overset{r}{\leftleftarrows}} I$$

with

$$f \circ s = \boldsymbol{id}_I \qquad r \circ f = \boldsymbol{id}_F$$

and hence the usual argument gives $s = r$, to show that f is an isomorphism. ■

2.4.3 Exercise 1.2.3 shows that $(\mathbb{N}, \mathrm{succ}, 0)$ is the initial object of ***Pno***. This fact is equivalent to the Peano axioms (first discovered by Dedekind).

The trivial object, with just one element, is final. ■

2.4.4 The trivial group is both initial and final in ***Grp***.

The ring $\mathbb{Z}$ of integers is initial in ***Rng***. The trivial ring, with $1 = 0$, is final.

The ring $\mathbb{Z}$ of integers is initial in ***Idm***. There is no final object (assuming that $1 \neq 0$ must hold in an integral domain).

There is neither an initial object not a final object in ***Fld***. However, if we fix the characteristic then there is an initial object. ■

2.4.5 In ***Set*** the final object $\mathsf{1}$ is the singleton set. It doesn't matter what its unique element is, so let

$$\mathsf{1} = \{\star\}$$

here.

Each function

$$\mathsf{1} \xrightarrow{\ \alpha\ } A$$

is uniquely determined by its only value

$$\alpha(\star)$$

which is an element of A, and every element is the unique value of some such function. Thus we have a bijection between

$$\boldsymbol{Set}[\mathsf{1}, A] \qquad A$$

as required.

Now consider any composite

$$\mathsf{1} \xrightarrow{\ \alpha\ } A \xrightarrow{\ f\ } B$$

where α corresponds to the element $a \in A$, that is $\alpha(\star) = a$. The composite $f \circ \alpha$ corresponds to the element

$$(f \circ \alpha)(\star) \in B$$

and this is just

$$f(\alpha(\star)) = f(a)$$

as required. ■

2.4.6 (a) The presheaf with a singleton for each component set is the final object $\mathsf{1}$.

(b) A global element

$$\mathsf{1} \longrightarrow A$$

of a presheaf $A = (\mathsf{A}, \mathcal{A})$ selects an element

$$a(i) \in A(i)$$

from each component set. This choice function $a(\cdot)$ must satisfy

$$A(j,i)\big(a(i)\big) = a(j)$$

for each $j \leq i$. ■

B.5 Products and coproducts

2.5.1 The 'algebraic' categories are straight forward. In each case we take the cartesian product of the two carrying sets and then furnish this in a fairly obvious way.

You may not have seen products in ***Pos*** before but they are constructed in the obvious way using cartersian products.

You will have seen products on ***Top*** before, and you have probably been puzzled by the strange construction of the product topology. The categorical description explains this.

Let S and T be a pair of topological spaces. We require a space $S \times T$ and a pair

$$\begin{array}{ccc} & S \times T & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ S & & T \end{array}$$

of continuous maps where this ***Top***-wedge has a certain universal property.

We take the cartesian product $S \times T$ of the two sets. For the topological furnishings let's try the smallest topology on $S \times T$ for which both projections are continuous. Thus we take the smallest topology on $S \times T$ for which each inverse image

$$p^{\leftarrow}(U) \quad \text{for } U \in \mathcal{O}S \qquad q^{\leftarrow}(V) \quad \text{for } V \in \mathcal{O}T$$

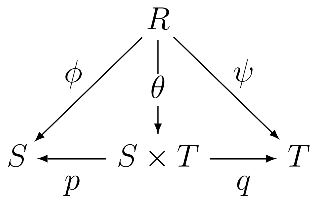
is open. This gives a subbase of the usual product topology.

Why does this give a product wedge in ***Top***?

Consider any wedge

$$\begin{array}{ccc} & R & \\ {\scriptstyle \phi} \swarrow & & \searrow {\scriptstyle \psi} \\ S & & T \end{array}$$

in ***Top***. Forget the topology for a moment. We have a wedge inh ***Set*** and a product wedge in ***Set***. Thus there is a *unique* function θ such that



commutes. It suffices to show that this function θ is continuous, for then we have a product wedge in ***Top***. To do that it suffices to show that $\theta^{\leftarrow}(W)$ is open in R for each subbasic open set W of $S \times T$.

There are two kinds of such sets, both are dealt with by the same argument. For instance consider $W = p^{\leftarrow}(U)$ for some $U \in \mathcal{O}S$. For each point $r \in R$ we have

$$r \in \theta^{\leftarrow}(W) \Longleftrightarrow \theta(r) \in W = p^{\leftarrow}(U) \Longleftrightarrow (p \circ \theta)(r) \in U \Longleftrightarrow \phi(r) \in U \Longleftrightarrow r \in \phi^{\leftarrow}(U)$$

so that

$$\theta^{\leftarrow}(W) = \phi^{\leftarrow}(U)$$

which is open in R. ■

2.5.2 For

Set, ***Pos***, ***Set***-R, ***Top***

each coproduct can be obtained as a furnished disjoint union with the obvious insertions.

For

CMon, ***AGrp***, ***Mod***-R

each coproduct can be obtained as a furnished cartesian product with the obvious insertions.

For

Mon, ***Grp***, ***CRng***, ***Rng***

a coproduct is formed by a more complicated construction.
`We may look at this later.` ■

2.5.3 For two elements a, b of a poset (with arrows pointing upwards) the

meet $a \wedge b$ join $a \vee b$

is the

product coproduct

of the pair. ■

2.5.4 For ***Set***$_\perp$-objects A and B the product in ***Set***$_\perp$ is given by the cartesian product $A \times B$ with the obvious projections. The distinguished element of $A \times B$ is $(\perp, \perp)$.

The proof of this is easier than, for instance, the ***Mon*** case.

The coproduct is more interesting. Let

$$A \amalg B = \big((A - \{\perp\}) + (B - \{\perp\}\big) \cup \{\perp\}$$

the disjoint union of the two point depleted sets with a point attached. This set has three kinds of elements

$$(a, 0) \text{ for } a \in A - \{\perp\} \qquad (b, 1) \text{ for } b \in B - \{\perp\} \qquad \perp$$

and, of course, $\perp$ is the distinguished point.

The function

$$\begin{array}{rcl} A & \xrightarrow{i} & A \amalg B \\ a & \longmapsto & (a, 0) \qquad \text{for } a \in A - \{\perp\} \\ \perp & \longmapsto & \perp \end{array}$$

is an arrow of $\boldsymbol{Set}_\perp$, and there is a similar arrow

$$B \xrightarrow{j} A \amalg B$$

from B. These furnish $A \amalg B$ as the coproduct. The proof is similar to that for ***Set***. ■

2.5.5 For convenience let ***SetD*** be the category of sets each with a distinguished subset.

The product is constructed in routine way using cartesian products. However, it is worth looking at some of the details.

This is one of the places where it is useful to distinguish between a structure and its carrying set. Thus let

$$\mathcal{A} = (A, X) \qquad \mathcal{B} = (B, Y)$$

be a pair of objects of ***SetD***. Let

$$\mathcal{A} \times \mathcal{B} = (A \times B, X \times Y)$$

so this is certainly an object of ***SetD***. Let

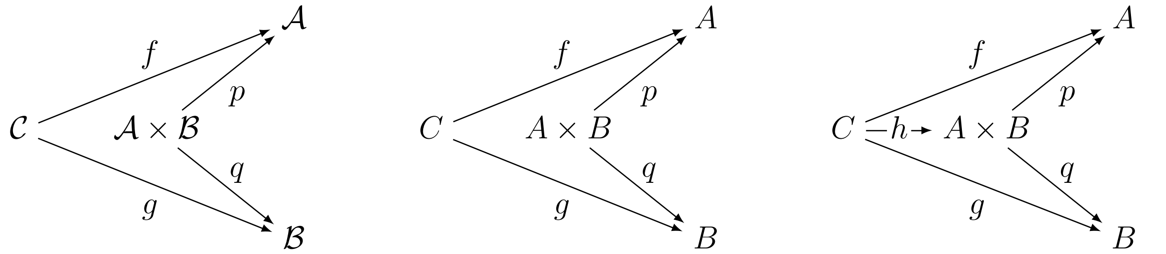
$$\begin{array}{ccc} & & A \\ & \nearrow p & \\ A \times B & & \\ & \searrow q & \\ & & B \end{array}$$

be the two projection *functions*. This is a wedge in ***Set***. In fact, it is a product wedge in ***Set***. We easily check that p and q are arrows of ***SetD***, so we have a wedge in ***SetD***.

$$\begin{array}{ccc} & & \mathcal{A} \\ & \nearrow p & \\ \mathcal{A} \times \mathcal{B} & & \\ & \searrow q & \\ & & \mathcal{B} \end{array}$$

We show this is a product wedge in ***SetD***.

Consider any object $\mathcal{C} = (C, Z)$ of ***SetD*** and wedge of ***SetD*** arrows, as on the left.



By forgetting the carried structure we obtain a wedge of ***Set*** arrows, as in the middle. But (p, q) are a product wedge in ***Set***, so we obtain a unique mediating ***Set***-arrow, a function h, as on the right. It suffices to show that h is a ***SetD*** arrow. That is a routine calculation.

The construction of the coproduct is not so obvious, but once we have seen the product construction we can dualize.

Let

$$\mathcal{A} = (A, X) \qquad \mathcal{B} = (B, Y)$$

be a pair of objects of ***SetD***. Recall in ***Set*** the coproduct

$$A + B = \big(A \times \{0\}\big) \cup \big(B \times \{1\}\big)$$

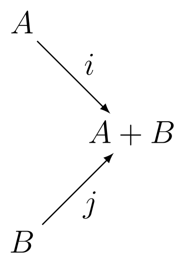
is the union of A and B where these sets have been tagged to make them disjoint. The union

$$X + Y = \big(X \times \{0\}\big) \cup \big(Y \times \{1\}\big)$$

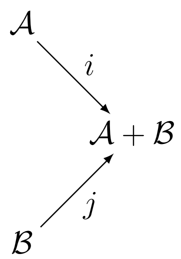
is a subset of $A + B$, and so

$$\mathcal{A} + \mathcal{B} = (A + B, X + Y)$$

is an object of ***SetD***. Let



be the two insertions. This is a coproduct wedge in ***Set***. It is easy to check that i and j are arrow of ***SetD***, so we have a wedge



in ***SetD***. By mimicking the proof for the product wedge with the arrows reversed, we see that we have a coproduct wedge in ***SetD***. ■

2.5.6 This is a teaser which almost everyone gets wrong the first time they see it.

For sets A and B the product and coproduct in ***RelA*** are both carried by the same set, but this is *not* the cartesian product $A \times B$. It is the disjoint union

$$A + B$$

of the sets.

Recall that the members of $A + B$ are tagged members of A and B. Thus $A + B$ has two kinds of elements

$$(a, 0) \text{ for } a \in A \qquad (b, 1) \text{ for } b \in B$$

where the tag records where the element came from. We set up four relations

$$A \underset{I}{\overset{P}{\rightleftarrows}} A + B \underset{J}{\overset{Q}{\rightleftarrows}} B$$

and show that P, Q form a product wedge, and I, J form a coproduct wedge.

With z ranging over $A + B$ we let

$$aPz \iff z = (a, 0) \iff zIa$$
$$bQz \iff z = (b, 1) \iff zJb$$

for $a \in A$ and $b \in B$.

To show that P, Q form a product wedge consider any wedge

$$\begin{array}{ccc} & & A \\ & \overset{F}{\nearrow} & \\ X & & \\ & \underset{G}{\searrow} & \\ & & B \end{array}$$

from an arbitrary set X. We require a pair of commuting triangles

$$\begin{array}{ccc} & & A \\ & \overset{F}{\nearrow} & \uparrow P \\ X & \xrightarrow{M} & A + B \\ & \underset{G}{\searrow} & \downarrow Q \\ & & B \end{array}$$

for some unique relation M.

Remembering that $z \in A + B$ can have only two forms, we see that

$$zMx \iff \begin{cases} (\exists a \in A)[z = (a, 0) \ \& \ aFx] \\ \text{or} \\ (\exists b \in B)[z = (b, 1) \ \& \ bGx] \end{cases}$$

gives a relation M of the correct type.

For $a \in A$ and $x \in X$ we have

$$a(P \circ M)x \iff (\exists z)[aPzMx] \iff (a,0)Mx \iff aFx$$

to show that

$$P \circ M = F$$

and hence the top triangle commutes. A similar argument shows that the bottom triangle commutes.

To show the uniqueness of this mediating relation consider any relation

$$X \xrightarrow{\;N\;} A + B$$

where both

$$P \circ N = F \qquad Q \circ N = G$$

hold. For $a \in A$ and $x \in X$ we have

$$(a,0)Nx \iff aP(a,0)Nx \iff (\exists z)[aPzNx] \iff a(P \circ N)x \iff aFx$$

and for $b \in B$ we have

$$(b,1)Nx \iff bGx$$

by a similar argument. This gives

$$N = M$$

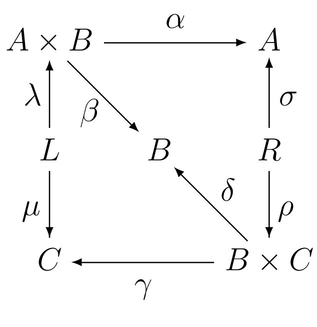
for the required uniqueness.

The verification that I, J form a coproduct wedge follows by a similar argument. ■

2.5.7 Let

$$L = (A \times B) \times C \qquad R = A \times (B \times C)$$

and consider the following diagram.



where each arrow is one of the structuring projections of one of the product wedges. There are no commuting cells in this diagram.

We insert four mediating arrows.

Firstly we obtain

$$L \xrightarrow{\;\eta\;} B \times C \qquad A \times B \xleftarrow{\;\zeta\;} R$$

with

$$\begin{array}{llll} (1) & \delta \circ \eta = \beta \circ \lambda & (3) & \beta \circ \zeta = \delta \circ \rho \\ (2) & \gamma \circ \eta = \mu & (4) & \alpha \circ \zeta = \sigma \end{array}$$

respectively. Notice that (1, 2) uniquely determine η, and (3, 4) uniquely determine ζ.

Secondly we obtain

$$L \xrightarrow{\ \phi\ } R \qquad\qquad L \xleftarrow{\ \psi\ } R$$

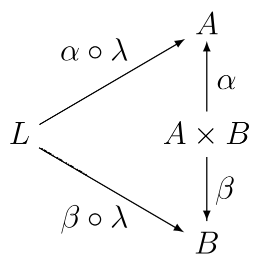
with

$$\begin{array}{llll} (5) & \sigma \circ \phi = \alpha \circ \lambda & (7) & \mu \circ \psi = \gamma \circ \rho \\ (6) & \rho \circ \phi = \eta & (8) & \lambda \circ \psi = \zeta \end{array}$$

respectively. Notice that (5,6) uniquely determine ϕ, and (7, 8) uniquely determine ψ.

We show that ϕ and ψ are an inverse pair of isomorphisms.

For the diagram



the unique mediator

$$L \longrightarrow A \times B$$

must be λ. But with

$$\xi = \psi \circ \phi$$

we have

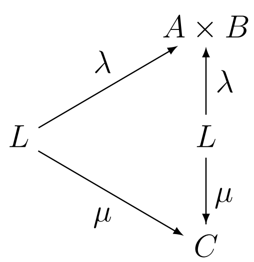
$$\begin{aligned} \alpha \circ \xi &= \alpha \circ \lambda \circ \psi \circ \phi = \alpha \circ \zeta \circ \phi = \sigma \circ \phi \qquad = \alpha \circ \lambda \\ \beta \circ \lambda \circ \xi &= \beta \circ \lambda \circ \psi \circ \phi = \beta \circ \zeta \circ \phi = \delta \circ \rho \circ \phi = \delta \circ \eta = \beta \circ \lambda \end{aligned}$$

using (8, 4, 5) on the top line, and (8, 3, 6, 1) on the bottom line. This shows that

$$\lambda \circ \xi = \lambda$$

since we have just verified that $\lambda \circ \xi$ has the required mediating property.

For the diagram



the unique mediator must be $\boldsymbol{id}_L$. But

$$\mu \circ \xi = \mu \circ \psi \circ \phi = \gamma \circ \rho \circ \phi = \gamma \circ \eta = \mu$$

using (7, 6, 2). This with the previous equality gives

$$\psi \circ \phi = \xi = \boldsymbol{id}_L$$

which is half of what we want.

The other required equality

$$\phi \circ \psi = \boldsymbol{id}_R$$

follows by a similar argument. ■

2.5.8 Let

$$L_1 = A \times C \qquad R_1 = A + B$$
$$L_2 = B \times C \qquad R_2 = C$$

so that

$$L = L_1 + L_2 \qquad R = R_1 \times R_2$$

are the two component objects.

Let

$$L_1 \xrightarrow{\alpha} A \qquad A \xrightarrow{\iota_A} R_1$$
$$L_1 \xrightarrow{\gamma_1} C \qquad B \xrightarrow{\iota_B} R_1$$
$$L_2 \xrightarrow{\beta} B$$
$$L_2 \xrightarrow{\gamma_2} C$$
$$R \xrightarrow{\rho_1} R_1 \qquad L_1 \xrightarrow{\lambda_1} L$$
$$R \xrightarrow{\rho_2} R_2 \qquad L_2 \xrightarrow{\lambda_2} L$$

be the

projections insertions

which structure the various objects as

products coproducts

respectively.

We can fit these arrows together in two ways. Let's look at both possibilities in parallel.

We have arrows

$$L_1 \xrightarrow{\delta_{11} = \iota_A \circ \alpha} R_1 \qquad L_2 \xrightarrow{\delta_{12} = \iota_B \circ \beta} R_1$$
$$L_1 \xrightarrow{\delta_{21} = \gamma_1} R_2 \qquad L_2 \xrightarrow{\delta_{22} = \gamma_2} R_2$$

which give commuting triangles

for $i, j \in \{1, 2\}$. In other words

We use the coproduct properties of L to produce a unique mediator μ_j

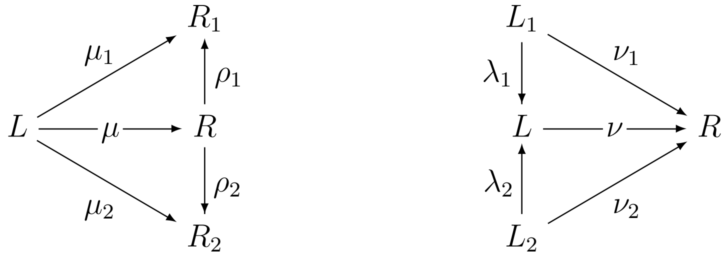
We use the product properties of R to produce a unique mediator ν_j

for the various cases. Observe that

$$\mu_j \circ \lambda_i = \delta_{ji} = \rho_j \circ \nu_i$$

(for $i, j \in \{1, 2\}$) uniquely determined μ_j and ν_i in terms of the δ_{ji}, and these in turn are determined by the given structuring arrows.

Next we interchange the roles of L and R to obtain commuting triangles



for unique mediators μ and ν. These are determined by

$$\mu_j = \rho_j \circ \mu \qquad \nu_i = \nu \circ \lambda_i$$

respectively. Either μ of ν does the required job.

In fact

$$\mu = \nu$$

as we now show.

We have

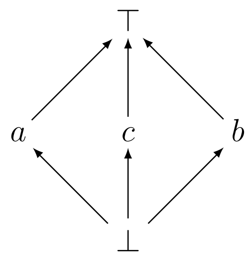
$$\rho_j \circ \mu \circ \lambda_i = \mu_j \circ \lambda_i = \delta_{ji}$$

for each i and j, so that

$$\nu_i = \mu \circ \lambda_i$$

for each i, and hence $\mu = \nu$.

For the counterexample consider the lantern poset



viewed as a category (with arrows pointing upwards). Then

$$a \wedge c = \bot = b \wedge c \qquad a \vee b = \top$$

to give

$$l = (a \wedge c) \vee (b \wedge c) = \bot \qquad r = (a \vee b) \wedge c = c$$

and hence $r \not\leq l$. ■

2.5.9 `Need to sort out with some decent non-commutative examples.` ■

B.6 Equalizers and coequalizers

2.6.1 The two parts of Lemma 2.17 are proved in the same way. Let's show that each equalizer is monic.

Let

$$\bullet \xrightarrow{\ m\ } \bullet$$

be the equalizer of the parallel pair

$$\bullet \underset{q}{\overset{p}{\rightrightarrows}} \bullet$$

so that

$$p \circ m = q \circ m$$

with the appropriate universal property. To show that m is monic consider any parallel pair

$$\bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet$$

with

$$m \circ f = h = m \circ g$$

where h is the common composite. We require $f = g$.

$$\bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet \xrightarrow{\ m\ } \bullet \underset{q}{\overset{p}{\rightrightarrows}} \bullet$$

From the diagram above we have

$$p \circ h = p \circ m \circ f = q \circ m \circ f = q \circ m \circ g = q \circ h$$

so that the universal property of m gives

$$h = m \circ k$$

for some unique arrow k. This uniqueness ensure that $f = g$.

The proof of the equalizer version of Lemma 2.18 is the mirror image of the coequalizer version. The proof can be obtained from the coequalizer version by changing one or two words and remembering that arrows now point the other way.

The arrow l makes equal f and g. The arrow k is the equalizer of f and g. Thus there is a unique mediator m satisfying (1).

By reversing the roles of l and k we see there is a unique mediator n satisfying (2).

From (1, 2) we have

$$k \circ m \circ n = l \circ n = k = k \circ \boldsymbol{id}_T$$

and hence

$$m \circ n = \boldsymbol{id}_T$$

since k is monic. Similarly

$$n \circ m = \boldsymbol{id}_S$$

to show that m and n are an inverse pair of isomorphisms. ■

2.6.2 We have a particular insertion

$$S \xhookrightarrow{\;\;i\;\;} A$$

which automatically staisfies

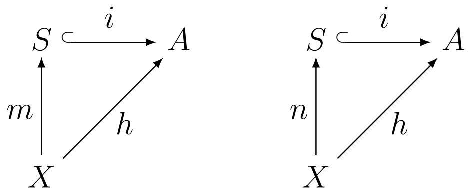
$$i(s) = s$$

for each $s \in S$. For the given function

$$X \xrightarrow{\;\;h\;\;} A$$

we have set up a triangle, as on the left

$$m(x) = h(x)$$



for a certain function m as indicated. Trivially, for $x \in X$ we have

$$i(m(x)) = m(x) = h(x)$$

so the triangle does commute.

Conversely, suppose we have some function n to make the triangle commute, as on the right. Then for each $x \in X$ we have

$$n(x) = i(n(x)) = h(x)$$

to show that $n = m$, and hence m has the required uniqueness. ■

2.6.3 (a) Making use of Example 2.19 we have

$$E \xhookrightarrow{\;\;j\;\;} S \xhookrightarrow{\;\;i\;\;} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

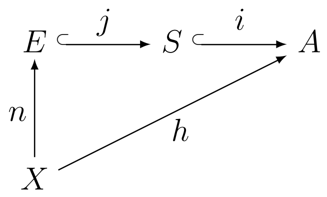
where $i \circ j$ is the equalizer of the pair f, g in $\boldsymbol{Set}$, where i is an injection and hence monic, and where i is a group morphism.

Consider any group morphism

$$X \xrightarrow{\;\;h\;\;} A$$

which does make equal f and g.

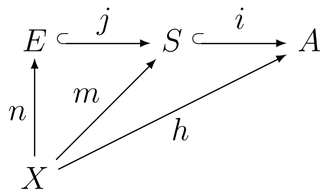
Working first in ***Set*** we have a commuting triangle



for some unique function n. Let

$$m = j \circ n$$

to obtain



where by construction the left hand triangle commutes in ***Set***, and

$$h = i \circ j \circ n = i \circ m$$

so the right hand triangle commutes, again in ***Set***. We show that m is a group morphism, so that the right hand triangle commutes in ***Grp***.

Consider any $x, y \in X$. We require

$$m(xy) = m(x)m(y)$$

in A. But, i and h are group morphisms so that

$$i\big(m(xy)\big) = h(xy) = h(x)h(y) = i(m(x))i(m(y)) = i\big(m(x)m(y)\big)$$

and i is an injection (monic in ***Grp***), to give the required result.

This shows that h does factorize through i via some group morphism m. We must show that this is the only possible factorization. Thus suppose

$$h = i \circ k$$

for some group morphism k. Then

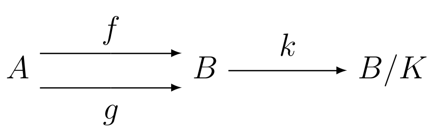
$$i \circ k = h = i \circ j \circ n = i \circ m$$

so that

$$k = m$$

since i is monic in ***Grp***.

(b) We have a diagram



where k is the canonical quotient. For each $a \in A$ we have

$$k\big(f(a)g(a)^{-1}\big) = k(1) = 1$$

which leads to

$$k(f(a)) = k(g(a))$$

and hence k does make equal f and g. We show that k is the coequalizer of f and g.

Consider any group morphism

$$B \xrightarrow{\quad h \quad} X$$

which does make equal f and g. For each $a \in A$ we have

$$h(f(a)) = h(g(a))$$

so that

$$h\big(f(a)g(a)^{-1}\big) = h(1) = 1$$

and hence

$$f(a)g(a)^{-1} \in \ker(h)$$

to show

$$F \subseteq \ker(h)$$

and hence

$$K \subseteq \ker(h)$$

by the construction of K.

This shows there is a unique morphism m for which the triangle

$$\begin{array}{ccc} B & \xrightarrow{\quad h \quad} & X \\ & {}_{k}\searrow \qquad \nearrow_{m} & \\ & B/K & \end{array}$$

commutes, and this is precisely the mediating property we require. ■

2.6.4 Since the function σ is surjective there can be at most one one function $h^\sharp$ to make the duagram commute.

For $s_1, s_2 \in S$ we have

$$[s_1] = [s_2] \Longrightarrow s_1 \sim s_2 \Longrightarrow h(s_1) = h(s_2)$$

to show that the suggested function $h^\sharp$ is well defined.

For $s \in S$ we have

$$(h^\sharp \circ \sigma)(s) = h^\sharp\big([s]\big) = h(s)$$

to show that the triangle commutes. ■

2.6.5 Let

$$\begin{array}{ccc} B & \xrightarrow{\quad \sigma \quad} & B/{\sim} \\ b & \longmapsto & [b] \end{array}$$

be the constucted quotient.

Consider any $a \in A$ and let

$$b_1 = f(a) \qquad b_2 = g(a)$$

to obtain to elements of b with $b_1 \rightsquigarrow b_2$. In particlar we have

$$b_1 \sim b_2$$

so that

$$(\beta \circ f)(a) = \beta(b_1) = [b_1] = [b_2] = \beta(b_2) = (\beta \circ g)(a)$$

to show that β does make equal f and g.

Consider any function h which does make equal f and g. We must show tha h factorizes unquely through β. Since β is surjective, there can be at most one such factorizaton, so it suffices to show that one does exist.

We use Example 2.20 and Exercise 2.6.4. Suppose

$$b_1 \rightsquigarrow b_2$$

for $b_1, b_2 \in B$. Then

$$b_1 = f(a) \qquad b_2 = g(a)$$

for some $a \in A$. This gives

$$h(b_1) = h(f(a)) = h(g(a)) = h(b_2)$$

and hence Example 2.20 gives us a function $h^\sharp$ for the required factorization. ■

2.6.6 For the given continuous maps

$$S \underset{\psi}{\overset{\phi}{\rightrightarrows}} T$$

we let

$$T \xrightarrow{\ \alpha\ } A$$

be the coequalizer of the two functions ϕ and ψ in ***Set***. Recall that α is surjective. We furnish A with the smallest topology $\mathcal{O}A$ for which α is continuous. This is precisely the set of all $W \subseteq A$ for which $\alpha^{\leftarrow}(W) \in \mathcal{O}T$. (You should check this. The topology is sometimes called the final topology or the quotient topology on A.)

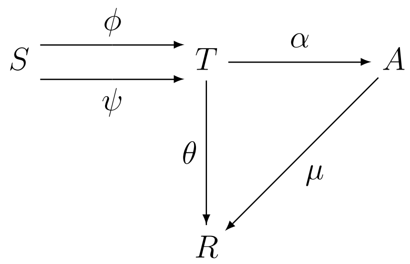
We show that this continuous map α is the coequalizer of the two maps ϕ and ψ in ***Top***.

Trivially, α does make equal ϕ and ψ.

Consider any continuous map

$$T \xrightarrow{\ \theta\ } R$$

which makes equal ϕ and ψ. At the ***Set*** level there is a unique function μ such that the triangle



commutes. It suffices to show that μ is continuous.

Consider any $U \in \mathcal{O}R$. We require

$$\mu^{\leftarrow}(U) \in \mathcal{O}A$$

that is

$$(\alpha^{\leftarrow} \circ \mu^{\leftarrow})(U) = \alpha^{\leftarrow}(\mu^{\leftarrow}(U)) \in \mathcal{O}T$$

(by the definition of $\mathcal{O}A$). But

$$\alpha^{\leftarrow} \circ \mu^{\leftarrow} = (\mu \circ \alpha)^{\leftarrow} = \theta^{\leftarrow}$$

and $\theta^{\leftarrow}(U) \in \mathcal{O}T$ since θ is continuous. ■

2.6.7 (a) Since the comparison $\leq$ is refexive, the defied relation $\sim$ is relexive.

By rephrasinng the definition as

$$a \sim b \iff a \leq b \text{ and } b \leq a$$

we see that $\sim$ is symmetric.

If

$$a \sim b \sim c$$

then

$$a \leq b \leq c \text{ and } c \leq b \leq a$$

so that

$$a \leq c \text{ and } c \leq a$$

to give

$$a \sim c$$

to show that $\sim$ is transitive.

The preset S is a poset precisely when

$$a \leq b \leq a \Longrightarrow a = b$$

that is

$$a \sim b \Longrightarrow a = b$$

and the converse implication always holds.

(b) To show that the comparison on $S/{\sim}$ is well-defined suppose

$$[a_1] = [a_2] \qquad [b_1] = [b_2]$$

for elements $a_1, a_2, b_1, b_2 \in S$. We require

$$a_1 \leq b_1 \iff a_2 \leq b_2$$

and clearly, by symmetry, a proof of one of the implications will do.

From the two assumed equalities we have

$$a_1 \sim a_2 \qquad b_1 \sim b_2$$

and hence

$$a_1 \leq b_1 \Longrightarrow a_2 \leq a_1 \leq b_1 \leq b_2 \Longrightarrow a_2 \leq b_2$$

as required.

This shows that

$$\begin{array}{ccc} S & \xrightarrow{\ \eta\ } & S/{\sim} \\ a & \longmapsto & [a] \end{array}$$

is well-defined and, trivially, it is monotone.

(c) Consider a monotone map

$$S \xrightarrow{\ f\ } T$$

from the preset S to a poset T.

For $a, b \in S$ we have

$$a \sim b \Longrightarrow a \leq b \leq a \Longrightarrow f(a) \leq f(b) \leq f(a) \Longrightarrow f(a) = f(b)$$

where the last step holds since T is a poset.

Since η is surjective there is at most one monotone map $f^\sharp$ such that

$$\begin{array}{ccc} S & \xrightarrow{\ f\ } & T \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle f^\sharp} & \\ & S/{\sim} & \end{array}$$

commutes.

Thus it suffices to show that

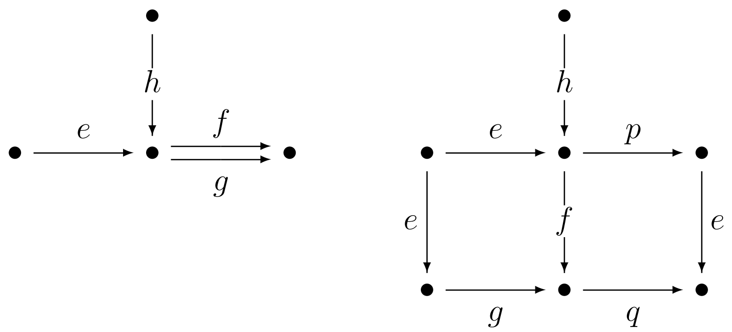
$$f^\sharp([a]) = f(a)$$

(for $a \in S$) gives a well-defined monotone function.

The previous sequence of implications shows that $f^\sharp$ is well-defined, and a similar argument shows that f is monotone.

This universal property induces the required functor. For the general argument see Solution 3.3.18. ■

2.6.8 We are given that e does make equal f and g. Consider any other arrow h which makes equal f and g, as on the left. We must show that h factorizes unquely through e.



Let

$$m = p \circ h$$

and consier the right had diagram. We have

$$e \circ m = e \circ p \circ h = q \circ f \circ h = q \circ g \circ h = h$$

to show that h does factorize through e.

Conversely, suppose

$$h = e \circ n$$

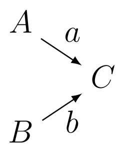
for some arrow n. Then

$$n = p \circ e \circ n = p \circ h = m$$

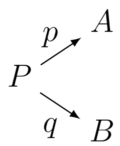
to show the required unqueness. ■

B.7 Pullbacks and pushouts

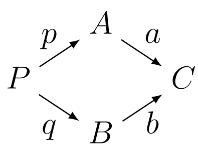
2.7.1 (a) Consider any wedge in $\boldsymbol{C}$.



Let



be the product wedge of the two objects A, B, and consider the square



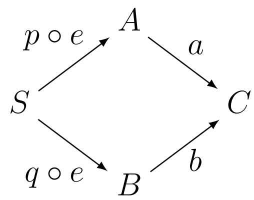
which, of course, need not commute. Let

$$S \xrightarrow{\ e\ } P$$

be the equalizer of the parallel pair

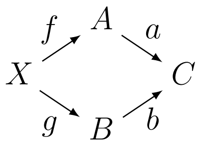
$$P \overset{a \circ p}{\underset{b \circ q}{\rightrightarrows}} C$$

obtained form the square. We show that

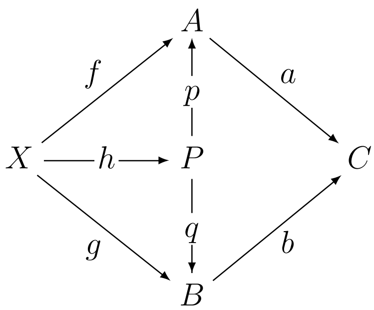


is a pullback square.

Consider any commuting square



where the right hand side is the given wedge. Using the product property we have



$$f = p \circ h \qquad g = q \circ h$$

for some *unique* arrow h. But now

$$a \circ p \circ h = a \circ f = b \circ g = b \circ q \circ h$$

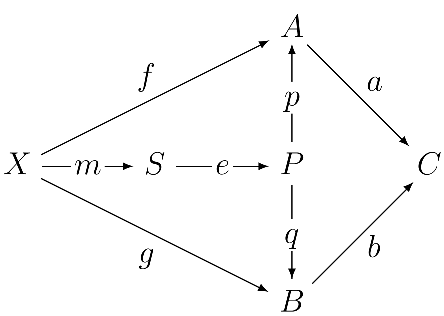
to show that h makes equal the parallel pair, and hence

$$h = e \circ m$$

for some unique arrow

$$X \xrightarrow{\ m\ } S$$

by the equalizing property. In particular, we have a commuting diagram



to show that the arbitrary square from X does factorize via m through the constructed square from S. We must show that this is the only possible factorization.

Suppose

$$f = p \circ e \circ n \qquad g = q \circ e \circ n$$

for some arrow

$$X \xrightarrow{\ n\ } S$$

in place of m. Then

$$e \circ n = h$$

by he uniqueness of h, and hence

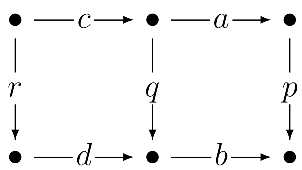
$$n = m$$

by the uniqueness of m.

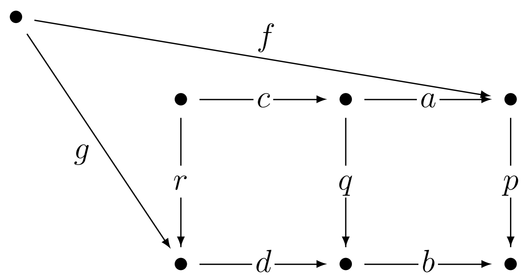
(b) This follows by a dual argument to that of (a). ■

2.7.2 Assuming the arrows point up he poset, it has all pushouts precisely when it has joins of those pairs of elements which have a lower bound. ■

2.7.3 For the first part we are given a pair of pullbacks

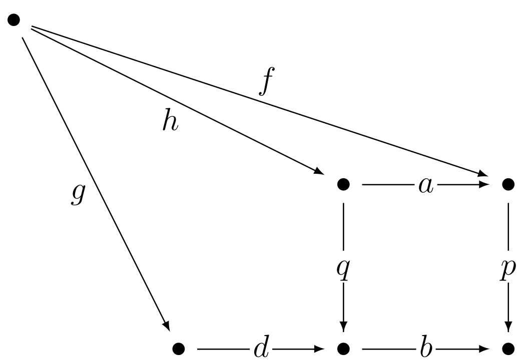


where for convenience we have labelled he arrows. Consider a pair of arrows



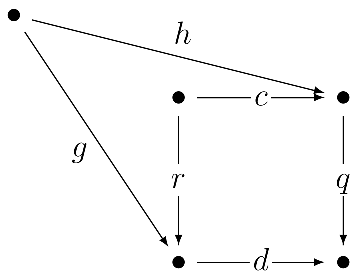
$$p \circ f = b \circ d \circ g$$

with the indicating commuting properties. Using the right hand pullback there is a *unique* arrow h for which



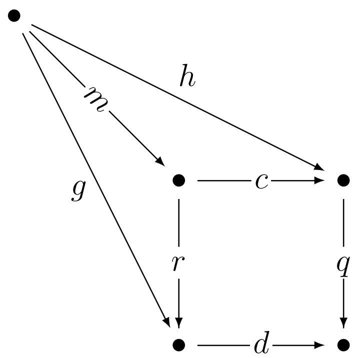
$$f = a \circ h \qquad d \circ g = q \circ h$$

commutes, as indicated. This gives us a commuting diagram



$$d \circ g = q \circ h$$

and the left hand pullback provides a *unique* arrow m for which



$$h = c \circ m \qquad g = r \circ m$$

commutes, as indicated.

From equalities we have

$$f = a \circ h = a \circ c \circ m \qquad g = r \circ m$$

to show that we have produced a factorization o f and g through a common arrow m. It remains to show that m is the only arrow that does this job.

Consider any arrow n for which

$$f = a \circ c \circ n \qquad g = r \circ n$$

holds. Then

$$f = a \circ c \circ n \qquad d \circ g = d \circ r \circ n = q \circ c \circ n$$

and hence

$$c \circ n = h$$

by the uniqueness of h. but now

$$c \circ n = h \qquad r \circ n = g$$

to give

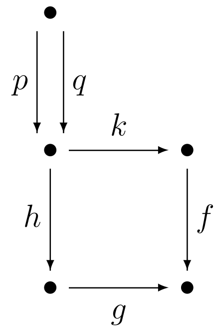
$$n = m$$

by the uniqueness of m. ■

2.7.4 Consider any parallel pair p, q of arrows that h makes equal

$$h \circ p = h \circ q = l$$

say. We must show that $p = q$.



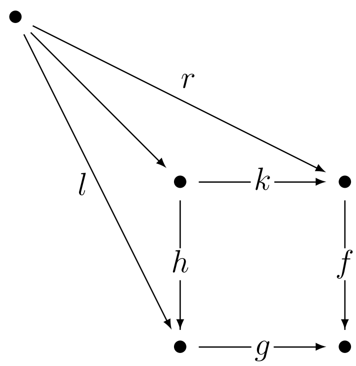
By traipsing round the square we find that

$$f \circ k \circ p = f \circ k \circ q$$

and hence, since f is monic, we have

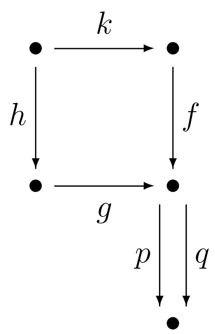
$$k \circ p = k \circ q = r$$

say. This show that both p and q make

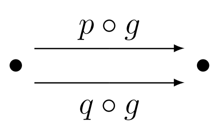


commutes, and hence $p = q$ since the given square is a pullback. ■

2.7.5 Suppose that f is the equalizer of the paralel pair p, q as indicated.

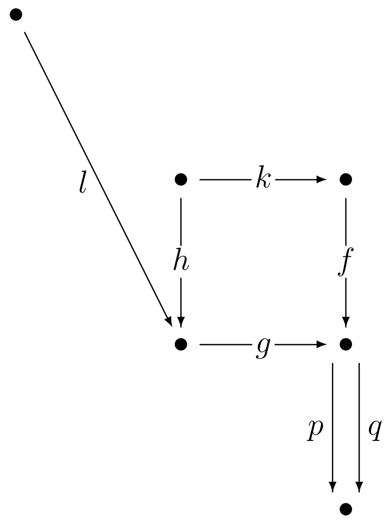


Since the given square does commute, we see that h makes equal the parallel pair

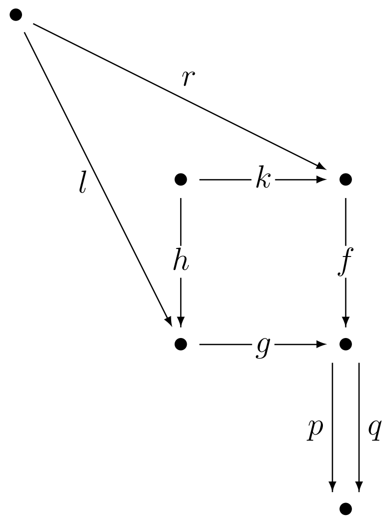


and we show that h actualy equalizes this pair.

Consider any other arrow l which does make equal this pair.

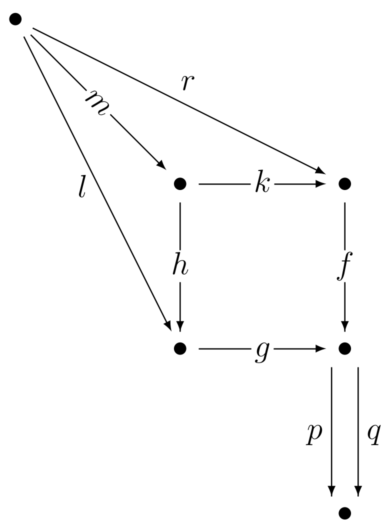


Since f is the equalizer of p, q there is a *unique* arrow r which makes the following diagram commute.



$$g \circ l = f \circ r$$

Since the given square is a pullback there is a *unique* arrow m which makes the following diagram commute



$$l = h \circ m \qquad r = k \circ m$$

and in particlar

$$l = h \circ m$$

holds. It suffices to show that this is the only possible factorization of l through h.

Consider any arrow n for which

$$l = h \circ n$$

holds. It suffices to show

$$r = k \circ n$$

for then $n = m$ by the uniqueness of m. Since the given square commutes we have

$$f \circ k \circ n = g \circ h \circ n = g \circ l$$

and hence the uniqueness of r gives the required result. a ■

B.8 Using the opposite category

2.8.1 No solution needed? ■

C
Functors and natural tansformations

C.1 Functors defined

3.1.1 A covariant functor from S to T is simply a monoid morphism from S to T.

A contravariant functor f from S to T is monoid 'morphism' that flips the elements, that is

$$f(rs) = f(s)f(r)$$

for $r, s \in S$. ■

3.1.2 A covariant functor from S to T is simply a monotone map.

A contravariant functor f from S to T is an antitone map, that is

$$r \leq s \Longrightarrow f(s) \leq f(r)$$

for $r, s \in S$. ■

3.1.3 Consider a covariant functor

$$\boldsymbol{Src}^{\mathrm{op}} \xrightarrow{\ F\ } \boldsymbol{Trg}$$

using the opposite on the source. Consider any arrow

$$A \xrightarrow{\ f\ } B$$

of $\boldsymbol{Src}$. This is an arrow

$$B \xrightarrow{\ f\ } A$$

of $\boldsymbol{Src}^{\mathrm{op}}$, and the functor F sends it to an arrow

$$FB \xrightarrow{\ F(f)\ } FA$$

of $\boldsymbol{Trg}$. Thus F has flipped the direction of f. The other required properties (preservation of composition and identity arrows) are immediate, to show that

$$\boldsymbol{Src} \xrightarrow{\ F\ } \boldsymbol{Trg}$$

is a contravariant functor.

The other part is just as easy. ■

3.1.4 The composite of two functors of the same variance produces a covariant functor.

The composite of two functors of opposite variance produces a contravariant functor.
■

C.2 Some simple functors

3.2.1 The functors S and T select the source and target of the arrow, respectively. The diagonal functor Δ send each object A to the identity arrow $\boldsymbol{id}_A$. ■

3.2.2 The ***Set***-valued contravariant functors on the poset S are precisely the presheaves on S. ■

3.2.3 The

covariant   contravariant

Set-valued functors from R are precisely the

left   right

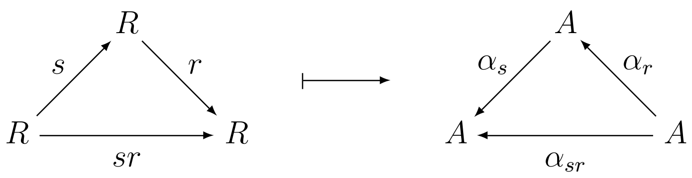
R-sets. It is worth looking at the details of this, and the contravariant case is potentially more interesting.

Consider such a contravariant functor. This must send each object of R to some set. But since R (when viewed as a category) has just one object, this object assignment produces a set, A say.

The functor must send each arrow of R (element of R) to a function from A to A.

$$R \xrightarrow{\ r\ } R \qquad \longmapsto \qquad A \xleftarrow{\ \alpha_r\ } A$$

Let α_r be the 1-placed operation on A assigned to $r \in R$. The contravariance



gives

$$\alpha_{sr} = \alpha_s \circ \alpha_r \qquad \alpha_1 = \boldsymbol{id}_A$$

for each $r, s \in R$, where the right hand equality is the identity requirement.

We now write each operation α as a right action

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha_r\ } & A \\ a & \longmapsto & ar \end{array}$$

to get

$$a(sr) = (as)r \qquad a1 = a$$

and so produce a right R-set. ■

3.2.4 This is more or less proved by Exercise 1.3.5. ■

3.2.5 Remember that $\boldsymbol{C}[-,-]$ is contravariant in the left argument and covariant in the right argument. So what we have here is really a covariant functor

$$\boldsymbol{C}^{\mathrm{op}} \times \boldsymbol{C} \xrightarrow{\ H\ } \boldsymbol{Set}$$

from the 'twisted product' category. In detail, for arrows

$$B \xrightarrow{f} A \qquad\qquad S \xrightarrow{g} T$$

of $\boldsymbol{C}$ we have

$$H(f,g) = g \circ - \circ f$$

for the arrow behaviour of H. ■

C.3 Some less simple functors

C.3.1 Three power set functors

3.3.1 Only $\forall(\cdot)$ is liable to cause trouble.
Given monotone maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

between sets, for each $X \in \mathcal{P}A$ we have

$$\begin{aligned}
\big(\forall(g) \circ \forall(f)\big)(X) &= \forall(g)\big(\forall(f)(X)\big) \\
&= \forall(g)\big(f[X']'\big) \\
&= g[f[X']'']' \\
&= g[f[X']]' \\
&= (g \circ f)[X']' \qquad = \forall(g \circ f)(X)
\end{aligned}$$

as required. A similar proof can be carried using the quantifier characterization. ■

3.3.2 This is a simpler case of the result of Exercise 1.3.9. Each set is a discrete poset. ■

C.3.2 Spaces, presets, and posets

3.3.3 For a preset A the specialization order of $\Uparrow A$ is the given comparison on A. ■

3.3.4 (a) Consider a monotone map

$$A \xrightarrow{f} B$$

between two presets, and an upper section $V \in \Upsilon B$ of the target. We require $f^{\leftarrow}(V) \in \Upsilon A$. To this end consider any $x \leq y$ in A with $x \in f^{\leftarrow}(V)$. Then

$$f(x) \leq f(y) \qquad f(x) \in V$$

so that $f(y) \in V$ and hence $y \in f^{\leftarrow}(V)$.
We have an object and an arrow assignment

$$\boldsymbol{Pre} \xrightarrow{\Uparrow} \boldsymbol{Top}$$

which is trivial on arrows, so we do have a functor.

(b) Consider a continuous map

$$S \xrightarrow{\ \phi\ } T$$

between spaces, and consider a comparison $x \le y$ in S. we require $\phi(x) \le \phi(y)$ in T.

Consider $V \in \mathcal{O}T$ with $\phi(x) \in V$. We required $\phi(y) \in V$. But

$$x \in \phi^{\leftarrow}(V) \in \mathcal{O}S \qquad x \le y$$

so that $y \in \phi^{\leftarrow}(V($, as required.

We have an object and an arrow assignment

$$\boldsymbol{Top} \xrightarrow{\ \Downarrow\ } \boldsymbol{Pre}$$

which is trivial on arrows, so we do have a functor. ■

3.3.5 Suppose first that θ is monotone and consider any $U \in \mathcal{O}S$. We require $\theta^{\leftarrow}(U) \in \upsilon A$. Consider elements x, y of A with

$$x \in \theta^{\leftarrow}(U) \qquad x \le y$$

so that $y \in \theta^{\leftarrow}(U)$ is required. We have

$$\theta(x) \in U \qquad \theta(x) \le \theta(y)$$

(since θ is monotone), and hence

$$y \in \theta(U)$$

since each open set of S is an upper section of S.

Secondly, suppose that θ is continuous and consider elements $x \le y$ of A. We require $\theta(x) \le \theta(y)$. For each $U \in \mathcal{O}S$ we have

$$\theta(x) \in U \Longrightarrow x \in \theta^{\leftarrow}(U) \Longrightarrow y \in \theta^{\leftarrow}(U) \Longrightarrow \theta(y) \in U$$

where the central implication holds since θ is continuous and hence $\theta^{\leftarrow}(U) \in \Upsilon A$.

These two implications show that the hom-sets

$$\boldsymbol{Pre}[A, \Downarrow S] \qquad \boldsymbol{Top}[\Uparrow A, S]$$

contain exactly the same functions. Thus there is a trivial bijection between the two sets. ■

3.3.6 For $U, V \in \mathcal{O}S$ with $U \subseteq V$ we require $\mathcal{O}(\phi)(U) \subseteq \mathcal{O}(\phi)(V)$. But for $t \in T$ we have

$$t \in \mathcal{O}(\phi)(U) \Longrightarrow \phi(t) \in U \subseteq V \Longrightarrow \phi(t) \in V \Longrightarrow t \in \mathcal{O}(\phi)(V)$$

for the required result.

For each pair of continuous maps

$$T \xrightarrow{\ \phi\ } S \xrightarrow{\ \psi\ } R$$

we require

$$\mathcal{O}(\psi \circ \phi) = \mathcal{O}(\phi) \circ \mathcal{O}(\psi)$$

that is

$$(\psi \circ \phi)^{\leftarrow}(U) = (\phi^{\leftarrow} \circ \psi^{\leftarrow})(U)$$

for $U \in \mathcal{O}R$. But for $t \in T$ we have

$$\begin{aligned} t \in (\psi \circ \phi)^{\leftarrow}(U) &\iff (\psi \circ \phi)(t) \in U \\ &\iff \psi(\phi(t)) \in U \\ &\iff \phi(t) \in \phi^{\leftarrow}(U) \\ &\iff t \in \psi^{\leftarrow}(\phi^{\leftarrow}(U)) \quad \iff (\phi^{\leftarrow} \circ \psi^{\leftarrow})(U) \end{aligned}$$

for the required result. ■

Observe that a character

$$p : S \longrightarrow \mathsf{2}$$

is continuous precisely when

$$p^{\leftarrow}(\{1\})$$

is open in S. This is because

$$p^{\leftarrow}(\emptyset) = \emptyset \qquad p^{\leftarrow}(\mathsf{2}) = S$$

and these are open in S.

For each continuous map ϕ an continuous character p on the target we have

$$\Xi(\phi)(p) = p \circ \phi$$

and this is continuous (since continuous maps are closed under composition).

To show that $\Xi(\phi)$ is monotone consider continuous characters p, q of S with $p \leq q$. Then for each $t \in T$ we have

$$\Xi(\phi)(p)(t) = p(\phi(t)) \leq q(\phi(t))\Xi(\phi)(q)(t)$$

to show

$$\Xi(\phi)(p) \leq \Xi(\phi)(q)$$

as required.

Finally, for each pair of continuous maps

$$T \xrightarrow{\ \ \phi\ \ } S \xrightarrow{\ \ \psi\ \ } R$$

we require

$$\Xi(\psi \circ \phi) = \Xi(\phi) \circ \Xi(\psi)$$

that is

$$\Xi(\psi \circ \phi)(r) = \big(\Xi(\phi) \circ \Xi(\psi)\big)(r)$$

for each continuous character r of R. But for such an r we have

$$\Xi(\psi \circ \phi)(r) = r \circ (\psi \circ \phi) = (r \circ \psi) \circ \phi = \Xi(\psi)(r) \circ \phi = \Xi(\phi)\big(\Xi(\psi)(r)\big) = \big(\Xi(\phi) \circ \Xi(\psi)\big)(r)$$

as required. ■

3.3.7 To show $\chi_S(U)$ is continuous (for $U \in \mathcal{O}S$) we require

$$\chi_S(U)^{\leftarrow}(W) \in \mathcal{O}S$$

for each $W \in \mathcal{O}2$. Trivially we have

$$\chi_S(U)^{\leftarrow}(\emptyset) = \emptyset \qquad \chi_S(U)^{\leftarrow}(2) = S$$

so it suffices to deal with $W = \{1\}$.

For each $s \in S$ we have

$$s \in \chi_S(U)^{\leftarrow}(\{1\}) \iff \chi_S(U)(s) \in (\{1\}) \iff \chi_S(U)(s) = 1 \iff s \in U$$

to give the required result.

For $p \in \Xi S$ with

$$U = p^{\leftarrow}(\{1\}) \in OS$$

we have

$$p = \chi_S(U)$$

and hence

$$\mathcal{O}S \xrightarrow{\chi_S} \Xi S$$

is a bijection. To show it is a poset isomorphism we require

$$U \subseteq V \iff \chi_S(U) \leq \chi_S(V)$$

for $U, V \in \mathcal{O}S$. But we have

$$\begin{aligned}
\chi_S(U) \leq \chi_S(V) &\iff (\forall s \in S)[\chi_S(U)(s) \leq \chi_S(V)(s)] \\
&\iff (\forall s \in S)[\chi_S(U)(s) = 1 \implies \chi_S(V)(s) = 1] \\
&\iff (\forall s \in S)[s \in (U \implies s \in V] \qquad \iff U \subseteq V
\end{aligned}$$

as required. ■

C.3.3 Functors from products

3.3.8 Let $F = - \times R$.

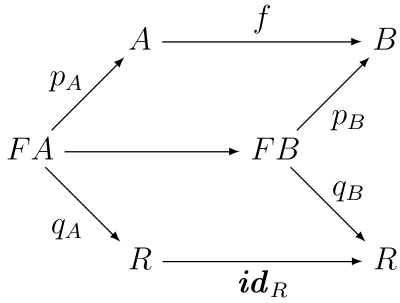
By construction, for each arrow

$$A \xrightarrow{f} B$$

the arrow

$$FA \xrightarrow{F(f)} FB$$

is the unique arrow for which



commutes.

For arrows

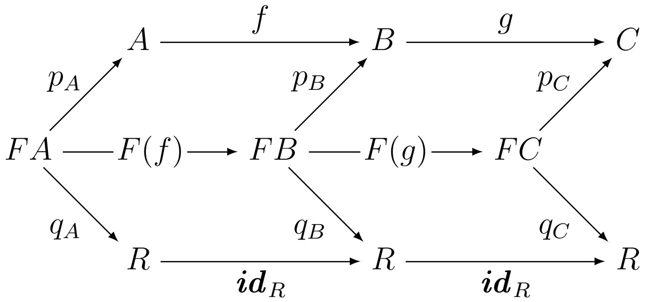
$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C$$

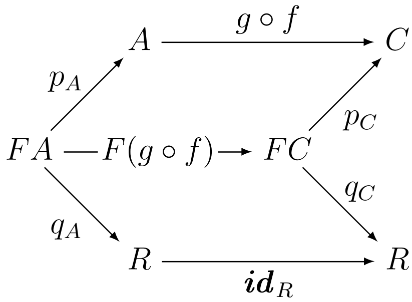
we require

$$F(g \circ f) = F(g) \circ F(f)$$

(together with a trivial observation to give $F(\boldsymbol{id}) = \boldsymbol{id}$).

We have several commuting cells





and the uniqueness properties of the central arrow give the required result. ■

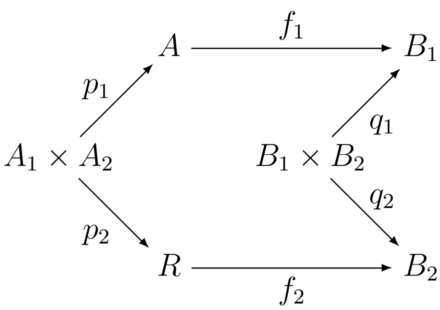
3.3.9 We have an object assignment

$$\begin{array}{ccc} \boldsymbol{C} \times \boldsymbol{C} & \longrightarrow & \boldsymbol{C} \\ A_1, A_2 & \longmapsto & A_1 \times A_2 \end{array}$$

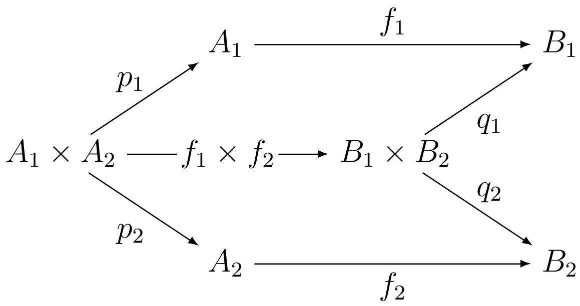
so we now require a companion arrow assignment. Consider any arrow of $\boldsymbol{C} \times \boldsymbol{C}$, in other words a pair

$$\begin{array}{c} A_1 \xrightarrow{\;f_1\;} B_1 \\ A_2 \xrightarrow[\;f_2\;]{} B_2 \end{array}$$

of arrows of $\boldsymbol{C}$. We have a diagram



where p_1, p_2, q_1, q_2 are the structuring projections. The product property of the right hand wedge gives a commuting diagram



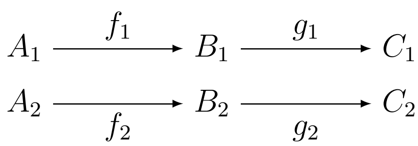
for some *unique* central arrow. This is often written $f_1 \times f_2$, as shown, and then

$$(f_1, f_2) \longmapsto f_1 \times f_2$$

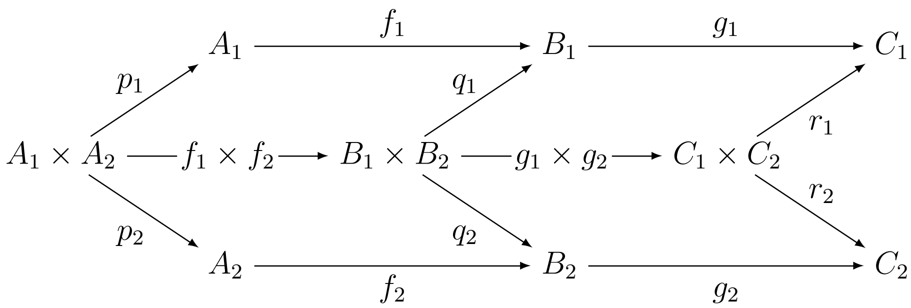
is the arrow assignment.

To verify that we have a functor we need to show that the arrow construction passes across composition. As usual, it is the uniqueness that gives this.

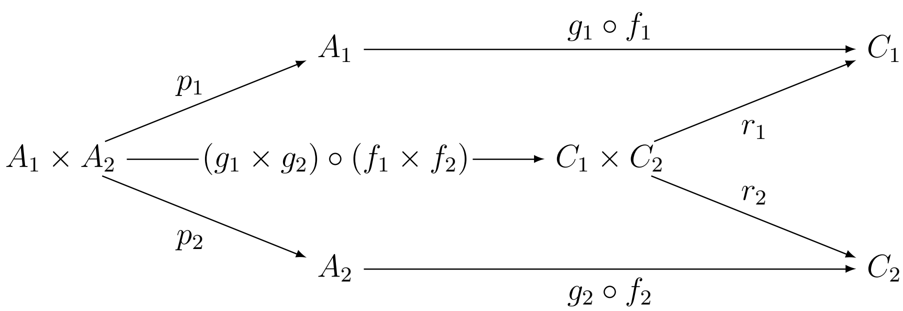
Consider a composible pair



of arrows of $\boldsymbol{C} \times \boldsymbol{C}$. The commuting diagram



ensures that



commutes, and hence

$$(g_1 \times g_2) \circ (f_1 \times f_2) = (g_1 \circ f_1) \times (g_2 \circ f_2)$$

by the uniqueness of the mediators. ■

C.3.4 Comma category

3.3.10 Let's check that arrows of $(U \downarrow L)$ do compose in an associative manner. consider two arrows of $(U \downarrow L)$

$$\begin{array}{ccccc} UA_U & & UB_U & & UC_U \\ \downarrow & & \downarrow & & \downarrow \\ \alpha & \xrightarrow{\quad f \quad} & \beta & \xrightarrow{\quad g \quad} & \gamma \\ \downarrow & & \downarrow & & \downarrow \\ LA_L & & LB_L & & LC_L \end{array}$$

which ought to compose. In more detail we have

$$\begin{array}{ccccc} A_U & \xrightarrow{f_U} & B_U & \xrightarrow{g_U} & C_U \\ UA_U & \xrightarrow{U(f_U)} & UB_U & \xrightarrow{U(g_U)} & UC_U \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ LA_L & \xrightarrow[L(f_L)]{} & LB_L & \xrightarrow[L(g_L)]{} & LC_L \\ A_L & \xrightarrow[f_L]{} & B_L & \xrightarrow[g_L]{} & C_L \end{array}$$

where the two squares commute. Now consider composite arrows

$$\begin{array}{cccc} A_U & \xrightarrow{h_U = g_L \circ f_U} & C_U & \qquad \boldsymbol{U} \\ A_L & \xrightarrow[h_L = g_L \circ f_L]{} & C_L & \qquad \boldsymbol{L} \end{array}$$

in the indicated categories. From above and using the functorial properties of U and L we see that the square

$$\begin{array}{ccc} A_U & \xrightarrow{h_U} & C_U \\ UA_U & \xrightarrow{U(h_U)} & UC_U \\ \downarrow \alpha & & \downarrow \gamma \\ LA_L & \xrightarrow[L(h_L)]{} & LC_L \\ A_L & \xrightarrow[h_L]{} & C_L \end{array}$$

commutes, and so we have an arrow

$$\begin{array}{ccc} UA_U & & UC_U \\ \downarrow & & \downarrow \\ \alpha & \xrightarrow{\quad h \quad} & \gamma \\ \downarrow & & \downarrow \\ LA_L & & LC_L \end{array}$$

of $(U \downarrow L)$. We take this as the composite

$$g \circ f$$

of the two given arrows. Routine diagram chasing now shows the category axioms do hold. ∎

3.3.11 (a) $\left(\boldsymbol{Id_C} \downarrow \boldsymbol{Id_C}\right) = \boldsymbol{C}^{\downarrow}$
(b) Using

$$\boldsymbol{C} \xrightarrow{\boldsymbol{Id_C}} \boldsymbol{C} \xleftarrow{K} \boldsymbol{C} \qquad\qquad \boldsymbol{C} \xrightarrow{K} \boldsymbol{C} \xleftarrow{\boldsymbol{Id_C}} \boldsymbol{C}$$

where K is the constant functor with

$$KA = S$$

for each $\boldsymbol{C}$-object A, we have

$$(\boldsymbol{C} \downarrow S) = (\boldsymbol{Id_C} \downarrow K) \qquad (S \downarrow \boldsymbol{C}) = (K \downarrow \boldsymbol{Id_C})$$

respectively. ∎

3.3.12 For the three cases the object

$$\begin{array}{c} A_U \\ \alpha \\ A_L \end{array}$$

of $\boldsymbol{Com}$ is sent to the object

$$\begin{array}{ccc} \boldsymbol{U} & \boldsymbol{C}^{\downarrow} & \boldsymbol{L} \\ & UA_U & \\ UA_U & \alpha\downarrow & LA_L \\ & LA_L & \end{array}$$

of the indicated category. The arrow

$$\begin{array}{ccc} UA_U & & UB_U \\ \alpha\downarrow & \xrightarrow{\quad f \quad} & \downarrow\beta \\ LA_L & & LB_L \end{array}$$

of $\boldsymbol{Com}$ is sent to the arrow

$$\begin{array}{ccc} \boldsymbol{U} & \boldsymbol{C}^{\downarrow} & \boldsymbol{L} \\ & \begin{array}{ccc} UA_U & \xrightarrow{U(f_U)} & UB_U \\ \alpha\downarrow & & \downarrow\beta \\ LA_L & \xrightarrow[L(f_L)]{} & LB_L \end{array} & \\ A_U \xrightarrow{\ f_U\ } B_U & & A_L \xrightarrow{\ f_L\ } B_L \end{array}$$

of the indicated category. The various composition requirements are easy to verify. ∎

C.3.5 Other examples

3.3.13 Consider an arbitrary group A. We assume it is written multiplicatively. A commutator of A is an element

$$[x, y] = xyx^{-1}y^{-1}$$

for arbitrary $x, y \in A$. Thus A is abelian precisely when the unit 1 is the only commutator.

Observe that

$$[x, y]^{-1} = [y, x]$$

so the set of products of commutators is a subgroup δA of A. In particular, A is abelian precisely when δA is the trivial subgroup.

To show that the object assignment

$$A \longmapsto \delta A$$

fills out to a functor we do a little bit more. We show there is a unique commuting square

$$\begin{array}{ccc} \delta A & \xrightarrow{\iota_A} & A \\ {\scriptstyle \delta(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \delta B & \xrightarrow[\iota_B]{} & B \end{array}$$

for each group morphism f. Here ι_A and ι_B are the two embeddings.

If there is such a morphism $\delta(f)$ then it can only be

$$f|_{\delta A}$$

the restriction of f to δA. We remember that

$$f(x^{-1}) = f(x)^{-1}$$

for each $x \in A$, and hence

$$f([x, y]) = [f(x), f(y)]$$

for each $x, y \in A$, so that

$$a \in \delta A \Longrightarrow f(a) \in \delta B$$

which is what we want.

The uniqueness in the construction of $\delta(\cdot)$ ensures that it passes across composition of morphisms. For each pair of morphisms

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

we have a commuting diagram, as on the left

$$\begin{array}{ccc} \delta A & \xrightarrow{\iota_A} & A \\ {\scriptstyle \delta(f)}\downarrow & & \downarrow{\scriptstyle f} \\ \delta B & \xrightarrow{\iota_B} & B \\ {\scriptstyle \delta(g)}\downarrow & & \downarrow{\scriptstyle g} \\ \delta C & \xrightarrow[\iota_C]{} & C \end{array} \qquad \begin{array}{ccc} \delta A & \xrightarrow{\iota_A} & A \\ {\scriptstyle \delta(g)\circ\delta(f)}\downarrow & & \downarrow{\scriptstyle g\circ f} \\ \delta C & \xrightarrow[\iota_C]{} & C \end{array}$$

to give a commuting square as on the right. Thus

$$\delta(g) \circ \delta(f) = \delta(g \circ f)$$

by the uniqueness of $\delta(g \circ f)$.

For the second part we show there is a unique commuting square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A/\delta A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f/\delta} \\ B & \xrightarrow[\eta_B]{} & B/\delta B \end{array}$$

for each group morphism f. Here η_A and η_B are the two canonical quotient morphisms.

The notation

$$f/\delta$$

is not to be taken seriously.

Since η_A is surjective (epic in ***Grp***) there can be at most one such morphism f/δ. There is such a morphism precisely when

$$\mathrm{Ker}(\eta_A) \subseteq \mathrm{Ker}(\eta_B \circ f)$$

that is

$$\delta A \subseteq \mathrm{Ker}(\eta_B \circ f)$$

that is

$$[x, y] \in \mathrm{Ker}(\eta_B \circ f)$$

for each $x, y \in A$. But we know

$$f([x, y]) = [f(x), f)y)] \in \delta B$$

which gives the required result.

The uniqueness in the diagram ensures that we have a functor. ■

3.3.14 (a) Given an R-set A we require

$$(a \star s) \star t = a \star (st)$$

for $s, t \in S$. But ϕ is a monoid morphism, so that

$$\phi(s)\phi(t) = \phi(st)$$

and hence

$$(a \star s) \star t = (a \cdot \phi(s)) \cdot \phi(t) = a \cdot (\phi(s)\phi(t)) = a \cdot \phi(st) = a \star st$$

as required.

(b) We must show that for each R-morphism

$$A \xrightarrow{f} B$$

the function f is also an S-morphism, that is

$$f(a \star s) = f(a) \star s$$

for each $a \in A$ and $s \in S$. But

$$f(a \star s) = f(a \cdot \phi(s)) = f(a) \cdot \phi(s) = f(a) \star s$$

for the required result. ■

3.3.15 We have an object assignment and an arrow assignment

$$\begin{array}{ccc} \boldsymbol{Mon} & & \boldsymbol{MON} \\ R & \longmapsto & \boldsymbol{Set}\text{-}R \\ \phi & \longmapsto & \Phi \end{array}$$

so it suffices to check that the arrow assignment passes across composition. But for each ***Mon***-arrow ϕ the resulting functor Φ is trivial on objects and arrows, so the requirement is satisfied. ■

3.3.16 In Solution 1.2.7 we set up an inverse pair of translations

$$\boldsymbol{Pfn} \underset{M}{\overset{L}{\rightleftarrows}} \boldsymbol{Set}_{\perp}$$

on both objects and arrows. We now check these each of these is a functor.

We deal first with L, and it is only the passage across composition that requires much thought.

Consider a composible pair of arrows

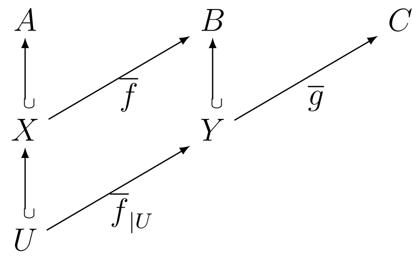
$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

in ***Pfn***. Thus

$$L(g \circ f) = L(g) \circ L(f)$$

is required.

The arrow composite $g \circ f$ is determined by



where X and Y are the respective domains of definition of f and g, and

$$U = \overline{f}^{\leftarrow}(Y)$$

that is

$$a \in U \Longleftrightarrow a \in X \text{ and } \overline{f}(a) \in Y$$

for $a \in A$.

We adjoin a bottom $\perp$ to each of A, B, C, and then set

$$L(f)(a) = \begin{cases} \overline{f}(a) & \text{if } a \in X \\ \perp & \text{if } a \notin X \end{cases}$$

for each $a \in A$. There are similar descriptions of $L(g)$ and $L(g \circ f)$ which we write down in a moment. In all cases $\perp$ is sent to $\perp$.

Observe that

$$L(f)(a) \in Y \iff a \in X \text{ and } \overline{f}(a) \in Y \iff a \in U$$

for each $a \in A$.

We have

$$L(g)(b) = \begin{cases} \overline{g}(b) & \text{if } b \in Y \\ \perp & \text{if } b \notin Y \end{cases}$$

for each $b \in B$. In particular, we have

$$L(g \circ f)(a) = \left\{\begin{matrix} \overline{g} \circ \overline{f}_{|U}(a) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{matrix}\right\} = \begin{cases} \overline{g}(\overline{f}(a)) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{cases}$$

for each $a \in A$.

Finally, for each $a \in A$ we have

$$\begin{aligned} L(g)\big(L(f)(a)\big) &= \left\{\begin{matrix} \overline{g}\big(L(f)(a)\big) & \text{if } L(f)(a) \in Y \\ \perp & \text{if } L(f)(a) \notin Y \end{matrix}\right\} \\ &= \left\{\begin{matrix} \overline{g}\big(\overline{f}(a)\big) & \text{if } a \in X \text{ and } \overline{f}(a) \in Y \\ \perp & \text{if not} \end{matrix}\right\} = \begin{cases} \overline{g}\big(\overline{f}(a)\big) & \text{if } a \in U \\ \perp & \text{if } a \notin U \end{cases} \end{aligned}$$

where the second equality follows by the observation above.

This shows that L passes across composition in the required fashion.

To show that M passes across composition consider a pair of arrows

$$R \xrightarrow{\ \psi\ } S \xrightarrow{\ \phi\ } T$$

in $\boldsymbol{Set}_\perp$. We remove the bottom from each of R, S, T to obtain sets MR, MS, MT and we let

$$M(\psi) = \psi_{|W} \qquad M(\phi) = \phi_{|X}$$

where these domains of definition are given by

$$r \in W \iff \psi(r) \neq \perp \qquad s \in X \iff \phi(s) \neq \perp$$

for $r \in R$ and $s \in S$. Similarly we have

$$M(\phi \circ \psi) = (\phi \circ \psi)_{|U}$$

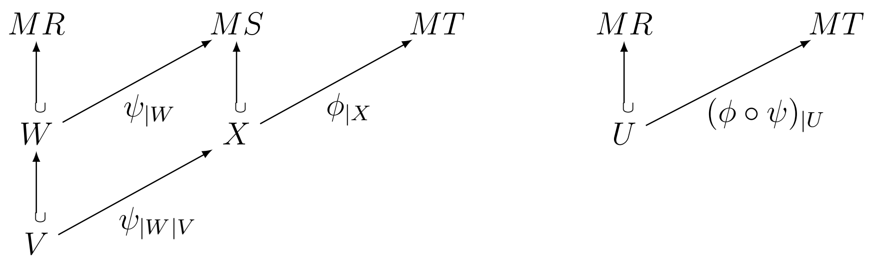
where U is given by

$$r \in U \iff \phi(\psi(r)) \neq \perp \iff \psi(r) \in X$$

for each $r \in R$.

These constructions give us two arrows in $\boldsymbol{Pfn}$.

$$MR \xrightarrow{M(\phi)\circ M(\psi)} MT \qquad\qquad MR \xrightarrow{M(\phi\circ\psi)} MT$$



The left hand one is a composite in $\boldsymbol{Pfn}$, whereas the right hand one is the image of a composite in $\boldsymbol{Set}_\perp$.

The domain of definition V is given by

$$r \in V \iff r \in W \text{ and } \psi_{|W}(r) \in X$$

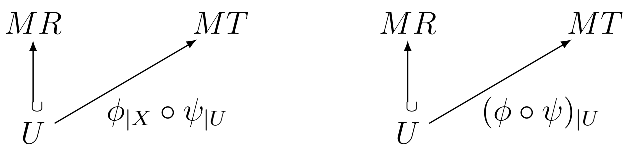
for $r \in R$. Since $\phi(\perp) = \perp$, for each $r \in R$ we have

$$\psi(r) = \perp \Longrightarrow \phi(\psi(r)) = \perp$$

and hence

$$\psi(r) \in X \Longrightarrow r \in W$$

to show that $V = U$. Thus the two arrows are



which, since $U \subseteq X$, show that they are equal. ∎

3.3.17 Of course, this exercise extends the earlier Exercise 2.6.7.

(a) To show that the comparison on $S/{\sim}$ is well-defined suppose

$$[s_1] = [s_1'] \qquad [s_2] = [s_2']$$

for elements $s_1, s_1', s_2, s_2' \in S$. We require

$$s_1 \le s_2 \iff s_2' \le s_2'$$

and clearly, by symmetry, a proof of one of the implications will do.

From the two assumed equalities we have

$$s_1 \sim s_1' \qquad s_2 \sim s_2'$$

and hence

$$s_1 \le s_2 \Longrightarrow s_1' \le s_1 \le s_2 \le s_2' \Longrightarrow s_1' \le s_2'$$

as required.

This shows that

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & S/{\sim} \\ s & \longmapsto & [s] \end{array}$$

is well-defined and, trivially, it is monotone.

(b) Consider a monotone map

$$S \xrightarrow{\ f\ } T$$

from a preset S to a poset T. We check there is a commuting triangle

$$\begin{array}{ccccc} S & & \xrightarrow{\quad f \quad} & & T \\ & \searrow^{\eta_S} & & \nearrow_{f^\sharp} & \\ & & S/{\sim} & & \end{array}$$

for some *unique* monotone map $f^\sharp$.

Since η_S is surjective there is at most one such map $f^\sharp$. Thus it suffices to show that

$$f^\sharp([s]) = f(s)$$

(for $s \in S$) gives a well-defined monotone function.

For $s_1, s_2 \in S$, since f is monotone, we have

$$[s_1] = [s_2] \Longrightarrow s_1 \le s_2 \le s_1 \Longrightarrow f(s_1) \le f(s_2) \le f(s_1) \Longrightarrow f(s_1) = f(s_2)$$

where the last step holds since T is a poset. This shows that f is well-defined. A similar argument shows that f is monotone.

This universal property induces the required functor. For the general argument see Solution 3.3.18. ■

3.3.18 (a) Consider an arbitrary arrow

$$A \xrightarrow{\ f\ } B$$

of ***Src***. We must produce an arrow

$$FA \xrightarrow{F(f)} FB$$

and then check that the two assignments form a functor.

Consider the composite arrow

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\eta_B} (\text{¿} \circ F)B$$

of ***Src***. Applying the universal property to this arrow gives a commuting square

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ (\text{¿} \circ F)A & \xrightarrow[\text{¿}(!)]{} & (\text{¿} \circ F)B \end{array}$$

for some unique arrow

$$FA \xrightarrow{\quad ! \quad} FB$$

of ***Trg***. We take this arrow for $F(f)$. Thus

$$F(f) = (\eta_B \circ f)^\sharp$$

in terms of the given notation.

To show that F passes across composition consider a pair of arrows

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

of ***Src*** together with the three commuting squares

$$\begin{array}{ccccc} A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_C} \\ (\text{¿} \circ F)A & \xrightarrow[\text{¿}(F(f))]{} & (\text{¿} \circ F)B & \xrightarrow[\text{¿}(F(g))]{} & (\text{¿} \circ F)C \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\ g \circ f\ } & C \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_C} \\ (\text{¿} \circ F)A & \xrightarrow[\text{¿}(F(g \circ f))]{} & (\text{¿} \circ F)C \end{array}$$

which determine $F(f), F(g)$ and $F(g \circ f)$. From the two left hand squares we see that

$$\begin{array}{ccc} A & \xrightarrow{\quad g \circ f \quad} & C \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ (\text{¿} \circ F)A & \xrightarrow[\text{¿}\left(F(g) \circ F(f)\right)]{} & (\text{¿} \circ F)B \end{array}$$

commutes, and hence

$$F(g \circ f) = F(g) \circ F(f)$$

by the given uniqueness in the original construction.

The required identity property is almost trivial.

(b) For each arrow

$$FA \xrightarrow{\ g\ } S$$

of ***Trg*** let

$$A \xrightarrow{\ g_\flat\ } \text{¿}S$$

be the composite

$$A \xrightarrow{\ \eta_A\ } (\text{¿} \circ F)A \xrightarrow{\ \text{¿}(g)\ } \text{¿}S$$

of ***Src***.

Trivially, the triangle

$$\begin{array}{ccc} A & \xrightarrow{g_\flat} & \wr S \\ & \searrow^{\eta_A} \quad \nearrow_{\wr(g)} & \\ & (\wr \circ F)A & \end{array}$$

commutes, to show that

$$g_\flat{}^\sharp = g$$

by the uniqueness property of the $(\cdot)^\sharp$ construction.

Similarly, for each arrow

$$A \xrightarrow{\ f\ } \wr S$$

of ***Src*** we have

$$f^\sharp{}_\flat = f$$

by the given commuting triangle.

This sets up an inverse pair

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Src}[A, \wr S] & & \boldsymbol{Trg}[FA, S] \\ g_\flat & \longleftarrow\!\!\dashv & g \end{array}$$

of bijections, and it is not too hard to show that each is natural for variations of A and S. ■

C.4 Natural transformations defined

3.4.1 (a) Consider the category

$$(\downarrow) \qquad \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}$$

where the two identity arrows have been omitted from the picture. Let $\boldsymbol{C}$ be an arbitrary category. A covariant functor

$$(\downarrow) \longrightarrow \boldsymbol{C}$$

must select two objects A_0, A_1 of $\boldsymbol{C}$ and an arrow

$$\begin{array}{c} A_0 \\ \big\downarrow \alpha \\ A_1 \end{array}$$

between them. (It also selects the identity arrows on these two objects, but that is not going to cause a problem.) Notice that there are no non-trivial composition properties to worry about here. Thus such a functor is precisely an object of $\boldsymbol{C}^\downarrow$.

Consider two such functors

$$\begin{array}{ccc} A_0 & & B_0 \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ A_1 & & B_1 \end{array}$$

that is, two objects of $\boldsymbol{C}^{\downarrow}$. There are just two source objects, namely 0 and 1, so a natural transformation between these functors must select two arrows

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ A_1 & \xrightarrow[f_1]{} & B_1 \end{array}$$

of $\boldsymbol{C}$. The naturality requires that the square

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ A_1 & \xrightarrow[f_1]{} & B_1 \end{array}$$

commutes. In other words, a natural transformation is just an arrow of $\boldsymbol{C}^{\downarrow}$.

(b) If you re-read Solution 1.3.10 you will find that the objects of $\boldsymbol{C}^{\nabla}$ are esentially the 'functors' from the graph $\nabla = (N, E)$, and the arrows of $\boldsymbol{C}^{\nabla}$ are essentially the 'natural transformations' between these 'functors'. The only problem is that ∇ is not a category and there are no composition requirements. Don't worry about that. Compsition can be dealt with later. ■

3.4.2 A presheaf on S is a contravariant functor

$$S \longrightarrow \boldsymbol{Set}$$

and an arrow between presheaves is a natural transformation between these functors. ■

3.4.3 We look at the contravariant case.

Let A and B be a pair of (right) R-sets viewed as functors from the 1-object category R. Since R has just one object a natural transformation will be a single function

$$A \xrightarrow{\ f\ } B$$

subject to certain conditions. For each arrow r of R (element of R) the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha_r}\big\uparrow & & \big\uparrow{\scriptstyle\beta_r} \\ A & \xrightarrow[f]{} & B \end{array}$$

must commute. Here α_r and β_r are the 1-placed operations on A and B selected by r. Thus

$$f \circ \alpha_r = \beta_r \circ f$$

is the required condition. In terms of elements and actions this is

$$f(ar) = f(\alpha_r(a)) = (f \circ \alpha_r)(a) = (\beta_r \circ f)(a) = \beta_r(f(a)) = f(a)r$$

so that f is just an arrow of ***Set***-R. ■

3.4.4 We must show that for each Src-arrow

$$A \xrightarrow{\ f\ } B$$

the induced Trg-square

$$\begin{array}{ccc} GA & \xrightarrow{\sigma_A} & FA \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ GB & \xrightarrow[\sigma_B]{} & Fb \end{array}$$

commutes. To do that consider the following diagram.

$$\begin{array}{ccccc} FA & \xrightarrow{\tau_A} & GA & \xrightarrow{\sigma_A} & FA \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} & & \downarrow{\scriptstyle F(f)} \\ FB & \xrightarrow[\tau_B]{} & GB & \xrightarrow[\sigma_B]{} & FB \end{array}$$

The left hand square does commute, since we are given that $\tau_\bullet$ is natural.

The outer cell commutes since we are given

$$\sigma_\bullet \circ \tau_\bullet = \boldsymbol{id}_\bullet$$

and hence both trips from FA to FB are equal to $F(f)$.

But now

$$F(f) \circ \sigma_A \circ \tau_A = \sigma_B \circ \tau_B \circ F(f) = \sigma_B \circ G(f) \circ \tau_A$$

and hence since

$$\tau_\bullet \circ \sigma_\bullet = \boldsymbol{id}_\bullet$$

we have

$$F(f) \circ \sigma_A = F(f) \circ \sigma_A \circ \tau_A \circ \sigma_A = \sigma_B \circ G(f) \circ \tau_A \circ \sigma_A = \sigma_B \circ G(f)$$

for the required result. ■

C.5 Examples of natural transformations

3.5.1 Given an arrow

$$L \xrightarrow{\;k\;} K$$

we certainly have a function

$$[A, L] \xrightarrow{\;k \circ -\;} [A, K]$$

for each object A. For this to be natural we require the square

$$\begin{array}{ccc} A & \qquad & \begin{array}{ccc} [A, L] & \xrightarrow{\;k \circ -\;} & [A, K] \\ {\scriptstyle - \circ f}\downarrow & & \downarrow{\scriptstyle - \circ f} \\ [B, L] & \xrightarrow[\;k \circ -\;]{} & [B, K] \end{array} \\ f\uparrow & & \\ B & & \end{array}$$

to commute for each arrow f. This is trivially satisfied. ∎

3.5.2 Each of F and G is a composite of contravariant functors, and hence each is a covariant functor.

We need an explicit description of the behaviour on arrows.

Consider an arrow

$$A \xrightarrow{\;f\;} B$$

of $\boldsymbol{C}$. This must produce a function

$$\boldsymbol{Set}[\boldsymbol{C}[A, P], R] \xrightarrow{\;F(f)\;} \boldsymbol{Set}[\boldsymbol{C}[B, P], R]$$

mapping functions to functions. In other words, for each input function

$$\boldsymbol{C}[A, P] \xrightarrow{\;l\;} R$$

an output function

$$\boldsymbol{C}[B, P] \longrightarrow R$$

is required. Thus we set

$$F(f)(l)(b) = l(b \circ f)$$

for each *arrow*

$$B \xrightarrow{\;b\;} P$$

of $\boldsymbol{C}$.

There is a similar description of G.

For the natural transformation we require a function

$$\boldsymbol{Set}[\boldsymbol{C}[A,P],R] \xrightarrow{\tau_A} \boldsymbol{Set}[\boldsymbol{C}[A,Q],S]$$

for each object A of $\boldsymbol{C}$. In other words, for each input function

$$\boldsymbol{C}[A,P] \xrightarrow{l} R$$

an output function

$$\boldsymbol{C}[A,Q] \longrightarrow S$$

is required. Thus we set

$$\tau_A(l)(a) = (s \circ l)(p \circ a)$$

for each *arrow*

$$A \xrightarrow{a} Q$$

of $\boldsymbol{C}$.

Finally, we show that the square

$$\begin{array}{ccc} A & \quad & FA \xrightarrow{\tau_A} GA \\ f\big\uparrow & & F(f)\big\downarrow \qquad \big\downarrow G(f) \\ B & & FB \xrightarrow[\tau_B]{} GB \end{array}$$

commutes for each arrow f of $\boldsymbol{C}$. In other words we require

$$\big(\tau_B \circ F(f)\big)(l) = \big(G(f) \circ \tau)A\big)(l)$$

for each member

$$\boldsymbol{C}[A,P] \xrightarrow{l} R$$

of FA. Thus we require

$$\big(\tau_B \circ F(f)\big)(l)(b) = \big(G(f) \circ \tau)A\big)(l)(b)$$

for each *arrow*

$$B \xrightarrow{b} P$$

of $\boldsymbol{C}$.

But we have

$$\begin{aligned} \big(\tau_B \circ F(f)\big)(l)(b) &= \tau_B\big(F(f)(l)\big)(b) \\ &= \big(s \circ F(f)(l)\big)(p \circ b) \\ &= s\big(F(f)(l)(p \circ b)\big) \\ &= s\big((l(p \circ b \circ f)\big) \qquad = (s \circ l)(p \circ b \circ f) \end{aligned}$$

and

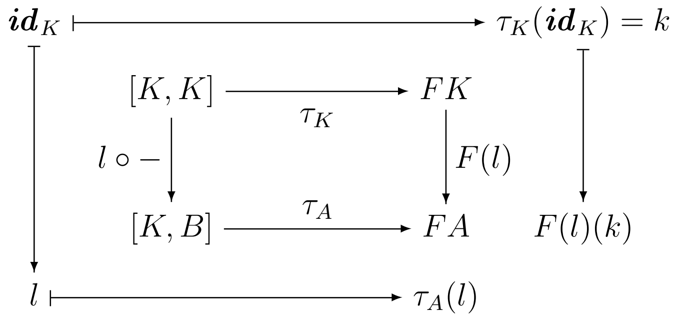
$$\big(G(f) \circ \tau_A\big)(l)(b) = G(f)\big(\tau_A(l)\big)(b) = \tau_A(l)(b \circ f) = (s \circ l)(p \circ b \circ f)$$

to give the required result. ■

3.5.3 It remains to check the last bit of part (b), namely that

$$\tau_A(l) = F(l)(k)$$

holds. To do that we track $\boldsymbol{id}_K$ round both sides of the given commuting square



for the required result. ■

3.5.4 The particular case where the category $\boldsymbol{C}$ is a poset is dealt with in Example 1.12. ■

3.5.5 This is the contravariant version of the first part of Example 3.7. ■

3.5.6 This is the contravariant version of the second part of Example 3.7. ■

3.5.7 For each set A we set

$$\eta^{\exists}_A(a) = \{a\} \qquad \eta^{\forall}_A(a) = \{a\}'$$

for each $a \in A$. ■

3.5.8 We require

$$q = p \circ f$$

but since each of these functions has target $\mathsf{2}$ is suffices to show they output 1 at the same inputs. For each $b \in B$ we have

$$\begin{aligned} q(b) = 1 &\iff \chi_B(f^l a(X))(b) = 1 \\ &\iff b \in f^{\leftarrow}(X) \\ &\iff f(b) \in X \\ &\iff \chi_A(X)(f)(b) = 1 \qquad \iff (p \circ f)(b) = 1 \end{aligned}$$

for the required result. ■

3.5.9 (a) From the diagram in Example 3.10 we require

$$\eta_B(f(a)) = \Pi(f)(\eta_A(a))$$

for each $a \in A$. Each side of this equality is a member of $a\mathcal{P}^2 B$. Thus, for $Y \in \mathcal{P}B$, we have

$$\begin{aligned} Y \in \Pi(f)(\eta_A(a)) &\iff f^{\leftarrow}(Y) \in \eta_A(a) \\ &\iff a \in f^{\leftarrow}(Y) \\ &\iff f(a) \in Y \qquad\qquad \iff Y \in \eta_B(f(a)) \end{aligned}$$

to give the required equation.

(b) As in the question let I be the inverse power set endofunctor on $\boldsymbol{Set}$, and also let J be $\boldsymbol{Set}[-,2]$, the endo-hom-functor on $\boldsymbol{Set}$.

We know that the arrow behaviour of J is by composition. Thus we have

$$\begin{array}{c} A \\ \Big\downarrow f \\ B \end{array} \qquad \begin{array}{c} \mathsf{J}A \\ \Big\uparrow \mathsf{J}(f) \\ \mathsf{J}B \end{array} \quad \begin{array}{c} q \circ f \\ \Big\uparrow \\ q \end{array} \qquad \begin{array}{c} \mathsf{J}^2A \\ \Big\downarrow \mathsf{J}^2(f) \\ \mathsf{J}^2B \end{array} \quad \begin{array}{c} \pi \\ \Big\downarrow \\ \pi \circ \mathsf{J}(f) \end{array}$$

for each arrow f of $\boldsymbol{Set}$, each $q : B \longrightarrow 2$, and each $\pi : [A, 2] \longrightarrow 2$. In particular, we have

$$\mathsf{J}^2(f)(\pi)(q) = \pi(q \circ f)$$

for each such f, π, q.

For each set A consider the 'evaluation' function

$$A \xrightarrow{\ \phi_A\ } \mathsf{J}^2 A$$

given by

$$\phi_A(a)(p) = p(a)$$

for each $p : A \longrightarrow 2$ and $a \in A$. We show this is a natural transformation. In other words we show that the square

$$\begin{array}{ccc} A & \xrightarrow{\ \phi_A\ } & \mathsf{J}^2A \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{J}^2(f)} \\ B & \xrightarrow[\ \phi_B\ ]{} & \mathsf{J}^2B \end{array} \qquad\qquad \phi_B \circ f = \mathsf{J}^2(f) \circ \phi_A$$

commutes. For each $a \in A$ and $q : B \longrightarrow 2$ we have

$$(\phi_B \circ f)(a)(q) = \phi_B(f(a))(q) = q(f(a)) = (q \circ f)(a)$$

and

$$\big(\mathsf{J}^2(f) \circ \phi_A\big)(a)(q) = \mathsf{J}^2(f)\big(\phi_A(a)\big)(q) = \phi_A(a)(q \circ f) = (q \circ f)(a)$$

to give the required result.

(c) On the whole the use of characters rather than subsets does lead to neater results.

The statement in part (b) of the question that 'I^2 is naturally isomorphic to J^2' is a bit glib. It is true, but not entirely obvious. It can be justified using horizontal and vertical composition of natural transformations. This is not something we can go into here, but we can give a hint of what it is about.

We know we have an inverse pair of natural transformations

$$\mathsf{I} \underset{\xi_\bullet}{\overset{\chi_\bullet}{\rightleftarrows}} \mathsf{J}$$

given by

$$\begin{array}{ll} \chi_A(X)(a) = 1 \iff a \in X & a \in \xi_A(p) \iff p(a) = 1 \\ \chi_A(X)(a) = 0 \iff a \notin X & a \notin \xi_A(p) \iff p(a) = 0 \end{array}$$

for each set $A, X \in \mathcal{P}A$, $p : A \longrightarrow \mathsf{2}$, and $a \in A$. We modify these in several ways.

For each set A we may hit each of the arrows

$$\mathsf{I}A \xrightarrow{\chi_A} \mathsf{J}A \qquad\qquad \mathsf{I}A \xleftarrow{\xi_A} \mathsf{J}A$$

with each of the two contravariant functors I and J, and take particular instances (by replacing the set A).

$$\begin{array}{ll} \mathsf{I}^2A \xleftarrow{\mathsf{I}(\chi_A)} (\mathsf{I}\circ\mathsf{J})A & (\mathsf{J}\circ\mathsf{I})A \xrightarrow{\mathsf{J}(\xi_A)} \mathsf{J}^2A \\ \mathsf{I}^2A \xrightarrow{\chi_{\mathsf{I}A}} (\mathsf{J}\circ\mathsf{I})A & \mathsf{I}^2A \xleftarrow{\xi_{\mathsf{I}A}} (\mathsf{J}\circ\mathsf{I})A \\ (\mathsf{I}\circ\mathsf{J})A \xrightarrow{\chi_{\mathsf{J}A}} \mathsf{J}^2A & (\mathsf{I}\circ\mathsf{J})A \xleftarrow{\xi_{\mathsf{J}A}} \mathsf{J}^2A \end{array}$$

We can combine these in various ways. In particular, we may form

$$\mathsf{I}^2 \underset{\Delta_A = \mathsf{I}(\chi_A)\circ\xi_{\mathsf{J}A}}{\overset{\Gamma_A = \mathsf{J}(\xi_A)\circ\chi_{\mathsf{I}A}}{\rightleftarrows}} \mathsf{J}^2$$

going via $(\mathsf{J}\circ\mathsf{I})A$ for Γ and $(\mathsf{I}\circ\mathsf{J})A$ for Δ. We show that these are natural transformations, and are inverses.

For an arbitrary arrow

$$A \xrightarrow{f} B$$

consider the following two commuting squares.

$$\begin{array}{ccc} \mathsf{I}^2A & \xrightarrow{\chi_{\mathsf{I}A}} & (\mathsf{J}\circ\mathsf{I})A \\ {\scriptstyle \mathsf{I}^2(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{J}(\mathsf{I}(f))} \\ \mathsf{I}^2B & \xrightarrow[\chi_{\mathsf{I}B}]{} & (\mathsf{J}\circ\mathsf{I})B \end{array} \qquad \begin{array}{ccc} \mathsf{I}A & \xrightarrow{\xi_A} & \mathsf{J}A \\ {\scriptstyle \mathsf{I}(f)}\downarrow & & \downarrow{\scriptstyle \mathsf{J}(f)} \\ \mathsf{I}B & \xrightarrow[\xi_B]{} & \mathsf{J}B \end{array}$$

The right hand square is an instance of the naturality of $\xi_\bullet$ across the arrow f. The left hand square is an instance of the naturality of $\chi_\bullet$ across the arrow $\mathsf{I}(f)$.

After hitting the right hand square with J we obtain a commuting diagram

$$\begin{array}{ccccc}
\mathsf{I}^2A & \xrightarrow{\chi_{\mathsf{I}A}} & (\mathsf{J}\circ\mathsf{I})A & \xrightarrow{\mathsf{J}(\xi_A)} & \mathsf{J}^2A \\
\downarrow{\scriptstyle \mathsf{I}^2(f)} & & \downarrow{\scriptstyle (\mathsf{J}\circ\mathsf{I})(f)} & & \downarrow{\scriptstyle \mathsf{J}(f)} \\
\mathsf{I}^2B & \xrightarrow[\chi_{\mathsf{I}B}]{} & (\mathsf{J}\circ\mathsf{I})B & \xrightarrow[\mathsf{J}(\xi_B)]{} & \mathsf{J}^2B
\end{array}$$

to show that $\Gamma_\bullet$ is natural.

A similar argument shows that $\Delta_\bullet$ is natural.

To show that

$$\mathsf{I}^2 \xrightarrow{\Delta_A\circ\Gamma_A} \mathsf{I}^2$$

is the identity consider any

$$\mathcal{X}\in\mathsf{I}^2A=\mathcal{P}^2A \qquad X\in\mathsf{I}A=\mathcal{P}A$$

and let

$$Y=\chi_A(X)$$

to use in the calculation. We have

$$\begin{aligned}
X\in\big(\Delta_A\circ\Gamma_A\big)(\mathcal{X}) &\iff X\in\big(\mathsf{I}(\chi_A)\circ\xi_{\mathsf{J}A}\circ\Gamma_A\big)(\mathcal{X})\\
&\iff X\in\mathsf{I}(\chi_A)\Big(\big(\xi_{\mathsf{J}A}\circ\Gamma_A\big)(\mathcal{X})\Big)\\
&\iff X\in\chi_A^{\leftarrow}\Big(\big(\xi_{\mathsf{J}A}\circ\Gamma_A\big)(\mathcal{X})\Big)\\
&\iff Y=\chi_A(X)\in\big(\xi_{\mathsf{J}A}\circ\Gamma_A\big)(\mathcal{X})\\
&\iff Y\in\xi_{\mathsf{J}_A}\big(\Gamma_A(\mathcal{X})\big)\\
&\iff \big(\Gamma_A(\mathcal{X})\big)(Y)=1\\
&\iff \big(\mathsf{J}(\xi_A)\circ\chi_{\mathsf{J}A}\big)(\mathcal{X})(Y)=1\\
&\iff \mathsf{J}(\xi_A)\big(\chi_{\mathsf{J}A}(\mathcal{X})\big)(Y)=1\\
&\iff \big(\chi_{\mathsf{J}A}(\mathcal{X})\circ\xi_A\big)(Y)=1\\
&\iff \chi_{\mathsf{J}A}(\mathcal{X})\big(\xi_A(Y)\big)=1\\
&\iff \xi_A(Y)\in\mathcal{X} \qquad\qquad \iff \big(\xi_A\circ\chi_A\big)(X)\in\mathcal{X}
\end{aligned}$$

and now we remember that

$$\big(\xi_A\circ\chi_A\big)(X)=X$$

to give

$$\big(\Delta_A\circ\Gamma_A\big)(\mathcal{X})=\mathcal{X}$$

as required.

A similar argument deals with $\Gamma_A\circ\Delta_A$.

In the same way we can show how $\eta_\bullet$ and $\phi_\bullet$ are related.

Consider the following composite.

$$A \xrightarrow{\ \phi_A\ } \mathsf{J}^2A \xrightarrow{\ \Delta_A\ } \mathsf{I}^2A$$

For each $a \in A$ and $X \in \mathsf{I}A$ we have

$$\begin{aligned} X \in (\Delta_A \circ \phi_A)(a) &\iff X \in \big(\mathsf{I}(\chi_A) \circ \xi_{\mathsf{J}A} \circ \phi_A\big)(a) \\ &\iff X \in \mathsf{I}(\chi_A)\big((\xi_{\mathsf{J}A} \circ \phi_A)(a)\big) \\ &\iff \chi_A(X) \in (\xi_{\mathsf{J}A} \circ \phi_A)(a) \\ &\iff \chi_A(X) \in \xi_{\mathsf{J}A}\big(\phi_A(a)\big) \\ &\iff \phi_A(a)\big(\chi_A(X)\big) = 1 \\ &\iff \chi_A(X)(a) = 1 \\ &\iff a \in X \qquad\qquad\qquad \iff X \in \eta_A(a) \end{aligned}$$

and hence

$$\Delta_A \circ \phi_A = \eta_A$$

to show how $\phi_\bullet$ determines $\eta_\bullet$. A similar argument gives

$$\Gamma_A \circ \eta_A = \phi_A$$

and hence $\phi_\bullet$ and $\eta_\bullet$ are equivalent. ■

3.5.10 *To be done* ■

3.5.11 *To be done* ■

3.5.12 By Exercises 3.3.7 we have a bijection

$$\mathcal{O}S \xrightarrow{\ \chi_S\ } \Xi S$$

for each space S. It suffices to show that χ_S is natural for variation of S.

To this end consider the square

$$\begin{array}{ccccc} S & \qquad & \mathcal{O}S & \xrightarrow{\ \chi_S\ } & \Xi S \\ \phi\Big\uparrow & & \Big\downarrow \phi^{\leftarrow} & & \Big\downarrow - \circ \phi \\ T & & \mathcal{O}T & \xrightarrow[\ \chi_T\ ]{} & \Xi T \end{array}$$

induced by a continuous map ϕ, as on the left. We must show that the square commutes. Thus we required

$$\chi_T(\phi^{\leftarrow}(U)) = \chi_S(U) \circ \phi$$

for each $U \in \mathcal{O}S$. These two functions can return only 0 and 1 as a value. For each $s \in S$ we have

$$\chi_T(\phi^{\leftarrow}(U))(s) = 1 \iff s \in \pi^{\leftarrow}(U) \iff \phi(s) \in U \iff \chi_S(\phi(U)) = 1$$

which gives the required equality. ■

3.5.13 Let's consider the more complicated version, that dealt with by Exercise 3.3.8.
For a category $\boldsymbol{C}$ we have two functors

$$\boldsymbol{C}\times\boldsymbol{C} \underset{G}{\overset{F}{\rightrightarrows}} \boldsymbol{C}$$

where

$$F(A_1, A_2) = A_1 \times A_2 \qquad F(A_1, A_2) = A_1$$

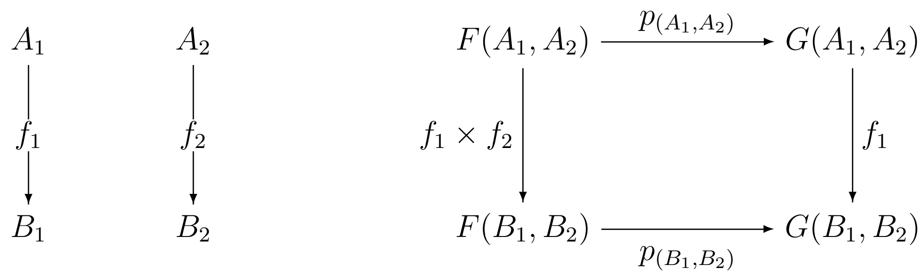
are the two object assignments. (One thing you should be wary of here is the two different uses of '×'. The first use gives the cartesian product of the category $\boldsymbol{C}$ with itself, and the second gives the internal product object in $\boldsymbol{C}$.)

For each pair (A_1, A_2) of objects we have an arrow

$$F(A_1, A_2) \xrightarrow{p_{(A_1,A_2)}} G(A_1, A_2)$$

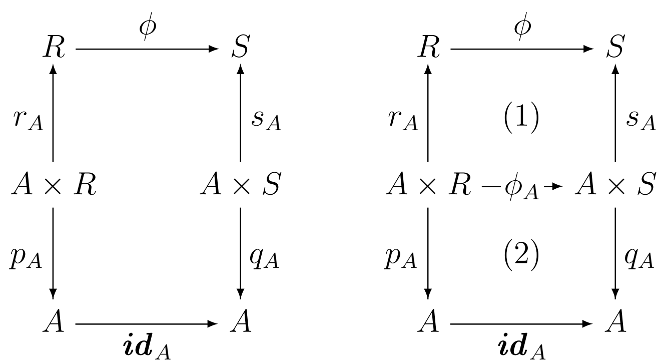
namely the projection arrow. We must show this is natural for variation of (A_1, A_2).

Consider an arrow of $\boldsymbol{C}\times\boldsymbol{C}$, that is a pair of arrows of $\boldsymbol{C}$ as indicated on the left of the following diagram.



We must show that the square on the right commutes. But this is just one of the commuting squares that determines the projection, as in Solution 3.3.8 (in a slightly different notation). ■

3.5.14 We must first set up ϕ_A for an arbitrary object A of $\boldsymbol{C}$. To do that consider the diagram on the left.



This consists of two product wedges with the associated projections $r_A.p_A, s_A, q_A$. The generating arrow ϕ has also been inserted. Using the product property of $A \times S$ we see there is a unique arrow ϕ_A to produce a pair of commuting squares as on the right. This is just $\boldsymbol{id}_A \times \phi$ in product notation. We have labelled the two squares for later use.

We must show that $\phi_\bullet$ is natural for variation of the object. Let's set up that problem.

Consider an arbitrary arrow f as on the left. We must show that the square on the right commutes.

$$\begin{array}{ccccc} A & \qquad & A \times R & \xrightarrow{\phi_A} & A \times S \\ f\big\downarrow & & f \times \boldsymbol{id}_R \big\downarrow & (?) & \big\downarrow f \times \boldsymbol{id}_S \\ B & & B \times R & \xrightarrow[\phi_B]{} & B \times S \end{array}$$

How are we going to do this? Consider the diagram on the left.

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ r_A \big\uparrow & & \big\uparrow s_B \\ A \times R & & B \times S \\ p_A \big\downarrow & & \big\downarrow q_B \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{\phi} & S \\ r_A \big\uparrow & (rs) & \big\uparrow s_B \\ A \times R & \xrightarrow{\psi} & B \times S \\ p_A \big\downarrow & (pq) & \big\downarrow q_B \\ A & \xrightarrow[f]{} & B \end{array}$$

This is not the same as the first diagram. We have now varied the object A along the arrow f. However, in the same way the product property of $B \times S$ ensures there is a unique arrow ψ that makes both squares commute, that is

$$\begin{array}{ll} (rs) & s_B \circ \psi = \phi \circ r_A \\ (pq) & q_B \circ \psi = f \circ p_A \end{array}$$

for some unique arrow ψ. In the square (?) we show that both trips from $A \times R$ to $B \times S$

$$(f \times \boldsymbol{id}_S) \circ \phi_A \qquad \phi_B \times (f \times \boldsymbol{id}_R)$$

satisfy (rs) and (pq), and hence must be equal.

We need a property of the product construction, namely that projections are natural.

$$\begin{array}{ccc} R & \xrightarrow{\boldsymbol{id}_R} & R \\ r_A \big\uparrow & (3) & \big\uparrow r_B \\ A \times R & \xrightarrow{f \times \boldsymbol{id}_R} & B \times R \\ p_A \big\downarrow & (4) & \big\downarrow p_B \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\boldsymbol{id}_S} & S \\ sa_A \big\uparrow & (5) & \big\uparrow s_B \\ A \times R & \xrightarrow{f \times \boldsymbol{id}_S} & B \times S \\ q_A \big\downarrow & (6) & \big\downarrow q_B \\ A & \xrightarrow[f]{} & B \end{array}$$

Thus all four of these squares commute.

We are now ready to do the several small calculations.

With

$$\psi = (f \times \boldsymbol{id}_S) \circ \phi_A$$

a use of $(6, 2)$ and then a use of $(5, 1)$ gives

$$\begin{array}{ll} (rs) & s_B \circ \psi = s_B \circ (f \times \boldsymbol{id}_S) \circ \phi_A = \quad s_A \circ \phi_A \quad = \phi \circ r_A \\ (pq) & q_B \circ \psi = q_B \circ (f \times \boldsymbol{id}_S) \circ \phi_A = f \circ a_A \circ \phi_A = f \circ p_A \end{array}$$

to show that this ψ satisfies the two required conditions.

With

$$\psi = \phi_B \circ (f \times \boldsymbol{id}_R)$$

a use of $(1, 3)$ and then a use of $(2, 4)$, with (3) and (2) in the B version, gives

$$\begin{array}{ll} (rs) & s_B \circ \psi = \ s_B \circ \phi_B \circ (f \times \boldsymbol{id}_R) = \phi \circ r_B \circ (f \times \boldsymbol{id}_R) = \phi \circ r_A \\ (pq) & q_B \circ \psi = q_B \circ \phi_B \circ (f \times \boldsymbol{id}_R) \ = \ \ p_B \circ (f \times \boldsymbol{id}_R) \ \ = f \circ p_A \end{array}$$

to show that this ψ satisfies the two required conditions.

This completes the proof.

This and the next solution are a nice illustration of why reading and writing proofs in category theory can be a bit tricky. Often many small diagrams have to be looked at, and there is a tendency to combine these into one big diagram, and so make it incomprehensible. ■

3.5.15 If we fix two of the three inputs then each L and each R is a composite of various known functors. However, let's see if we can make sense of the 3-placed version.

Let

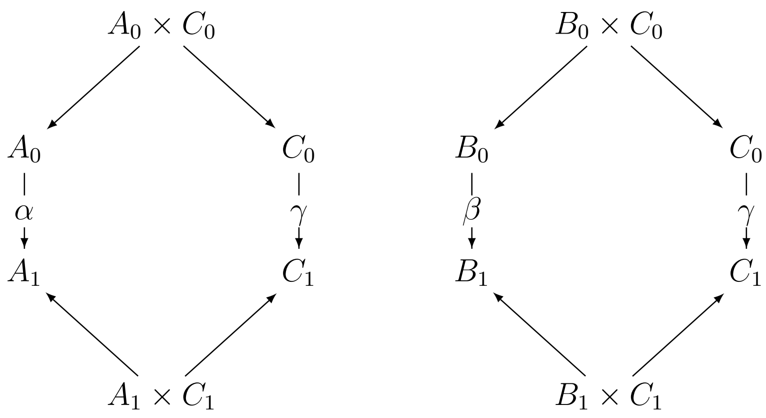
$$\begin{array}{ccc} A_0 & B_0 & C_0 \\ \Big\downarrow \alpha & \Big\downarrow \beta & \Big\downarrow \gamma \\ A_1 & B_1 & C_1 \end{array}$$

be a triple of arrows in $\boldsymbol{C}$. What are the resulting arrows

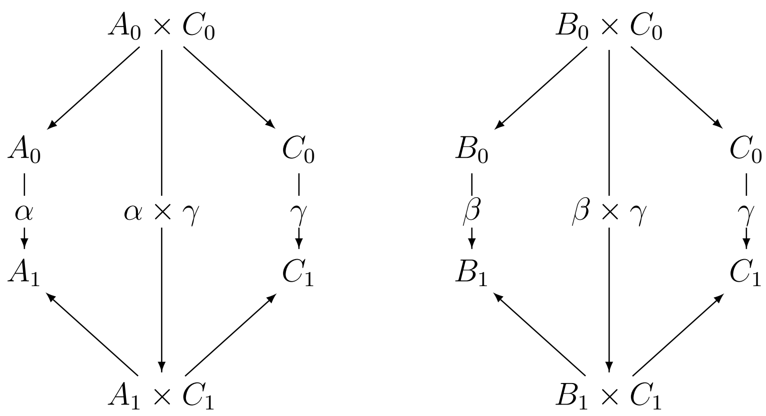
$$\begin{array}{cc} L(A_0, B_0, C_0) & R(A_0, B_0, C_0) \\ \Big\downarrow L(\alpha, \beta, \gamma) & \Big\downarrow R(\alpha, \beta, \gamma) \\ L(A_1, B_1, C_1) & R(A_1, B_1, C_1) \end{array}$$

in $\boldsymbol{C}$.

We look at L first. Consider the cells

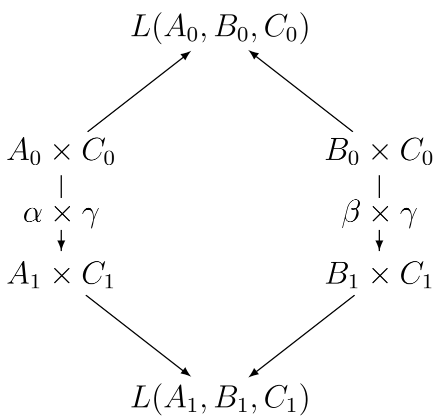


where the unnamed arrows are the projections. The product property provides two unique arrows

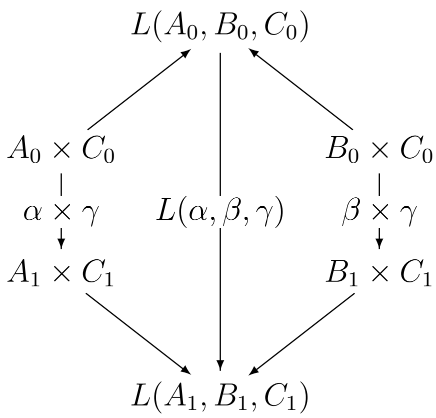


to makes the diagrams commute. This is just the functorial property of the binary product.

Now consider the cell

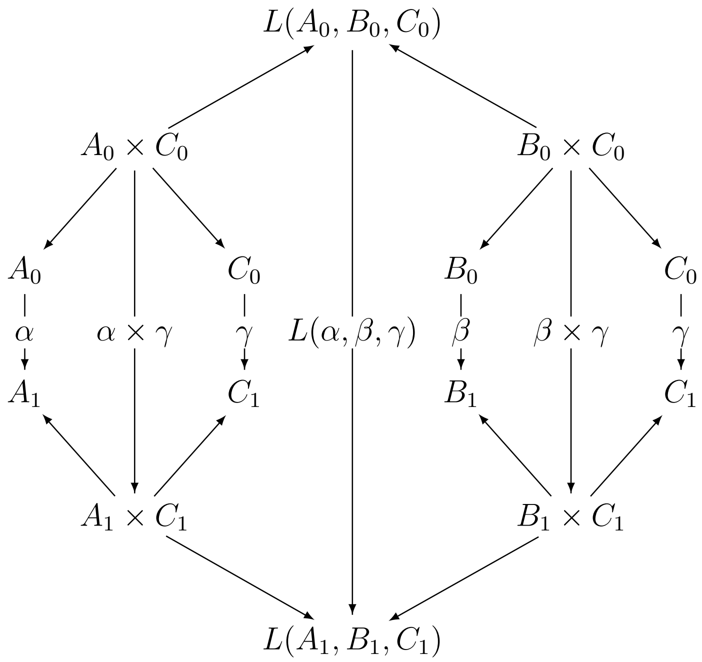


where the unnamed arrows are the insertions. The coproduct property provides a unique arrow



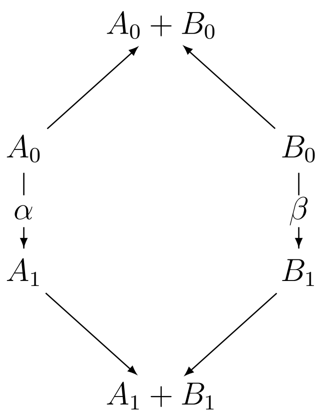
for which the diagram commutes.

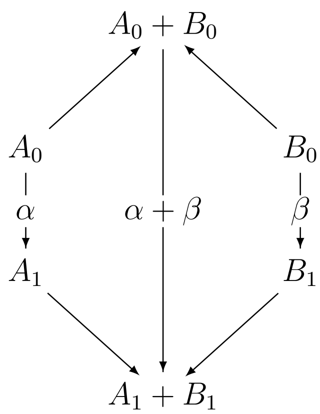
Here is the full diagram.



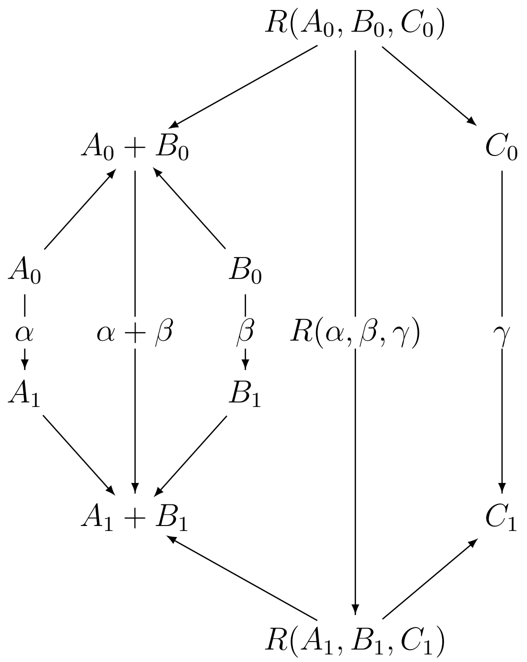
This shows how the arrow $L(\alpha, \beta, \gamma)$ is obtained. The functorial property is induced by the uniqueness of $\alpha \times \beta, \beta \times \gamma$, and $L(\alpha, \beta, \gamma)$.

The behaviour of R on arrows is obtained in a similar way. Starting from three arrows α, β, γ, as above, we use the cell on the left





to obtain the unique arrow $\alpha+\beta$ and commuting diagram on the right. Here the unnamed arrows are the insertions. We now introduce the arrow γ to obtain a unique arrow $R(\alpha,\beta,\gamma)$ and commuting diagram



where the new unnamed arrows are projections. The various uniquenesses ensure that R passes across composition in the appropriate manner to be a functor.

For the next part let

$$L = L(A,B,C) \qquad R = R(A,B,C)$$

for some arbitrary objects A, B, C. We produce an arrow

$$L \longrightarrow R$$

which, in due course, we show is natural for variation of the three objects.

So far we have managed without naming the various projections and insertion, but now we have to. Thus let

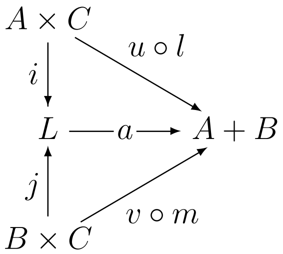
$$\begin{array}{ll}
A\times C \xrightarrow{\ l\ } A \xrightarrow{\ u\ } A+B & \qquad A\times C \xrightarrow{\ r\ } C \\
B\times C \xrightarrow{\ m\ } B \xrightarrow{\ v\ } A+B & \qquad B\times C \xrightarrow{\ s\ } C \\
A\times C \xrightarrow{\ i\ } L & \qquad R \xrightarrow{\ p\ } A+B \\
B\times C \xrightarrow{\ j\ } L & \qquad R \xrightarrow{\ q\ } C
\end{array}$$

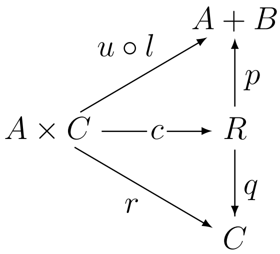
be these various arrows.

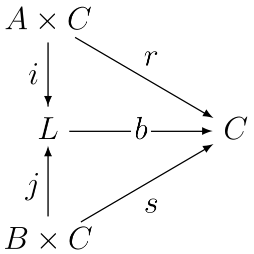
The

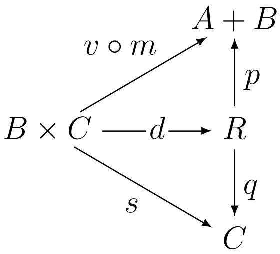
coproduct property of L product property of R

produce unique arrows a, b, c, d such that





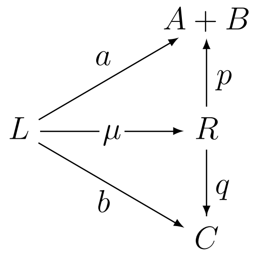


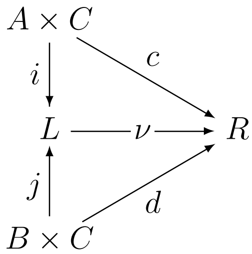


commute. With these the

product property of R coproduct property of L

produce unique arrows μ, ν such that





commute. We first show that $\mu = \nu$.

Using the characterizing properties of μ of ν it suffices to show that either

$$(\mu a) \quad p \circ \nu = a \qquad (\nu c) \quad \mu \circ i = c$$

or

$$(\mu b) \quad q \circ \nu = b \qquad (\nu d) \quad \mu \circ j = d$$

for then

$$\nu = \mu \quad \text{or} \quad \mu = \nu$$

respectively. For these, using characterizing properties of a and b or c and d it suffices to show

$$(\mu ai) \quad p \circ \nu \circ i = u \circ l \qquad (\nu cp) \quad p \circ \mu \circ i = u \circ l$$
$$(\mu aj) \quad p \circ \nu \circ j = v \circ m \qquad (\nu cq) \quad q \circ \mu \circ i = r$$

or

$$(\mu bi) \quad q \circ \nu \circ i = r \qquad (\nu dp) \quad p \circ \mu \circ j = v \circ m$$
$$(\mu bj) \quad q \circ \nu \circ j = s \qquad (\nu dq) \quad q \circ \mu \circ j = s$$

respectively. All of these follows by the previous six diagrams. For instance

$$p \circ \nu \circ i = p \circ c = u \circ l$$

gives (μai).

Let us write μ for this arrow. It remains to show that μ is natural for variation of A, B, C.

To do that consider three arrows α, β, γ, as above. Let

$$\lambda = L(\alpha, \beta, \gamma) \qquad \rho = R(\alpha, \beta, \gamma)$$

so that we must show that the square

$$\begin{array}{ccc} L(A_0, B_0, C_0) & \xrightarrow{\mu_0} & R(A_0, B_0, C_0) \\ \lambda \downarrow & & \downarrow \rho \\ L(A_1, B_1, C_1) & \xrightarrow[\mu_1]{} & R(A_1, B_1, C_1) \end{array}$$

commutes. Here μ_0 and μ_1 are the two version of μ for the triple of objects indicated by the index. We also use the various projections and insertions for the two triples with a indexed version of the notation above.

To show

$$\mu_1 \circ \lambda = \rho \circ \mu_0$$

we invoke the coproduct property of $L(0)$ to observe that the pair of equalities

$$\begin{aligned} \mu_1 \circ \lambda \circ i_0 &= \rho \circ \mu_0 \circ i_0 \\ \mu_1 \circ \lambda \circ j_0 &= \rho \circ \mu_0 \circ j_0 \end{aligned}$$

will suffice. To prove these we invoke the product property of $R(1)$ to observe that the four equalities

$$\begin{aligned} p_1 \circ \mu_1 \circ \lambda \circ i_0 &= p_1 \circ \rho \circ \mu_0 \circ i_0 \\ q_1 \circ \mu_1 \circ \lambda \circ i_0 &= q_1 \circ \rho \circ \mu_0 \circ i_0 \\ p_1 \circ \mu_1 \circ \lambda \circ j_0 &= p_1 \circ \rho \circ \mu_0 \circ j_0 \\ q_1 \circ \mu_1 \circ \lambda \circ j_0 &= q_1 \circ \rho \circ \mu_0 \circ j_0 \end{aligned}$$

will suffice. All four of these are proved in the same way. Let's look at the proof of the first.

Using various commuting cells and remembering that $\mu_1 = \nu_1$ we have

$$\begin{aligned} p_1 \circ \mu_1 \circ \lambda \circ i_0 &= p_1 \circ \mu_1 \circ i_1 \circ (\alpha \times \gamma) = p_1 \circ c_1 \circ (\alpha \times \gamma) = u_1 \circ l_1 \circ (\alpha \times \gamma) \\ p_1 \circ \rho \circ \mu_0 \circ i_0 &= (\alpha + \beta) \circ p_0 \circ \mu_0 \circ i_0 = (\alpha + \beta) \circ a_0 \circ i_0 = (\alpha + \beta) \circ u_0 \circ l_0 \end{aligned}$$

and hence it suffices to show that the diagram

$$\begin{array}{ccccc} A_0 \times C_0 & \xrightarrow{\alpha \times \gamma} & A_1 \times C_1 & \xrightarrow{l_1} & A_1 \\ l_0 \downarrow & & & & \downarrow u_1 \\ A_0 & \xrightarrow[u_0]{} & A_0 + B_0 & \xrightarrow[\alpha + \beta]{} & A_1 + B_1 \end{array}$$

commutes. To do this simply observe that the arrow α, as an upwards diagonal, makes the two resulting cells commute. ■

3.5.16 If you are a bit confused it's probably because you have forgotten the forgetful functor.

Let ***Sgp*** and ***Mon*** be the the categories of semigroups and monoids. We are concerned with two functors

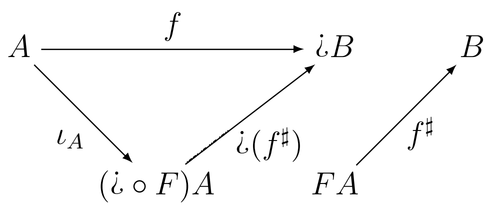
$$\boldsymbol{Sgp} \underset{¿}{\overset{F}{\rightleftarrows}} \boldsymbol{Mon}$$

where ¿ is the given forgetful functor and F is the functor we are trying to produce. (Technically, F is a left adjoint to ¿, but that's for later.) Let's insert ¿ where it should appear.

(c) For each ***Sgp*** arrow

$$A \xrightarrow{\ f\ } ¿B$$

where B is monoid, there is a commuting triangle



for some unique ***Mon*** arrow $f^\sharp$, as indicated. There is only one possibility for $f^\sharp$.

$$f^\sharp{}_{|A} = f \qquad f^\sharp(\omega) = \text{unit of } B$$

The rest is now standard category theory where semigroups and monoids need not be mentioned.

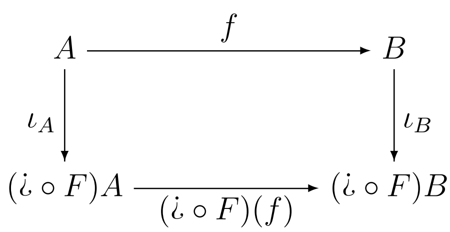
(d) For each ***Sgp*** arrow

$$A \xrightarrow{\ f\ } B$$

there is a unique ***Mon*** arrow

$$FA \xrightarrow{F(f)} FB$$

such that



commutes. This follows from part (c) by setting

$$F(f) = (\iota_B \circ f)^\sharp$$

for the given arrow f. a By consider a composite

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in ***Sgp*** with the induced commuting squares and remembering the uniqueness of $F(\cdot)$, we see that

$$F(g) \circ F(f) = F(g \circ f)$$

which more or less shows that F is a functor.

(e) The commuting square above shows the naturality of ι.

(f) If A already has a unit then this is forgotten and a new unit is adjoined. The two are *not* coalesced. ■

3.5.17 (a) Concatenation is associative, but not commutative. The unit is the empty word.

(b) The functor

$$\textbf{\textit{Set}} \xrightarrow{F} \textbf{\textit{Mon}}$$

goes from the category of sets to the category of monoids.

We need to describe the action

$$A \xrightarrow{f} B \qquad \longmapsto \qquad FA \xrightarrow{F(f)} FB$$

on arrows. Given a ***Set*** arrow f, a function between sets, as above let

$$F(f) : FA \longrightarrow GB$$

be the function given by

$$F(f)(\mathsf{a}) = [f(a_1), \ldots, f(a_l)]$$

for each list

$$\mathsf{a} = [a_1, \ldots, a_l]$$

in FA. Almost trivially this is a monoid morphism, and the required functorial properties are just as easy. Thus we do have a functor, as above.

(c) We have a forgetful functor

$$\textbf{\textit{Set}} \xleftarrow[\text{¿}]{} \textbf{\textit{Mon}}$$

which sends each monoid to its carrying set. We show that the insertion

$$\begin{array}{rcl} A & \xrightarrow{\iota_A} & (\text{¿} \circ F)A \\ a & \longmapsto & [a] \end{array}$$

is natural for variation on A. In other words, for each function f, as on the left,

$$\begin{array}{ccccc} A & & A & \xrightarrow{\iota_A} & (¿ \circ F)A \\ \big\downarrow f & & \big\downarrow f & & \big\downarrow (¿ \circ F)(f) \\ B & & B & \xrightarrow[\iota_B]{} & (¿ \circ F)B \end{array}$$

the ***Set*** square on the right commutes. This is a trivial calculation. Both trips round the square send each element $a \in A$ to $[f(a)] \in (¿ \circ F)B$.

(d) We show that for each function

$$A \xrightarrow{\ f\ } B$$

from a set to a monoid, there is a unique monoid morphism

$$FA \xrightarrow{\ f^\sharp\ } B$$

such that the triangle commutes.

$$\begin{array}{ccccc} A & \xrightarrow{\ \ f\ \ } & ¿B & & B \\ & {}_{\iota_A}\searrow & \nearrow_{¿(f^\sharp)} & & \nearrow_{f^\sharp} \\ & (¿ \circ F)A & & FA & \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\ \ f\ \ } & B & & B \\ & {}_{\iota_A}\searrow & \nearrow_{f^\sharp} & & \nearrow_{f^\sharp} \\ & FA & & FA & \end{array}$$

On the left we have the official version, and on the right we have the way it is usually written with the forgetful functor forgotten.

Given a function f we have to show two things: there is at most one fill-in morphism $f^\sharp$, and there is at least one fill-in morphism $f^\sharp$. Almost always with this kind of problem these two parts are dealt with separately. Here is a useful way to handle the first part.

We show that ι_A is 'as epic as it can be'. We show that for each parallel pair of monoid morphism

$$FA \overset{g}{\underset{h}{\rightrightarrows}} C$$

we have

$$g \circ \iota_A = h \circ \iota_A \Longrightarrow g = h$$

and hence there is at most one fill-in morphism.

Consider any element $\mathsf{a} \in FA$. With

$$\mathsf{a} = [a_1, \ldots, a_l]$$

we have

$$\mathsf{a} = \iota_A(a_1) \frown \cdots \frown \iota_A(a_l)$$

where $\cdot \frown \cdot$ is the operation on FA, that is concatenation. a Consider any pair of morphism g, h, as above. We have

$$\begin{aligned} g(\mathsf{a}) &= g\big(\iota_A(a_1) \frown \cdots \frown \iota_A(a_l)\big) = (g \circ \iota_A)(a_1) \star \cdots \star (g \circ \iota_A)(a_l) \\ h(\mathsf{a}) &= h\big(\iota_A(a_1) \frown \cdots \frown \iota_A(a_l)\big) = (h \circ \iota_A)(a_1) \star \cdots \star (h \circ \iota_A)(a_l) \end{aligned}$$

where $\star$ is the operation of C. Thus if

$$g \circ \iota_A = h \circ \iota_A$$

then

$$g(\mathsf{a}) = h(\mathsf{a})$$

and so $g = h$.

It remains to show that there is at least one morphism $f^\sharp$ that make the triangle commute. To do that we simple set

$$f^\sharp(\mathsf{a}) = f(a_1) * \cdots * f(a_l)$$

for each $\mathsf{a} \in FA$ (as above) where $*$ is the operation on B.

(e) If A is already a monoid then this structure is forgotten and a much bigger monoid is produced. Even when A is the 1-element monoid, the constructed monoid FA is infinite. ■

3.5.18 (a) Solution 3.3.13 show that we have two functors

$$\boldsymbol{Grp} \underset{F}{\overset{G}{\rightrightarrows}} \boldsymbol{AGrp}$$

given by

$$GA = \delta A \qquad FA = A/\delta A$$

for each group A. The diagrams in that solution show that ι and η are natural.

(b) When B is abelian the subgroup δB is trivial. Thus the construction of $f^\sharp$ is a particular case of the construction of f/δ given in the latter part of Solution 3.3.13. ■

3.5.19 As in Solution 3.3.18, the construction of the arrow assignment ensures that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ (¿ \circ F)A & \xrightarrow[(¿ \circ F)A]{} & (¿ \circ F)B \end{array}$$

commutes for each arrow

$$A \xrightarrow{\quad f \quad} B$$

of ***Src***. This show that η is a natural transformation from the identity endo-functor on ***Src*** to $¿ \circ F$. ■

3.5.20 Consider three objects of $\boldsymbol{C}^{\nabla}$ and two arrows which ought to be composible.

$$F \xrightarrow{\ \sigma\ } G \xrightarrow{\ \tau\ } H$$

Thus we have three functors and two natural transformations. We require an appropriate composite

$$F \xrightarrow{\ \tau \circ \sigma\ } H$$

of the two transformations.

For each object i of ∇ we have a composible pair of arrows

$$Fi \xrightarrow{\ \sigma_i\ } Gi \xrightarrow{\ \tau_i\ } Hi$$

of arrows of $\boldsymbol{C}$. We set

$$(\tau \circ \sigma)_i = \tau_i \circ \sigma_i$$

to obtain a ∇-indexed family of arrows of $\boldsymbol{C}$. We show this family is natural for variation of i.

Consider any arrow e of ∇ and suppose this starts at i and finishes at j. We know that the two squares on the left do commute.

$$\begin{array}{ccccc} Fi & \xrightarrow{\ \sigma_i\ } & Gi & \xrightarrow{\ \tau_i\ } & Hi \\ {\scriptstyle F(e)}\downarrow & & \downarrow{\scriptstyle G(e)} & & \downarrow{\scriptstyle H(e)} \\ Fj & \xrightarrow[\ \sigma_j\ ]{} & Gj & \xrightarrow[\ \tau_j\ ]{} & Hj \end{array} \qquad \begin{array}{ccc} Fi & \xrightarrow{\ (\tau\circ\sigma)_i\ } & Hi \\ {\scriptstyle F(e)}\downarrow & & \downarrow{\scriptstyle H(e)} \\ Fj & \xrightarrow[\ (\tau\circ\sigma)_j\ ]{} & Hj \end{array}$$

Hence so does the square on the right, to show the required naturality.

A similar argument shows that this composition is associative. ■

Warning: Sometimes the symbol '∘' is not used for the composition of natural transformations described in the previous solution, but it is used for the composition $\star$ described in the next solution.

3.5.21 The naturality of ρ ensures that the $\boldsymbol{B}$-square

$$\begin{array}{c} B_0 \\ {\scriptstyle g}\downarrow \\ B_1 \end{array} \qquad \begin{array}{ccc} KB_0 & \xrightarrow{\ \rho_0\ } & LB_0 \\ {\scriptstyle K(g)}\downarrow & & \downarrow{\scriptstyle L(g)} \\ KB_1 & \xrightarrow[\ \rho_1\ ]{} & LB_1 \end{array} \qquad \begin{array}{c} FA \\ \downarrow{\scriptstyle \lambda_A} \\ GA \end{array}$$

commutes for each $\boldsymbol{B}$-arrow g, as on the left. Each object A of $\boldsymbol{A}$ gives us an arrow λ_A of $\boldsymbol{B}$, as on the right. Taking this for g gives the required commuting $\boldsymbol{C}$-square. ■

3.5.22 For an arbitrary $\boldsymbol{A}$-object A consider the following diagram.

$$\begin{array}{ccccc}
(K\circ F)A & \xrightarrow{\rho_{FA}} & (L\circ F)A & \xrightarrow{\sigma_{FA}} & (M\circ F)A \\
{\scriptstyle K(\lambda_A)}\downarrow & & \downarrow{\scriptstyle L(\lambda_A)} & & \downarrow{\scriptstyle M(\lambda_A)} \\
(K\circ G)A & \xrightarrow{\rho_{GA}} & (L\circ G)A & \xrightarrow{\sigma_{GA}} & (M\circ G)A \\
{\scriptstyle K(\mu_A)}\downarrow & & \downarrow{\scriptstyle L(\mu_A)} & & \downarrow{\scriptstyle M(\mu_A)} \\
(K\circ H)A & \xrightarrow[\rho_{HA}]{} & (L\circ H)A & \xrightarrow[\sigma_{HA}]{} & (M\circ H)A
\end{array}$$

Each of the four small squares commutes. This is four instances of the result of Exercise 3.5.21. The diagonals of the top let and bottom right squares are

$$(\rho\star\lambda)_A \qquad (\sigma\star\mu)_A$$

respectively, and hence

$$\big((\sigma\star\mu)\circ(\rho\star\lambda)\big)_A$$

is the composite diagonal.

The outside square commutes, and this is just

$$\begin{array}{ccc}
(K\circ F)A & \xrightarrow{(\sigma\circ\rho)_{FA}} & (M\circ F)A \\
{\scriptstyle K\left((\mu\circ\lambda)_A\right)}\downarrow & & \downarrow{\scriptstyle M\left((\mu\circ\lambda)_A\right)} \\
(K\circ H)A & \xrightarrow[(\sigma\circ\rho)_{HA}]{} & (M\circ H)A
\end{array}$$

using the construction of the vertical composition. The diagonal of this square is

$$\big((\sigma\circ\rho)\star(\mu\circ\lambda)\big)_A$$

by the definition of horizontal composition.

Comparing the two descriptions of the full diagonal gives the required result. ■

Solutions 3.5.10 and 3.5.11 still to be done

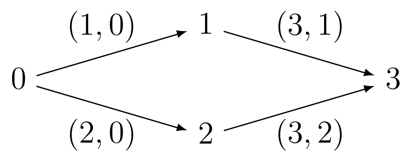
D
Limits and colimits in general

D.1 Template and diagram – a first pass

4.1.1 Sticking paths end to end is associative, so composition in $\boldsymbol{Pth}(\nabla)$ is associative.

The only problem is to ensure that each object (node of ∇) has an identity arrow on $\boldsymbol{Pth}(\nabla)$. Each node has an associated path of length zero, and sticking such a path on the end of some other path doesn't change that second path. Thus the required identity arrows are the paths of length zero. ■

4.1.2 (a) Let us label the four edges as follows.



There are just 10 possible paths.

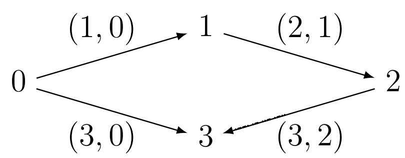
Paths of length		Number of such paths
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1, 0), (2, 0), (3, 1), (3, 2)$	4
2	The two formal composites $(3, 1) \circ (1, 0) \quad (3, 2) \circ (2, 1)$	2

This graph generates a category of four objects and 10 arrows. This category is not a poset since there are two distinct arrows

$$(3, 1) \circ (1, 0) \qquad (3, 2) \circ (2, 1)$$

from 0 to 3.

The graph is

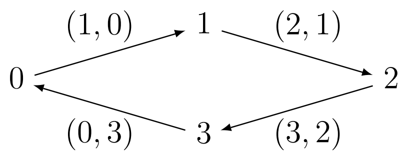


with the edges labelled. This generates a category

Paths of length		Number of such paths
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1,0), (2,1), (3,2), (3,0)$	4
2	The two formal composites $(2,1)\circ(1,0) \quad (3,2)\circ(2,1)$	2
3	The formal composite $(3,2)\circ(2,1)\circ(1,0)$	1

of four nodes an 11 arrows.

(c) The labelled graph is



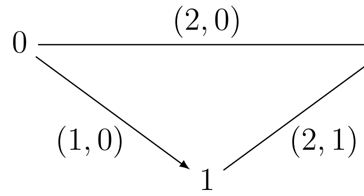
and this generates a category of four objects and infinitely many arrows.

Paths of length		Number of such paths
0	The four nodes $0, 1, 2, 3$	4
1	The four edges $(1,0), (2,1), (3,2), (0,3)$	4
2	The four formal composites $(2,1)\circ(1,0) \quad (3,2)\circ(2,1)$ $(0,3)\circ(3,2) \quad (1,0)\circ(0,3)$	4
3	The formal composites $(3,2)\circ(2,1)\circ(1,0)$ $(0,3)\circ(3,2)\circ(2,1)$ $(1,0)\circ(0,3)\circ(3,2)$ $(2,1)\circ(1,0)\circ(0,3)$	4
4	The formal composites $(0,3)\circ(3,2)\circ(2,1)\circ(1,0)$ $(1,0)\circ(0,3)\circ(3,2)\circ(2,1)$ $(2,1)\circ(1,0)\circ(0,3)\circ(3,2)$ $(3,2)\circ(2,1)\circ(1,0)\circ(0,3)$	4
4	The formal composites $(1,0\circ(0,3)\circ(3,2)\circ(2,1)\circ(1,0)$ $(2,1)\circ(1,0)\circ(0,3)\circ(3,2)\circ(2,1)$ $\vdots$	4

We can cycle round the four given edges for ever. ■

4.1.3 No! The generated category is bigger.

For example consider the category



of three objects, there identity arrows (not shown), and three oher arrows where

$$(2,1) \circ (1,0) = (2,0)$$

in the category. In the graph there are (at least) two paths from 0 to 2, namely

$$(2,0) \quad \text{and} \quad (1,0) \text{ followed by } (2,1)$$

and these are *not* the same path.

In general, the parent category is a quotent of the generated path category. ■

4.1.4 Let ∇ be the graph with nodes i and edges e. A ∇-diagram in the category $\boldsymbol{C}$ consists of

$$\begin{array}{cc} \text{objects} & \text{arrows} \\ A(i) & A(e) \end{array}$$

indexed by the

$$\begin{array}{cc} \text{nodes} & \text{edges} \end{array}$$

respectively. For each edge

$$i \xrightarrow{\;e\;} j$$

we require an arrow

$$A(i) \xrightarrow{A(e)} A(j)$$

but there are no requirements that certain triangles must commute.

Now consider a functor $\boldsymbol{Pth}(\nabla) \longrightarrow \boldsymbol{C}$ from the path category. This certainly gives a famly of objects of $\boldsymbol{C}$

$$A(i)$$

indexed by the objects of $\boldsymbol{Pth}(\nabla)$, the nodes of ∇. It also gives an arrow of $\boldsymbol{C}$

$$A(\pi)$$

for each arrow of $\boldsymbol{Pth}(\nabla)$, each path of through ∇. In particlar, we have an arrow in $\boldsymbol{C}$

$$A(e)$$

for each edge of ∇, each path in $\boldsymbol{Pth}(\nabla)$ of length 1. Thus there is a ∇-diagram embedded in the functor.

Each path π has a unique decomposition

$$i(0) \xrightarrow{e(1)} i(1) \xrightarrow{e(2)} i(2) \longrightarrow \cdots\cdots \xrightarrow{e(l)} i(l)$$

as a sequence of edges through ∇. The functorial ensure that

$$A(\pi) = A(e(l)) \circ \cdots \circ A(e(1))$$

to show that the functor is uniquely determined by the embedded ∇-diagram.

Observe that more or less the same proof shows that each ∇-diagram extendd to a functor. ■

D.2 Functor categories

4.2.1 *To be done* ■

4.2.2 We must show that each arrow

$$X \xrightarrow{\ f\ } Y$$

induces a natural transformation

$$\Delta X \xrightarrow{\ \Delta(f)_\bullet\ } \Delta Y$$

such that

$$\begin{array}{ccccc}
(\Delta X)(i) & \xrightarrow{\ \Delta(f)_i\ } & (\Delta Y)(i) & \qquad & i \\
{\scriptstyle (\Delta X)(e)}\big\downarrow & & \big\downarrow{\scriptstyle (\Delta Y)(e)} & & \big\downarrow{\scriptstyle e} \\
(\Delta X)(j) & \xrightarrow[\ \Delta(f)_j\ ]{} & (\Delta Y)(j) & & j
\end{array}$$

commutes for each edge e of ∇, as on the right.

When we insert the values of ΔX and ΔY we see that a commuting square

$$\begin{array}{ccc}
X & \xrightarrow{\ \Delta(f)_i\ } & Y \\
{\scriptstyle \boldsymbol{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle \boldsymbol{id}_Y} \\
X & \xrightarrow[\ \Delta(f)_j\ ]{} & Y
\end{array}$$

is required. Thus we set

$$\Delta(f)_i = f$$

for each node i. ■

4.2.3 We do both solutions in parallel.

A typical arrow in $\boldsymbol{C}^\nabla$

$$\Delta X \longrightarrow A \qquad\qquad A \longrightarrow \Delta X$$

is a natural transformation, a family of arrows of $\boldsymbol{C}$

$$(\Delta X)(i) \xrightarrow{\xi(i)} A(i) \qquad\qquad A(i) \xrightarrow{\xi(i)} (\Delta X)(i)$$

indexed by the nodes and such that

$$\begin{array}{ccc} (\Delta X)(i) & \xrightarrow{\xi(i)} & A(i) \\ {\scriptstyle (\Delta X)(e)}\downarrow & & \downarrow{\scriptstyle A(e)} \\ (\Delta X)(j) & \xrightarrow[\xi(j)]{} & A(j) \end{array} \qquad \begin{array}{c} i \\ \downarrow{\scriptstyle e} \\ j \end{array} \qquad \begin{array}{ccc} A(i) & \xrightarrow{\xi(i)} & (\Delta X)(i) \\ {\scriptstyle A(e)}\downarrow & & \downarrow{\scriptstyle (\Delta X)(e)} \\ A(j) & \xrightarrow[\xi(j)]{} & (\Delta X)(j) \end{array}$$

commutes for each edge e. When we insert the values of ΔX we see that we require a commuting triangle

$$\begin{array}{ccc} & & A(i) \\ & \overset{\xi(i)}{\nearrow} & \\ X & & \downarrow{\scriptstyle A(e)} \\ & \underset{\xi(i)}{\searrow} & \\ & & A(j) \end{array} \qquad \begin{array}{c} i \\ \downarrow{\scriptstyle e} \\ j \end{array} \qquad \begin{array}{ccc} A(i) & & \\ & \overset{\xi(i)}{\searrow} & \\ {\scriptstyle A(e)}\downarrow & & X \\ & \underset{\xi(i)}{\nearrow} & \\ A(j) & & \end{array}$$

for each edge e. In other words such an arrow is just a

left right

solution for the diagram A. ■

D.3 Problem and solution

4.3.1 Consider any ∇-diagram and the corresponding $\boldsymbol{Pth}(\nabla)$-diagram. These have the same family

$$A(i)$$

of objects indexed by the nodes i of ∇. Each edge

$$i \xrightarrow{e} j$$

of ∇ gives an arrow

$$A(i) \xrightarrow{A(e)} A(j)$$

of the ∇-diagram, and this is also an arrow of the $\boldsymbol{Pth}(\nabla)$-diagram. However, there are many more arrows in the $\boldsymbol{Pth}(\nabla)$-diagram. Each path

$$i \xrightarrow{p} ij$$

in $\boldsymbol{Pth}(\nabla)$ gives an arrow

$$A(i) \xrightarrow{A(\pi)} A(j)$$

in the $\boldsymbol{Pth}(\nabla)$-diagram.

In the ∇-diagram there are no requiements that certain triangles commute (for there is no notion of composition in the graph ∇).

In the $\boldsymbol{Pth}(\nabla)$-diagram each composite path requires that cerain triangles commute. For example let π be the composite path

$$i(0) \xrightarrow{e(1)} i(1) \xrightarrow{e(2)} i(2) \longrightarrow \cdots\cdots \xrightarrow{e(l)} i(l)$$

of l edges. Then then the two arrows

$$A(i(0)) \xrightarrow{A(e(1))} A(i(1)) \xrightarrow{A(e(2))} A(i(2)) \longrightarrow \cdots\cdots \xrightarrow{A(e(l))} A(i(l))$$

$$A(i(0)) \xrightarrow[A(\pi)]{\hspace{8cm}} A(e(l))$$

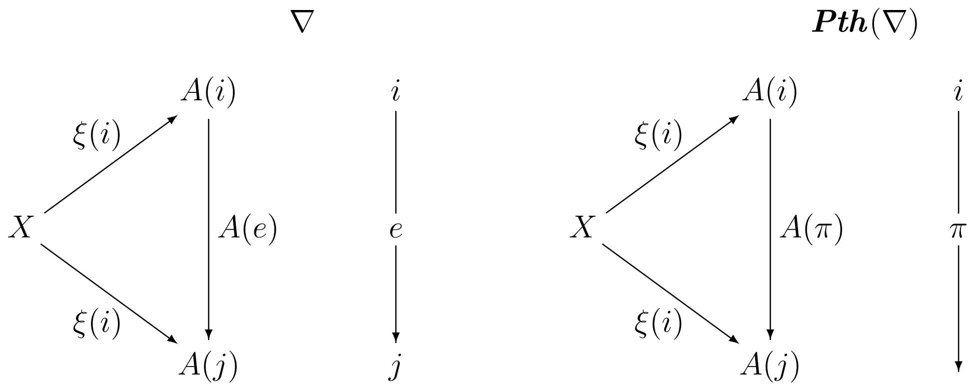
must agree. The ∇-diagram completely determines the $\boldsymbol{Pth}(\nabla)$-diagram. (You should also think of how the conditions on identity arrows are handled in the $\boldsymbol{Pth}(\nabla)$-diagram.)

At this point we have to decide whether we look at left solutions or right solutions. The two discussions are entirely symmerical, so let's look at left solutions.

Consider a solution of each diagram with the same apex X. Each is a family of arrows

$$X \xrightarrow{\xi(i)} A(i)$$

indexed by the nodes of ∇. There are certain commuting conditions. All the triangles



for each edge e of ∇ and each path π of $\boldsymbol{Pth}(\nabla)$.

■

D.4 Universal solution

4.4.1 Let

$$A(i) \xrightarrow{\sigma(i)} S$$

be a colimit of a diagram indexed by the nodes i of a template.

If

$$S \underset{\psi}{\overset{\theta}{\rightrightarrows}} X$$

is a parallel pair of arrows with

$$\theta \circ \sigma(i) = \psi \circ \sigma(i)$$

for each node i, then $\theta = \psi$.

If

$$S \xrightarrow{\ \epsilon\ } S$$

is an endo-arrow of S such that

$$\epsilon \circ \sigma(i) = \sigma(i)$$

for each node i, then $\epsilon = \boldsymbol{id}_S$.

If

$$A(i) \xrightarrow{\ \tau(i)\ } T$$

is also a limit of the diagram then there is a unique arrow

$$S \xrightarrow{\ \tau\ } T$$

such that

$$\tau(i) = \tau(i) \circ \sigma(i)$$

for each node i. Furthermore, τ is an isomorphism.

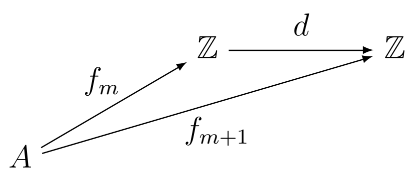
These are prove simply by reversing the arrows of the limit proofs. ■

D.5 A geometric limit and colimit

4.5.1 Consider any left solution of the diagram.

$$A \xrightarrow{\ f_m\ } \mathbb{Z}$$

This is a $\mathbb{Z}$-indexed family of arrows, as indicated, such that



commutes for each $m \in \mathbb{Z}$. Thus

$$f_{m+1}(a) = 2f_m(a)$$

for each $a \in A$ and $m \in \mathbb{Z}$. By a trivial induction this gives

$$f_{m+r}(a) = 2^r f_m(a)$$

for each $a \in A, m \in \mathbb{Z}, r \in \mathbb{N}$.

Each value $f_m(a)$ of f_m is in $\mathbb{Z}$. The above equality shows that the value is divisible by 2^r for arbitrarily large $r \in \mathbb{N}$. Thus

$$f_m(a) = 0$$

for each $a \in A$ and $m \in \mathbb{Z}$.

This shows that each left soloution has a rather simple structure.

Consider the singleton

$$L = \{*\}$$

funished with the constant functions

$$\begin{array}{ccc} L & \xrightarrow{\lambda_m} & \mathbb{Z} \\ * & \longmapsto & 0 \end{array}$$

for each $m \in \mathbb{Z}$. Since

$$\lambda_{m+1}(*) = 0 = 2 \times 0 = 2\lambda_m(*)$$

this certainly gives a left solution of the diagram.

To show that this is the limit consider any left solution, as above. We require a unique function

$$A \xrightarrow{\ h\ } L$$

such that

$$f_m = \lambda_m \circ h$$

for each $m \in \mathbb{Z}$. In fact, there is only one possible function h of the indicated type, namely that given by

$$h(a) = *$$

for each $a \in A$. But now

$$(\lambda_m \circ h)(a) = \lambda_m(*) = 0 = f_m(a)$$

to show that h is the required mediator.

Consider any right solution of the diagram.

$$\mathbb{Z} \xrightarrow{\ f_m\ } A$$

This is a $\mathbb{Z}$-indexed family of arrows, as indicated, such that

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\ d\ } & \mathbb{Z} & & \\ & \searrow^{f_m} & & \searrow^{f_{m+1}} & \\ & & & & A \end{array}$$

commutes for each $m \in \mathbb{Z}$. Thus

$$f_m(z) = f_{m+1}(2z)$$

for all $m, z \in \mathbb{Z}$. By a trivial induction this gives

$$f_m(z) = f_{m+r}(2^r z)$$

for all $m, z \in \mathbb{Z}$ and $r \in \mathbb{N}$. For later we need a refined version of this. We require

$$(\star) \quad 2^{-m}x = 2^{-n}y \Longrightarrow f_m(x) = f_n(y)$$

for all $m, n, x, y \in \mathbb{Z}$.

To prove this suppose

$$2^{-m}x = 2^{-n}y$$

so that

$$2^n x = 2^m y$$

holds. By symmetry we may suppose $m \leq n$, so that $n = m + r$ for some $r \in \mathbb{N}$. Thus

$$y = 2^r x$$

and hence

$$f_m(x) = f_{m+r}(2^r x) = f_n(y)$$

as required.

The dyadic rationals $\mathbb{D}$ consists of those rationals of the form

$$2^{-m}x$$

for $m, x \in \mathbb{Z}$. Of course, this representation is not unique (which is why we proved $(\star)$).

For each $m \in \mathbb{Z}$ consider the function

$$\mathbb{Z} \xrightarrow{\ \rho_m\ } \mathbb{D}$$

given by

$$\rho_m(x) = 2^{-m}x$$

for each $x \in \mathbb{Z}$. This gives a right soloution since

$$\rho_{m+1}(2x) = 2^{-m-1} \times 2x = 2^{-m}x = \rho_m(x)$$

for each $m, x \in \mathbb{Z}$. We show that this is the colimit of the diagram.

Consider any right solution, as above. We require a unique function

$$\mathbb{D} \xrightarrow{\ h\ } A$$

such that

$$f_m = h \circ \rho_m$$

for each $m \in \mathbb{N}$.

If there is such a function h then it must satisfy

$$h(2^{-m}x) = (h \circ \rho_m)(x) = f_m(x)$$

for each $m, x \in \mathbb{Z}$. Thus there is only one possible function h.

Consider and $d \in \mathbb{D}$. We may have

$$2^{-m}x = d = 2^{-n}y$$

for $m, n, x, y \in \mathbb{Z}$. The result $(\star)$ gives

$$f_m(x) = f_n(y)$$

and hence we may set

$$h(2^{-m}x) = f_m(x)$$

to obtain a well-defined function of the required type.

Finally, for each $m, x \in \mathbb{Z}$, we have

$$(h \circ \rho_m)(x) = h(2^{-m}x) = f_m(x)$$

so that h does the required mediating job. ■

D.6 How to calculate certain limits

D.6.1 Limits in *Set*

4.6.1 Let A be the set of threads. We furnish A with a distinguished subset R to obtain an object (A, R) of $\boldsymbol{Set}(D)$. We let

$$a \in R \Longleftrightarrow (\forall i \in \mathbb{I})[a(i) \in R(i)]$$

for each thread a.

We need to check various conditions.

For an arbitrary index i consider the conecting function

$$A \xrightarrow{\alpha(i)} A(i)$$

in $\boldsymbol{Set}$. We check that this function is an arrow

$$(A, R) \xrightarrow{\alpha(i)} (A(i), R(i))$$

of $\boldsymbol{Set}(D)$. In other words that

$$a \in R \Longrightarrow \alpha(i)(a) \in R(i)$$

holds. But

$$\alpha(i)(a) = a(i)$$

so this is an immediate consequence of the definition of R.

Next we observe that we have a solution of the diagram in ***Set***(D). This requires that certain triangles in ***Set***(D) commute. But we know that these triangles commute in ***Set***, so there is nothing to prove.

We check that this solution is a universal solution in ***Set***(D).

Consider any solution

$$(X, W) \xrightarrow{\xi(i)} (A(i), R(i))$$

in ***Set***(D). This is a ***Set***(D)-object (X, W) and an $\mathbb{I}$-indexed family of ***Set***(D)-arrows $\xi(i)$. We are given that certain triangles in ***Set***(D) commute. We must show that there is a unique mediator

$$(X, W) \xrightarrow{\mu} (A, R)$$

in ***Set***(D).

The trick is to forget the furnishings for a moment and drop down to ***Set***. We have a ∇-diagram in ***Set***, a solution of the diagram based on X, and a universal solution based on A. Thus if the ***Set***(D)-situation has a mediator, then it can only be the ***Set***-mediator, given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. In oher words, it suffices to show that this function μ is a ***Set***(D)-arrow, that is

$$x \in W \Longrightarrow \mu(x) \in R$$

for each $x \in X$.

Consider any $x \in W$. For arbitrary $i \in \mathbb{I}$, we are given that $\xi(i)$ is a ***Set***(D)-arrow. Thus

$$\mu(x)(i) = \xi(i)(x) \in R(i)$$

which, since i is arbitrary, gives $\mu(x) \in R$, as required. ■

D.6.2 Limits in ***Pos***

4.6.2 (a) An arrow

$$(A, \sim) \xrightarrow{f} (B, \approx)$$

is a function f from A to B for which

$$a_1 \sim a_2 \Longrightarrow f(a_1) \approx f(a_2)$$

for all $a_1, a_2 \in A$. Observe that this is aonly an implication, not an equivalence.

(b) Let A be the set of threads. We furnish A with an equivalence relation $\sim$ to obtain an object $(A, \sim)$ of ***Eqv***. We let

$$a \sim b \Longleftrightarrow (\forall i \in \mathbb{I})[a(i) \sim_i b(i)]$$

for each pair of thread a, b. Trivially, this is reflexive an symmetic, and a few moment's thought shows that it is transitive. Thus we do have an equivalence relation.

We need to check various conditions.

For an arbitrary index i consider the connecting function

$$A \xrightarrow{\alpha(i)} A(i)$$

in ***Set***. We check that this function is an arrow

$$(A, \sim) \xrightarrow{\alpha(i)} (A(i), \sim_i)$$

of ***Eqv***. In other words that

$$a \sim b \Longrightarrow \alpha(i)(a) \sim_i \alpha(i)(b)$$

for all $a, b \in A$. But

$$\alpha(i)(a) = a(i) \qquad \alpha(i)(b) = b(i)$$

so this is an immediate consequence of the definition of $\sim$.

Next we observe that we have a solution of the diagram in ***Eqv***. This requires that certain triangles in ***Set***(D) commute. But we know that these triangles commute in ***Set***, so there is nothing to prove.

We check that this solution is a universal solution in ***Eqv***.

Consider any solution

$$(X, \approx) \xrightarrow{\xi(i)} (A(i), \sim_i)$$

in ***Eqv***. This is a ***Eqv***-object $(X, \approx)$ and an $\mathbb{I}$-indexed family of ***Eqv***-arrows $\xi(i)$. We are given that certain triangles in ***Eqv*** commute. We must show that there is a unique mediator

$$(X, \approx) \xrightarrow{\mu} (A, \sim)$$

in ***Eqv***.

The trick is to forget the furnishings for a moment and drop down to ***Set***. We have a ∇-diagram in ***Set***, a solution of the diagram based on X, and a universal solution based on A. Thus if the ***Eqv***-situation has a mediator, then it can only be the ***Set***-mediator, given by

$$\mu(x)(i) = \xi(i)(x)$$

for each $x \in X$ and $i \in \mathbb{I}$. In oher words, it suffices to show that this function μ is a ***Eqv***-arrow, that is

$$x \approx y \Longrightarrow \mu(x) \sim \mu(y)$$

for each $x, y \in X$.

For each $x, y \in X$ we have

$$\begin{aligned} x \approx y &\Longrightarrow (\forall i \in \mathbb{I})[\xi(i)(x) \sim_i \xi(i)(y)] \\ &\Longrightarrow (\forall i \in \mathbb{I})[\mu(x)(i) \sim_i \mu(y)(i)] \Longrightarrow \mu(x) \sim \mu(y) \end{aligned}$$

to give the required result. ■

D.6.3 Limits in ***Mon***

4.6.3 (a) Each commutative monoid is a monoid with an extra property. The crucial observation is that any ***Mon***-arrow between commutative monoids is automatically a ***CMon***-arrow. (Technically this says that ***CMon*** is a full subcategory of ***Mon***.)

For a template ∇ consider a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in ***CMon***. Thus each $A(i)$ is a commutative monoid, and each $A(e)$ is a monoid morphism. By forgetting the commutative property we have a diagram in ***Mon***. We know this has a limit

$$A \xrightarrow{\alpha(i)} A(i)$$

carried by the set of threads. We show that A is commutative, and then it is automatically a limit in ***CMon***.

The operation $\star$ on A is given by

$$(a \star b)(i) = a(i)b(i)$$

for each $a, b \in A$ and $i \in \mathbb{I}$. Since $A(i)$ is commutative we have

$$(a \star b)(i) = a(i)b(i) = b(i)a(i) = (b \star a)(i)$$

which, since i is arbitrary, gives

$$a \star b = b \star a$$

to show that $\star$ is commutative.

(b) Each group is a monoid with some extra structure. The crucial observation, which takes a few moment's thought to justify, is that any ***Mon***-arrow between groups is automatically a ***Grp***-arrow. (Technically this says that ***Grp*** is a full subcategory of ***Mon***.)

For a template ∇ consider a ∇-diagram

$$\mathsf{A} = \big(A(i) \,|\, i \in \mathbb{I}\big) \qquad \mathcal{A} = \big(A(e) \,|\, e \in \mathbb{E}\big)$$

in ***Grp***. Thus each $A(i)$ is a group, and each $A(e)$ is a group. By forgetting the existence of inverses we have a diagram in ***Mon***. We know this has a limit

$$A \xrightarrow{\alpha(i)} A(i)$$

carried by the set of threads. We show that A is a group, and then it is automatically a limit in ***Grp***.

Consider any element $a \in A$. We produce an inverse of a in A, a thread $b \in A$ such that

$$a \star b = b \star a$$

that is

$$(a \star b)(i) = (b \star a)(i)$$

for each index i.

For each index i we have

$$a(i)b(i) = 1(i) = b(i)a(i)$$

for a unique element $b(i) \in A(i)$. In other words, $b(i)$ is the unique inverse of $a(i)$ in $A(i)$. This certainly gives us a choice function

$$b(\cdot) : \mathbb{I} \longrightarrow \bigcup \mathsf{A}$$

but we need to show that b is a thread, that is with

$$A(e)\big(b(i)\big) = b(j)$$

for each edge

$$i \xrightarrow{\ e\ } j$$

of ∇.

We remember that a is a thread and $A(e)$ is a group morphism. Thus

$$a(j)\big(A(e)(b(i))\big) = \big(A(e)(a(i))\big)\big(A(e)(b(i))\big) = A(e)(a(i)b(i) = A(e)(1(i)) = 1(j)$$

with

$$\big(A(e)(b(i))\big)a(j) = 1(j)$$

by a similar argument. But $a(j)$ has a unique inverse in $A(j)$, namely $b(j)$, and hence

$$\big(A(e)(b(i))\big) = b(j)$$

as required.

This shows that $b \in A$, and for each index i we have

$$(a \star b)(i) = a(i)b(i) = 1(i) = b(i)a(i) = (b \star a)(i)$$

and hence b is the inverse of a in A.

This shows that A is a group. The required arrow-theoretic properties to show that A is the limit of the diagram are immediate, since they hold in ***Mon***.

(c) *To be done* ■

4.6.4 Given a ∇-diagram in ***Pom*** in the usual notation, let A be the set of threads. We know that A can be furnished as a poset and a monoid by

$$x \le a \iff (\forall i)[x(i) \le a(i)] \qquad (ab)(i) = a(i)b(i)$$

for all $a, b, x \in A$ and index i. (We can now drop the use of $\star$ for the operation on A.) We show this is a pom.

Consider $a, b, x, y \in A$ with $x \le a$ and $y \le b$. Then

$$(\forall i)[(xy)(i) = x(i)y(i) \le a(i)b(i) = (ab)(i)]$$

to verify the required comparison property.

By now you should find the required arrow-theoretic properties routine. ■

D.6.4 Limits in ***Top*** - *to be done*

D.7 Confluent colimits in ***Set*** *No exs yet*

E
Adjunctions

E.1 Adjunctions defined

5.1.1 *To be done.* ■

5.1.2 Let

$$\textit{\textbf{Set}} \xleftarrow{\ U\ } \textit{\textbf{Pre}}$$

be the forgetfully functor. We produce a left adjoint and a right adjoint to this functor.

$$\textit{\textbf{Set}} \underset{I}{\overset{D}{\rightrightarrows}} \xleftarrow{\ U\ } \textit{\textbf{Pre}}$$

Each set X can be converted into a preset in two extreme ways.

$$DX = (X, =) \qquad IX = (X, \|)$$

On the left we use equality as the comparison. This gives a poset. On the right any two elements are comparable This is not a poset if X at least two elements. We call these the

discrete indiscrete

presets, respectively.

This gives us the object assignments of two functors.

Consider any function.

$$X \xrightarrow{\ f\ } Y$$

We observe that f is monotone relative to both the discrete and the indiscrete comparisons, that is

$$x = y \Longrightarrow f(x) = f(y) \qquad f(x) \parallel f(y)$$

for all $x, y \in X$.

This gives us the arrow assignments of two functors. We use the same function but view it as a monotone map in one of two ways.

We now check the two adjunctions

$$D \dashv U \qquad U \dashv I$$

separately.

For the adjunction $D \dashv U$ we require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \boldsymbol{Set}[X, US] & & \boldsymbol{Pre}[DX, S] \\ \phi_\flat & \longleftarrow & \phi \end{array}$$

for each set X and each preset S. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$DX \xrightarrow{f^\sharp} S$$

is monotone for each function f, as above. This is trivial.

Finally, for $D \dashv U$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate.

For the adjunction $U \dashv I$ we require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Pre}[S, IX] & & \boldsymbol{Set}[US, X] \\ f_\flat & \longleftarrow & f \end{array}$$

for each preset S and each set X. In fact both $(\cdot)^\sharp$ and $(\cdot)_\flat$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$S \xrightarrow{f_\flat} IX$$

is monotone. But since IX is indiscrete, this is trivial.

Finally, for $U \dashv I$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate. ■

5.1.3 This is more or less the same as Solution 5.1.2 except we now use topologies rather than pre-orders.

Let

$$\boldsymbol{Set} \xleftarrow{\;U\;} \boldsymbol{Top}$$

be the forgetfully functor. We produce a left adjoint and a right adjoint to this functor.

$$\boldsymbol{Set} \begin{array}{c} \xrightarrow{\;D\;} \\ \xleftarrow{\;U\;} \\ \xrightarrow[\;I\;]{} \end{array} \boldsymbol{Top}$$

Each set X can be converted into a topological space in two extreme ways.

$$DX = (X, \mathcal{P}X) \qquad IX = (X, \{\emptyset, X\})$$

On the left we use discrete topology in which each subset is open. On the right we use the indiscrete topology in which only the two extreme subsets are open. Naturally, we call these the

discrete indiscrete

spaced, respectively.

This gives us the object assignments of two functors.

Consider any function.

$$X \xrightarrow{\quad f \quad} Y$$

We observe that f is continuous relative to both the discrete and the indiscrete topologies. This gives us the arrow assignments of two functors. We use the same function but view it as a continuous map in one of two ways.

We now check the two adjunctions

$$D \dashv U \qquad U \dashv I$$

separately.

For the adjunction $D \dashv U$ we require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^{\sharp} \\ \boldsymbol{Set}[X, US] & & \boldsymbol{Top}[DX, S] \\ \phi_{\flat} & \longleftarrow\!\shortmid & \phi \end{array}$$

for each set X and each space S. In fact both $(\cdot)^{\sharp}$ and $(\cdot)_{\flat}$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$DX \xrightarrow{\quad f^{\sharp} \quad} S$$

is continuous for each function f, as above. This is trivial.

Finally, for $D \dashv U$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate.

For the adjunction $U \dashv I$ we require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^{\sharp} \\ \boldsymbol{Top}[S, IX] & & \boldsymbol{Set}[US, X] \\ f_{\flat} & \longleftarrow\!\shortmid & f \end{array}$$

for each space S and each set X. In fact both $(\cdot)^{\sharp}$ and $(\cdot)_{\flat}$ return the same function, so do form an inverse pair. The only thing we have to check is that

$$S \xrightarrow{\quad f_{\flat} \quad} IX$$

is continuous. But since IX is indiscrete, this is trivial.

Finally, for $U \dashv I$, we must check that the two assignments are natural. However, nothing much is happening as we pass across a functor or an assignment, so the naturality is immediate. ■

5.1.4 This is dealt with in Posets, completions, and related notions', specifically by Lemma 15 and Exercise 4.2. ■

5.1.5 As in Chapter 5 we let $i, j, \ldots$ range over the objects of the template category ∇ and refer to these as nodes. We use

$$i \xrightarrow{\quad e \quad} j$$

as a typical arrow of ∇ and refer to this as an edge.

An object of $\boldsymbol{C}^{\nabla}$, a ∇-diagram in $\boldsymbol{C}$ is a functor

$$\nabla \xrightarrow{F} \boldsymbol{C}$$

so consists of a family of objects of $\boldsymbol{C}$

$$F(i)$$

indexed by the nodes and a collection of arrows of $\boldsymbol{C}$

$$F(i) \xrightarrow{F(e)} F(j)$$

indexed by the edges. Various triangles in $\boldsymbol{C}$ are required to commute.

An arrow of $\boldsymbol{C}^{\nabla}$

$$F \xrightarrow{\eta\bullet} G$$

is just a natural transformation between the two functors. These are composed in the obvious way.

For each object A of $\boldsymbol{C}$ we set

$$(\Delta A)(i) = A \qquad (\Delta A)(e) = \boldsymbol{id}_A$$

for each node i and each edge e. This gives a constant diagram.

Each arrow of $\boldsymbol{C}$

$$A \xrightarrow{\eta} B$$

gives a 'constant' natural transformation

$$\Delta A \xrightarrow{\eta\bullet} \Delta B$$

in the obvious way.

This sets up a functor

$$\boldsymbol{C} \xrightarrow{\Delta} \boldsymbol{C}^{\nabla}$$

and we are interested in the existence or not of a right adjoint to Δ.

Consider first an object A of $\boldsymbol{C}$ and an object F of $\boldsymbol{C}^{\nabla}$. What does a member of

$$\boldsymbol{C}^{\nabla}[\Delta A, F]$$

look like?

It is a family of arrows

$$A \xrightarrow{\alpha(i)} F(i)$$

indexed by the nodes of ∇ such that

$$\begin{array}{ccccc} A & \xrightarrow{\alpha(i)} & F(i) & \qquad & i \\ {\scriptstyle \boldsymbol{id}}\downarrow & & \downarrow{\scriptstyle F(e)} & & \downarrow{\scriptstyle e} \\ A & \xrightarrow[\alpha(j)]{} & F(j) & & j \end{array}$$

commutes for each edge e, as indicated on the right.

In other words, this is nothing more than a left solution to the diagram F.

Suppose now that we fix a particular solution

$$S \xrightarrow{\sigma(i)} F(i)$$

and try to compare S with an arbitrary object A of $\boldsymbol{C}$. How are the two hom-sets

$$\boldsymbol{C}[A,S] \qquad \boldsymbol{C}^{\nabla}[\Delta A, F]$$

related?

There is an obvious assignment

$$\begin{array}{ccc} \mu & \longmapsto & \mu^{\sharp} \\ \boldsymbol{C}[A,S] & & \boldsymbol{C}^{\nabla}[\Delta A, F] \end{array}$$

in one direction. Given a $\boldsymbol{C}$-arrow

$$A \xrightarrow{\mu} S$$

for each node i we let $\mu^{\sharp}(i)$ be the composite

$$A \xrightarrow{\mu} S \xrightarrow{\sigma(i)} F(i)$$

to produce a left solution of F.

This gives us a family of assignments

$$\boldsymbol{C}[-,S] \xrightarrow{(\cdot)^{\sharp}} \boldsymbol{C}^{\nabla}[\Delta -, F]$$

as we let the object A vary through $\boldsymbol{C}$. Notice that we have a pair of contravariant functors

$$\boldsymbol{C} \longrightarrow \boldsymbol{Set}$$

and it is not to hard to show that $(\cdot)^{\sharp}$ is a natural transformation between the two.

When is this natural transformation a natural equivalence? Precisely when each solution

$$\Delta A \xrightarrow{\alpha(\bullet)} F$$

arises from a unique arrow

$$A \xrightarrow{\mu} S$$

as $\mu^\sharp$. This is simply saying that (S,σ) is a universal left solution, a limit of F.

Every ∇-diagram F in $\boldsymbol{C}$ has a limit precisely when there is an object assignment

$$S \longleftarrow F$$

picking out the object which carries the limit. The required functorial and adjunction properties follow by similar arguments. ■

5.1.6 We use various aspects of the gadgetry of the adjunction.

$$\boldsymbol{Src}[-,G-] \underset{(\cdot)_\flat}{\overset{(\cdot)^\sharp}{\rightleftarrows}} \boldsymbol{Trg}[F-,-]$$

Here F is the left adjoint of the given functor G.

Consider some diagram

$$\boldsymbol{T}D \qquad T(i) \qquad T(e)$$

in $\boldsymbol{Trg}$. Here i ranges over the nodes of the template and e ranges over the edges. In this diagram there will be certain triangles that are required to commute.

We use the functor G to transport these objects and arrows to $\boldsymbol{Src}$.

$$\boldsymbol{S}D \qquad GT(i) \qquad GT(e)$$

Since G is a functor this is a diagram of the same template in $\boldsymbol{Src}$.

Suppose

$$T \xrightarrow{\tau(i)} T(i)$$

is a limit of the diagram $\boldsymbol{T}D$ in $\boldsymbol{Trg}$. Thus T is a fixed object and there is an arrow $\tau(i)$ for each node i of the template. Various triangles are required to commute, those indexed by the edges e of the template. We use G to transport this to $\boldsymbol{Src}$

$$GT \xrightarrow{G(\tau(i))} GT(i)$$

and since G is a functor we certainly obtain a left solution of the diagram $\boldsymbol{S}D$ in $\boldsymbol{Src}$.

Consider any left solution

$$X \xrightarrow{\xi(i)} GT(i)$$

of the diagram $\boldsymbol{S}D$ in $\boldsymbol{Src}$. We must somehow produce a unique mediator

$$X \xrightarrow{\mu} GT$$

for which

$$\begin{array}{ccc} & & GT \\ & \overset{\mu}{\nearrow} & \\ X & (\lhd) & \downarrow G(\tau(i)) \\ & \underset{\xi(i)}{\searrow} & \\ & & GT(i) \end{array}$$

commutes for each node i.

We use the transpositions

$$\boldsymbol{Src}[X, GT(i)] \xrightarrow{(\cdot)^\sharp} \boldsymbol{Trg}[FX, T(i)]$$

to obtain a family of arrows

$$FX \xrightarrow{\xi(i)^\sharp} T(i)$$

in $\boldsymbol{Trg}$. Since $(\cdot)^\sharp$ is natural, this is a left solution of the diagram $\boldsymbol{T}D$ in $\boldsymbol{Trg}$. Thus, since we have a limit of this diagram, here is a unique arrow ν such that each triangle

$$\begin{array}{ccc} & & T \\ & \overset{\nu}{\nearrow} & \\ FX & & \Big\downarrow \tau(i) \\ & \underset{\xi(i)^\sharp}{\searrow} & \\ & & T(i) \end{array}$$

commutes.

Since the transposition

$$\boldsymbol{Src}[X, GT] \xrightarrow{(\cdot)^\sharp} \boldsymbol{Trg}[FX, T]$$

is a bijection we have

$$\nu = \mu^\sharp$$

for some unique arrow

$$X \xrightarrow{\mu} GT$$

of $\boldsymbol{Src}$. It suffices to show that each triangle ($\lhd$) commutes. But this follows by the naturality of $(\cdot)^\sharp$ (or strictly speaking, by the naturality of the inverse $(\cdot)_\flat$ of $(\cdot)^\sharp$. ■

E.2 Adjunctions illustrated

E.2.1 An algebraic example

5.2.1 We deal with Σ first.

We are given

$$\Sigma X = X + X = \{(x, i) \mid x \in X,\ i = 0, 1\}$$

for each set X. The carried involution fips the tag, that is

$$(x, i)^\bullet = (x, 1 - i)$$

for each $x \in X$ and tag $i \in \{0, 1\}$. Since

$$(x, i)^{\bullet\bullet} = (x, 1 - i)^\bullet = (x, i)$$

this does produce an involution algebra.

For each function

$$Y \xrightarrow{\ k\ } X$$

the only sensible asignment

$$\begin{array}{ccc} \Sigma Y & \xrightarrow{\ \Sigma(k)\ } & \Sigma X \\ (y,i) & \longmapsto & (k(y),i) \end{array}$$

is to leave the tag alone. Since

$$\Sigma(k)\big((y,i)^\bullet\big) = \Sigma(k)\big((y,1-i)\big) = (k(y),1-i) = (k(y),i)^\bullet = \big(\Sigma(k)(y,i)\big)^\bullet$$

we see that $\Sigma(k)$ is a morphism.

The functorial requirments are immediate.

Next we deal with Π.

We are given

$$\Pi X = X \times X = \{(x,y) \,|\, x,y \in X\}$$

for each set X. The carried involution swaps the components, that is

$$(x,y)^\bullet = (x,y)$$

for each $x,y \in X$. Trivially, this does produce an involution algebra.

For each function

$$Y \xrightarrow{\ k\ } X$$

the only sensible asignment

$$\begin{array}{ccc} \Pi Y & \xrightarrow{\ \Pi(k)\ } & \Pi X \\ (y,z) & \longmapsto & (k(y),k(z)) \end{array}$$

is to apply the function to bothy components. Since

$$\Pi(k)\big((y,z)^\bullet\big) = \Pi(k)\big((z,y)\big) = (k(z),k(y)) = (k(y),k(z))^\bullet = \big(\Pi(k)(y,i)\big)^\bullet$$

we see that $\Pi(k)$ is a morphism.

The functorial requirments are immediate. ■

5.2.2 We deal with Σ first.

We require an inverse pair of assignments

$$\begin{array}{ccc} f & \longmapsto & f^\sharp \\ \textbf{\textit{Set}}[X,UA] & & \textbf{\textit{Inv}}[\Sigma X,A] \\ \psi_\flat & \longleftarrow & \psi \end{array}$$

for each set X and each algebra A. We set

$$f^\sharp\big((x,i)\big) = f(x)^{(i)} \qquad \psi_\flat(x) = \psi(x,0)$$

for each $x \in X$ and tag i. There are some requirements we must check.

We need to show that $f^\sharp$ is a morphism, that is

$$f^\sharp\big((x,i)^\bullet\big) = \big(f^\sharp(x,i)\big)^\bullet$$

for each $x \in X$ and tag i. To do that we remember that

$$a^{(i)\bullet} = a^{(1-i)} = a^{\bullet(i)}$$

for each $a \in A$. With this we have

$$f^\sharp\big((^\bullet x,i)\big) = f^\sharp(x,1-i) = f(x)^{(1-i)} = f(x)^{(i)\bullet} = \big(f^\sharp(x,i)\big)^\bullet$$

as required.

For each $x \in X$ and tag i we have

$$f^\sharp{}_\flat(x) = f^\sharp(x,0) = f(x)^{(0)} = f(x) \qquad \psi_\flat{}^\sharp(x,i) = \psi_\flat(x)^{(i)} = \psi(x,0)^{(i)} = \psi(x,i)$$

to show that the two assignments form an inverse pair. The last step on the right hand side follows since ψ is a morphism.

Next we deal with Π.

We require an inverse pair of assignments

$$\begin{array}{ccc} \phi & \longmapsto & \phi^\sharp \\ \boldsymbol{Inv}[A,\Pi X] & & \boldsymbol{Set}[UA,X] \\ g_\flat & \longleftarrow\!\!\!\mid & g \end{array}$$

for each set X and each algebra A. We set

$$\phi^\sharp(a) = \phi(a)_0 \qquad g_\flat(a) = (g(a), g(a^\bullet))$$

for each $a \in A$. In $\phi^\sharp$ the $(\cdot)_0$ indicates that we select the left and component.

We need to show that $g_\flat$ is a morpisms, that is

$$g_\flat(a^\bullet) = g_\flat(a)^\bullet$$

for each $a \in A$. But, remembering how ΠX is structured, we have

$$g_\flat(a^\bullet) = \big(g(a^\bullet), g(a^{\bullet\bullet})\big) = \big(g(a^\bullet), g(a)\big) = \big(g(a), g(a^\bullet)\big)^\bullet = g_\flat(a)^\bullet$$

as required

To show that these two assignments form an inverse pair consider any $a \in A$.

Let

$$\phi(a) = (x,y)$$

where $x, y \in X$. Then, since ϕ is a morphism, we have

$$\phi(a^\bullet) = \phi(a)^\bullet = (y,x)$$

so that

$$\phi^\sharp(a) = x \qquad \phi^\sharp(b) = y$$

to give

$$\phi^{\sharp}{}_{\flat}(a) = \big(\phi^{\sharp}(a), \phi^{\sharp}(a^{\bullet})\big) = (x, y) = \phi(a)$$

for one of the required conditions.

For the other, since

$$g_{\flat}(a) = (g(a), -)$$

we have

$$g_{\flat}{}^{\sharp}(a) = g(a)$$

as required. ■

E.2.2 A set-theoretic example

5.2.3 For this and the next solution let us write L and R for the two endofunctors on ***Set***. Thus

$$LX = X \times I \qquad RY = I \Rightarrow Y$$

for all sets X and Y. The arrow assignments are given in the subsection.

We require an inverse pair of bijections

$$\begin{array}{ccc} f & \longmapsto & f^{\sharp} \\ \boldsymbol{Set}[X, RY] & & \boldsymbol{Set}[LX, Y] \\ g_{\flat} & \longleftarrow\!\!\dashv & g \end{array}$$

for arbitrary X, Y.

Each member f of $\boldsymbol{Set}[X, RY]$ is a 2-step function which first consumes an element $x \in X$ and then an index $i \in I$ to return an eventual value $f(x)(i)$ in Y.

Each member g of $\boldsymbol{Set}[LX, Y]$ is a function which consumes a pair (x, i) where $x \in X$ and $i \in I$ to return a value $g(x, i)$ in Y.

The two transpositions merely shuffle brackets around. We have

$$f^{\sharp}(x, i) = f(x)(i) \qquad g_{\flat}(x)(i) = g(x, i)$$

for $x \in X$ and $i \in I$.

Normally in Mathematics we would hardly distinguish between f and $f^{\sharp}$, nor between g and $g_{\flat}$. ■

5.2.4 We continue with the notation of Solution 5.2.3.

We require a pair of assignments

$$X \xrightarrow{\eta_X} (R \circ L)X \qquad (L \circ R)Y \xrightarrow{\epsilon_Y} Y$$

where

$$(R \circ L)X = I \Rightarrow (X \times I) \qquad (L \circ R)Y = (I \Rightarrow Y) \times I$$

for sets X and Y.

For each $x \in X$ the value $\eta_X(x)$ must be a function which consumes some $i \in I$ and returns a pair in $X \times I$. Thus

$$\eta_X(x)(i) = (x, i)$$

is the only sensible suggestion.

The function ϵ_Y must consume a pair (p, i) where $p : I \to Y$ and $i \in I$ to return a member of Y. Thus

$$\epsilon_Y(p, i) = p(i)$$

is the only sensible suggestion.

We show that each of these is natural.

Recall that the arrow assignments of L and R are given by

$$\begin{array}{ccccc} X_2 & & X_2 \times I & (x, i) \\ k \big\downarrow & \longmapsto & \big\downarrow L(k) & \big\downarrow \\ X_1 & & X_1 \times I & (k(x), i) \end{array} \qquad \begin{array}{ccccc} Y_1 & & I \Rightarrow Y_1 & p \\ \big\downarrow l & \longmapsto & \big\downarrow R(l) & \big\downarrow \\ Y_2 & & I \Rightarrow Y_2 & l \circ p \end{array}$$

respectively.

To deal with $\eta_\bullet$ we must show that

$$\begin{array}{ccc} X_2 & \xrightarrow{\eta_{X_2}} & (R \circ L)X_2 \\ k \big\downarrow & & \big\downarrow (R \circ L)(k) \\ X_1 & \xrightarrow[\eta_{X_1}]{} & (R \circ L)X_1 \end{array}$$

commutes for an arbitrary function k, as on the left. Thus we require

$$\eta_{X_1} \circ k = (R \circ L)(k) \circ \eta_{X_2}$$

equivalently

$$\eta_{X_1}\big(k(x)\big) = R\big(L(k)\big)\big(\eta_{X_2}(x)\big)$$

for each $x \in X_2$. Each side of this equality is a function

$$I \longrightarrow (X \times I)$$

so we evaluate both at an arbitrary $i \in I$.

We have

$$\eta_{X_1}\big(k(x)\big)(i) = (k(x), i)$$

be the definition of $\eta_\bullet$.

The behaviour of R gives

$$R\big(L(k)\big)\big(\eta_{X_2}(x)\big) = L(k) \circ \eta_{X_2}(x)$$

so that

$$R\big(L(k)\big)\big(\eta_{X_2}(x)\big)(i) = L(k)\big(\eta_{X_2}(x)(i)\big) = L(k)(x, i) = (k(x), i)$$

by the behaviour of L, to give the required result.

To deal with $\epsilon_\bullet$ we must show that

$$\begin{array}{ccc} (L\circ R)Y_1 & \xrightarrow{\epsilon_{Y_1}} & Y_1 \\ {\scriptstyle (L\circ R)(l)}\downarrow & & \downarrow{\scriptstyle l} \\ (L\circ R)Y_2 & \xrightarrow[\epsilon_{Y_2}]{} & Y_2 \end{array}$$

commutes for an arbitrary function l, as on the right. Thus

$$\epsilon_{Y_2}\circ\big((L\circ R)(l)\big) = l\circ\epsilon_{Y_1}$$

is the required equality.

A typical member of $(L\circ R)Y_1$ is a pair (p,i) where $p : I \to Y_1$ and $i \in I$. We have

$$\big(l\circ\epsilon_{Y_1}\big)(p,i) = l\big(\epsilon_{Y_1}(p,i)\big) = l\big(p(i)\big) = (l\circ p)(i)$$

for each such pair.

We also have

$$\big(L\circ R\big)(l)(p,i) = L\big(R(l)\big)(p,i) = \big(R(l)(p),i\big) = (l\circ p,i)$$

for each such pair. This gives

$$\Big(\epsilon_{Y_2}\circ\big((L\circ R)(l)\big)\Big)(p,i) = \epsilon_{Y_1}\Big(\big(L\circ R\big)(l)(p,i)\Big) = \epsilon_{Y_1}(l\circ p,i) = (l\circ p)(i)$$

as required. ■

E.2.3 A topological example

5.2.5 As in the subsection, it suffices to show that for a continuous map

$$Y_1 \xrightarrow{\psi} Y_2$$

between two spaces, the induced assignment

$$\begin{array}{ccc} (I\Rightarrow Y_1) & \xrightarrow{\Psi} & (I\Rightarrow Y_2) \\ \theta & \longmapsto & \psi\circ\theta \end{array}$$

is continuous where each of the two functions spaces carries the compact open topology. To do that we consider an arbitrary subbasic open set $\langle K,V\rangle$ of $(I\Rightarrow Y_2)$ where $K\in\mathcal{K}I$ and $V\mathcal{O}Y_2$, and show

$$\Psi^{\leftarrow}\big(\langle K,V\rangle\big) = \langle K,\psi^{\leftarrow}(V)\rangle$$

which is a subbasic open set of $(I\Rightarrow Y_1)$

For each function $\theta : I \to Y_1$ we have

$$\begin{aligned}\theta \in \Psi^{\leftarrow}\big(\langle K, V\rangle\big) &\iff \psi \circ \theta \in \langle K, V\rangle \\ &\iff (\forall i \in I)[i \in K \Rightarrow \psi(\theta(i)) \in V] \\ &\iff (\forall i \in I)[i \in K \Rightarrow \theta(i) \in \psi^{\leftarrow}(V)] \iff \theta \in \langle K, \psi^{\leftarrow}(V)\rangle\end{aligned}$$

for the required result. ■

5.2.6 As suggested in the partial proof of Lemma 5.6, we consider a typical subbasic open set

$$\langle K, V\rangle$$

of $I \Rightarrow Y$, and show that

$$\psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

is open in X. Here

$$K \in \mathcal{K}I \qquad V \in \mathcal{O}Y$$

are the two components of the consider subbasic.

We consider an arbitrary

$$s \in \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

and show that

$$s \in U \subseteq \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$

for some open $U \in \mathcal{O}X$.

For the considered point s we have

$$\psi_\flat(s) \in \langle K, V\rangle$$

that is

$$\psi(s, i) = \psi_\flat(s)(i) \in V$$

for each $i \in K$.

For each $i \in K$ we have

$$(s, i) \in \psi^{\leftarrow}(V)$$

so that, since ψ is continuous, we have

$$(s, i) \in U_i \times W_i \subseteq \psi^{\leftarrow}(V)$$

for some $U_i \in \mathcal{O}X$ and $W_i \in \mathcal{O}I$. As i ranges through K the sets W_i produce an open covering of K. Since K is compact this refines to a finite covering

$$W = W_1 \cup \cdots \cup W_m$$

of K indexed by some $i(1), \ldots, i(m) \in K$. Let

$$U = U_1 \cap \cdots \cap U_m$$

using the same indexes. We have

$$s \in U \qquad K \subseteq W$$

with
$$U \times W \subseteq \psi^{\leftarrow}(V)$$
by construction. Also, for each $x \in X$ we have
$$x \in U \Longrightarrow (\forall i \in K)[\psi(x,i) \in V] \Longrightarrow \psi_\flat(x)[K] \subseteq V \Longrightarrow \psi_\flat(x) \in \langle K, V\rangle \Longrightarrow \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$
to give
$$U \subseteq \psi_\flat^{\leftarrow}\big(\langle K, V\rangle\big)$$
for the required result. ■

5.2.7 We continue with the partial proof of Lemma 5.7.

We start from any
$$(s, r) \in \phi^{\sharp\leftarrow}(V)$$
where $V \in \mathcal{O}Y$,and produce
$$U \times W \subseteq \phi^{\sharp\leftarrow}(V)$$
such that both
$$s \in U \in \mathcal{O}X \qquad r \in W \in \mathcal{O}I$$
hold. We already have
$$r \in W \subseteq K \subseteq \phi(s)^{\leftarrow}(V)$$
for some $K \in \mathcal{K}I$ and $W \in \mathcal{O}I$.

Observe that for $i \in I$ we have
$$i \in K \Longrightarrow i \in \phi(s)^{\leftarrow}(V) \Longrightarrow \phi(s)(i) \in V$$
for each $i \in I$. This gives
$$\phi(s)[K] \subseteq V$$
so that
$$\phi(s) \in \langle K, V\rangle$$
and hence
$$s \in U \quad \text{where} \quad U = \phi^{\leftarrow}\big(\langle K, V\rangle\big)$$
with U open in X.

From the construction of U and W, for each $x \in X$and $i \in I$ we have
$$(x, i) \in U \times W \Longrightarrow (x, i) \in U \times K \Longrightarrow \phi^\sharp(x, i) = \phi(x)(i) \in V$$
and hence
$$U \times W \subseteq \phi^{\sharp\leftarrow}(V)$$
for the final requirement. ■

5.2.8 For spaces X, Y we require continuous maps
$$X \xrightarrow{\ \eta_X\ } \big(I \Rightarrow (X \times I)\big) \qquad \big((I \Rightarrow Y) \times I\big) \xrightarrow{\ \epsilon_Y\ } Y$$
to do a certain job. We use the idea of the set-theoretic example of Subsection 5.2.2. Thus we set
$$\eta_X(x)(i) = (x, i) \qquad \epsilon_Y(p, i) = p(i)$$

for each $x \in X, i \in I$, and $p : I \to Y$.

It now looks as though we have quite a bit of work to do, but this is an illusion.

Why are

$$\eta_X \qquad \epsilon_Y$$

continuous? Observe that

$$\eta_X = \psi_\flat \quad \text{where} \quad \psi = \boldsymbol{id}_{X\times I} \qquad \epsilon_Y = \phi^\sharp \quad \text{where} \quad \phi = \boldsymbol{id}_{I\Rightarrow Y}$$

and hence Lemmas 5.6 and 5.7 give the required continuity.

Why are

$$\eta_\bullet \qquad \epsilon_\bullet$$

natural? We require that certain squares commute. But these are squares in ***Set***, and we have already verified that they do commute for the set-theoretic example. ■

E.3 Adjunctions uncoupled

5.3.1 We use the notation of this section as in Table 5.1.

Letting only A vary is equivalent to taking $S = T$ with $l = \boldsymbol{id}_S$. For this case ($\sharp$) and ($\flat$) become ($\sharp\uparrow$) and ($\flat\downarrow$), and these are equivalent as in Lemma 5.9.

Letting only S vary is equivalent to taking $A = B$ with $k = \boldsymbol{id}_A$. For this case ($\sharp$) and ($\flat$) become ($\sharp\downarrow$) and ($\flat\uparrow$), and these are equivalent as in Lemma 5.9. ■

5.3.2 As explained in Solutions 5.2.1 and 5.2.2, the three functors D, U, I and the four transpositions all return the consumed function (but perhaps viewed in a different way). Thus the various squares commute trivially, since they commute down in ***Set***. ■

5.3.3 Let us use the notation

$$L = - \times I \qquad R- = I \Rightarrow -$$

of Solutions 5.2.3 and 5.2.4.

To show that $(\cdot)^\sharp$ we must check that the square

$$\begin{array}{ccccc}
X_1 & [X_1, RY_1] & \xrightarrow{(\cdot)^\sharp} & [LX_1, Y_1] & Y_1 \\
k \uparrow & \downarrow R(l)\circ - \circ k & & \downarrow l \circ - \circ L(k) & \downarrow l \\
X_2 & [X_2, RY_2] & \xrightarrow[(\cdot)^\sharp]{} & [LX_2, Y_2] & Y_2
\end{array}$$

commutes for each pair of functions k and l. Here we need not indicate ***Set*** since it is the only category involved. In terms of equations we must show that

$$\big(R(l) \circ f \circ k\big)^\sharp = \big(l \circ f^\sharp \circ L(k)\big)$$

for each function

$$f : X_1 \longrightarrow (I \longrightarrow Y_1)$$

in the top left hand corner of the diagram. To do that the abbreviation

$$g = f \circ k$$

will be useful.

Each side of this equation is a function that consumes a pair

$$(x, i) \in X_2 \times I = LX_2$$

to return a value in Y_2.

We have

$$\begin{aligned}\big(R(l) \circ f \circ k\big)^\sharp(x,i) &= \big(R(l) \circ g\big)(x)(i) \\ &= R(l)\big(g(x)\big)(i) \\ &= \big(l \circ g(x)\big)(i) \qquad = l\big(g(x)(i)\big)\end{aligned}$$

which evaluates the left hand side.

We also have

$$\begin{aligned}\big(l \circ f^\sharp \circ L(k)\big)(x,i) &= \big(l \circ f^\sharp\big)\big(L(k)(x,i)\big) \\ &= \big(l \circ f^\sharp\big)\big(k(x), i\big) \\ &= l\big(f^\sharp\big(k(x), i\big)\big) \\ &= l\big(f(k(x))(i)\big) \qquad = l\big(g(x)(i)\big)\end{aligned}$$

which evaluates the right hand side, and verifies the equality.

The diagram for the naturality of $(\cdot)_\flat$ is similar but with two

$$\xleftarrow{\quad(\cdot)_\flat\quad}$$

pointing the other way. We have to show that

$$\big(R(l) \circ g_\flat \circ k\big) = \big(l \circ g \circ L(k)\big)_\flat$$

for each function

$$g : X_1 \times I \longrightarrow Y_1$$

in the top right hand corner. To prove the equality the abbreviation

$$f = l \circ g$$

will be useful.

Each side of this equation is a 2-step function which first consumes $x \in X_2$ and then $i \in I$ to return a value in Y_2.

We have

$$\begin{aligned}\big(R(l) \circ g_\flat \circ k\big)(x)(i) &= R(l)\big(g_\flat(k(x))\big)(i) \\ &= \big(l \circ g_\flat(k(x)\big)(i) \\ &= l\big(g_\flat(k(x)(i)\big) \\ &= l\big(g(k(x), i)\big) \qquad = f\big(k(x), i\big)\end{aligned}$$

which evaluates the left hand side.

We also have

$$\begin{aligned}\big(l \circ g \circ L(k)\big)_\flat(x)(i) &= \big(f \circ L(k)\big)_\flat(x)(i)\\ &= \big(f \circ L(k)\big)(x,i)\\ &= f\big(L(k)(x,i)\big) \qquad = f\big(k(x),i\big)\end{aligned}$$

which evaluates the right hand side, and verifies the equality. ■

5.3.4 At the function level the two assignment $(\cdot)^\sharp$ and $(\cdot)_\flat$ are just the same as those used in the Set-theoretic example of Subsection 5.2.2. The naturality of these topological versions requires that certain squares in ***Set*** must commute. These are just the same as the Set-theoretic squares, and are dealt with in Solution 5.3.3. ■

E.4 The unit and the co-unit

5.4.1 We must show that the square commutes

$$\begin{array}{lccc} \epsilon_S = \big(\boldsymbol{id}_{GS}\big)^\sharp & (F\circ G)S & \xrightarrow{\ \epsilon_S\ } & S \\ & {\scriptstyle (F\circ G)(l)}\downarrow & & \downarrow{\scriptstyle l} \\ \epsilon_T = \big(\boldsymbol{id}_{GT}\big)^\sharp & (F\circ G)T & \xrightarrow[\ \epsilon_T\ ]{} & T \end{array}$$

for an arbitrary arrow l as on the right. In equational terms we must show that

$$\epsilon_T \circ (F\circ G)(l) = l \circ \epsilon_S$$

holds. To do that we use $(\sharp)$ twice.

We have

$$\begin{aligned}\epsilon_T \circ (F\circ G)(l) &= \big(\boldsymbol{id}_{GT}\big)^\sharp \circ F(G(l))\\ &= \boldsymbol{id}_T \circ \big(\boldsymbol{id}_{GT}\big)^\sharp \circ F(G(l))\\ &= \big(G(\boldsymbol{id}_T)\circ \boldsymbol{id}_{GT} \circ G(l)\big)^\sharp \ = G(l)^\sharp\end{aligned}$$

where the penultimate step is the first use of $(\sharp)$.

We also have

$$l \circ \epsilon_S = l \circ \big(\boldsymbol{id}_{GS}\big)^\sharp = l \circ \big(\boldsymbol{id}_{GS}\big)^\sharp \circ F(\boldsymbol{id}_{GS}) = \big(G(l)\circ \boldsymbol{id}_{GS}\circ \boldsymbol{id}_{GS}\big)^\sharp = G(l)^\sharp$$

where the penultimate step is the second use of $(\sharp)$. ■

5.4.2 For each arrow

$$FA \xrightarrow{\ g\ } S$$

we show that

$$g_\flat = G(g)\eta_A$$

and to to that we use ($\flat\downarrow$). Thus

$$G(g)\circ\eta)A = G(g)\circ\big(\boldsymbol{id}_{FA}\big)_\flat = \big(g\circ\boldsymbol{id}_{FA}\big)_\flat = g_\flat$$

where the penultimate step uses ($\flat\downarrow$). ■

5.4.3 We must first show that for an arbitrary arrow

$$FA \xrightarrow{\ g\ } S$$

the second transpose $g_\flat{}^\sharp$ is just g. To do that we use an instance of the naturality of $\epsilon_\bullet$ as given in Solution 5.4.1. We use the case $l = g$. Thus

$$g_\flat{}^\sharp = \big(G(g)\circ\eta_A\big)^\sharp = \epsilon_S\circ F\big(G(g)\circ\eta_A\big) = \epsilon_S\circ\big(F\circ G\big)(g)\circ F\big(\eta_A\big) = g\circ\epsilon_{FA}\circ F\big(\eta_A\big) = g$$

where the penultimate step uses the naturality of ϵ and the ultimate step uses one of the given conditions on η and ϵ.

It remains to verify ($\flat$). Using the notation of Table 5.1, a use of the definition of $(\cdot)_\flat$ gives

$$\big(l\circ g\circ F(k)\big)_\flat = G\big(l\circ g\circ F(k)\big)\circ\eta_B = G\big(l\circ g\big)\circ\big(G\circ F\big)(k)\big)\circ\eta_B = G\big(l\circ g\big)\circ\eta_A\circ k$$

where this last step use the naturality of η. Continuing we have

$$\big(l\circ g\circ F(k)\big)_\flat = G\big(l\circ g\big)\circ\eta_A\circ k = G(l)\circ G(g)\circ\eta_A\circ k = G(l)\circ g_\flat\circ k$$

using the definition of $g_\flat$. ■

5.4.4 These are dealt with in using a slightly different notation in the Notes Free and Cofree monoid sets.[1]. ■

5.4.5 These are dealt with in the Notes Free and Cofree monoid sets.[2]. ■

5.4.6 We use the notation of Solution 5.2.3. Thus we have

$$LX = X\times I \qquad RY = I\Rightarrow Y$$

for sets or spaces X, Y.

For the first part we must show that for functions

$$X\xrightarrow{\ f\ } RY \qquad\qquad LX\xrightarrow{\ g\ } Y$$

both

$$f^\sharp = \epsilon_Y\circ L(f) \qquad g_\flat = R(g)\circ\eta_X$$

hold. Thus we must evaluate the composites

$$LX\xrightarrow{L(f)}(L\circ R)Y\xrightarrow{\ \epsilon_Y\ }Y \qquad\qquad X\xrightarrow{\ \eta_X\ }(R\circ L)X\xrightarrow{R(g)}RY$$

[1]Available as MonSets.pdf. Eventually this will be part of Chapter 6.
[2]Available as MonSets.pdf

and remember that $(\cdot)^\sharp$ and $(\cdot)_\flat$ merely shuffle brackets about.

For $(x,i) \in LX$ we have

$$\big(\epsilon_Y \circ L(f)\big)(x,i) = \epsilon_Y\big(L(f)(x,i)\big) = \epsilon_Y\big(f(x),i\big) = f(x)(i)$$

as required.

For $x \in X$ we have

$$\big(R(g) \circ \eta_X\big)(x) = R(g)\big(\eta_X(x)\big) = g \circ \eta_X(x)$$

and then for $i \in I$ we have

$$\big(R(g) \circ \eta_X\big)(x)(i) = \big(g \circ \eta_X(x)\big)(i) = g\big(\eta_X(x)(i)\big) = g(x,i)$$

as required.

For the second part we must show that

$$\epsilon_{LX} \circ L(\eta_X) = \boldsymbol{id}_{X\times I} \qquad R(\epsilon_Y) \circ \eta_{RY} = \boldsymbol{id}_{I\Rightarrow Y}$$

for arbitrary sets X and Y.

For $(x,i) \in LX$ we have

$$\big(\epsilon_{LX} \circ L(\eta_X)\big)(x,i) = \epsilon_{LX}\big(L(\eta_X)(x,i)\big) = \epsilon_{LX}\big(\eta_X(x),i\big) = \eta_X(x)(i) = (x,i)$$

as required.

For each function $p : I \to Y$ we have

$$\big(R(\epsilon_Y) \circ \eta_{RY}\big)(p) = R(\epsilon_Y)\big(\eta_{RY}(p)\big) = \epsilon_Y \circ \big(\eta_{RY}(p)\big)$$

and this composite is a function $I \to Y$. For each $i \in Y$ we have

$$\big(R(\epsilon_Y) \circ \eta_{RY}\big)(p)(i) = \epsilon_Y\big(\eta_{RY}(p)(i)\big) = \epsilon_Y(p,i) = p(i)$$

to give the required result. ■

E.5 Free and co-free constructions —*to be inserted*

E.6 Contravariant adjunctions—*to be inserted*