Rings with only commuting nilpotents are Dedekind finite. A direct (elementary) proof.

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In [2] we can find

Lemma 2.1 (iv) *A ring with commuting nilpotents is Dedekind-finite.*

The one sentence proof is:

A Dedekind-infinite ring contains an infinite set of matrix units.

For reader's convenience we provide a (small steps)

Direct proof. Suppose $ab = 1$ in a ring R and denote $e := ba$. Then $e^2 = b(ab)a = e$ is an idempotent and $1 - e$ is its complementary idempotent.

Also denote $e_{ij} := b^i(1-e)a^j$.

1. By easy computation $a(1-e) = 0 = (1-e)b$.

2. $e_{12}^2 = b(1-e)a^2b(1-e)a^2 = b(1-e)\underline{a(1-e)}a^2 \overset{\mathbf{1}}{=} 0$ and similarly $e_{21}^2 = 0$, so both are nilpotents.

3. As in the ring R the nilpotents commute, $e_{12}e_{21} = e_{21}e_{12}$, that is,

$$b(1-e)a = b^2(1-e)a^2$$

(using the initial notations this actually is $e_{11} = e_{22}$).

4. By left multiplication with a and right multiplication with b we get

$$1 - e = b(1-e)a.$$

5. By left multiplication with $e = ba$ we get $0 = b(1-e)a$ and so $1 - e = 0$, i.e., $ba = 1$.

Moreover, recalling that a ring R is *NR* if the set of nilpotents $N(R)$ forms a subring, in [1] we can find the following refinement

Proposition 1.7 *Rings with commuting nilpotents are NR.*

Obvious: Let $a, b \in N(R)$. If $N(R)$ is commutative, $a - b \in N(R)$ and $ab \in N(R)$.

Proposition 1.4. *NR rings are Dedekind finite.*

Here again, the given proof uses the fact that a Dedekind-infinite ring contains an infinite set of matrix units.

Again, we provide a

Direct proof (for **P.1.4**: if $N(R)$ forms a subring then R is Dedekind finite).

We use the notations above and start again with

1. By easy computation $a(1-e)=0=(1-e)b$.

2. $e_{12}^2 = b(1-e)a^2b(1-e)a^2 = b(1-e)\underline{a(1-e)}a^2 \overset{\mathbf{1}}{=} 0$ and similarly $e_{21}^2 = 0$, so both are nilpotents.

Now

3'. Since $N(R)$ is supposed to be a subring, $e_{11} = e_{12}e_{21} \in N(R)$. Hence $e_{11} = e_{11}^2 = 0$, that is $b(1-e)a = 0$.

Now again

4. $1-e = b(1-e)a$, which gives $1-e=0$, i.e. $ba=1$.

References

[1] Y. Chun, Y. C. Jeon, S. Kang, K. N. Lee, Y. Lee *A concept unifying the Armendariz and NI conditions*. Bull. Korean Math. Soc. **48** (1) (2011), 115-127.

[2] D. Khurana, G. Marks, A. K. Srivastava *On unit-central rings*. Advances in ring theory, 205-212, Trends Math., Birkhauser-Springer Basel AG, Basel, 2010.