A theorem on tridimensional linkage of closed curves

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Abstract

[French existing in the original paper] On appelle enlacement d 'ordre m un système $(C_1, ..., C_m)$ de m courbes fermées simples, orientées, 2 à 2 disjointes, de l 'espace euclidien tridimensionel. Dans le cas d 'un enlacement (C_1, C_2), ou C_1 et C_2 ont en chaque point une tangente qui varie continûment sur la courbe, on appelle tangente d 'appui toute tangente à l 'une des courbes, qui rencontre l 'autre courbe. En appliquant la caractéristique de Kronecker, l 'auteur démontre le théorème suivant: Le nombre des tangentes d 'appui de l 'enlacement (C_1, C_2) est au moins égal au quadruple du coefficient d 'enlacement $G(C_1, C_2)$ donné par l 'intégrale de Gauss.

We call a tridimensional *linkage of order* m, a system $(C_1, ..., C_m)$ of m mutually disjoint oriented closed curves, without multiple points, in the 3-dimensional Euclidean space. In the sequel we discuss linkages of order 2, (C_1, C_2), such that each curve has a tangent in every point which varies continuous along the curve. Any tangent to one of these curves, which intersects the other curve is called *supporting tangent*. Such tangents do not necessarily exist if the curves are not linked, that is, by a continuous deformation of these, such that the curves remain disjoint, without multiple points, one of them (say C_1) can be moved in the interior of a sphere, the other one (C_2) remaining in the exterior. But when C_1 and C_2 are linked, the existence of the supporting tangents seems to be necessary.

In what follows we establish a theorem which shows that if the linking coefficient $G(C_1, C_2)$, given by the double integral of Gauss, is not zero, the supporting tangents do exist, and their number is at least $4G(C_1, C_2)$. It is known that if $G(C_1, C_2) = 0$, the curves C_1 and C_2 still can be linked, so in this case the theorem below does not establish the existence of the supporting tangents, although these tangents seem to exist with the only condition that C_1 and C_2 are linked.

Let s_1 and s_2 be the length of the arc on C_1 and C_2, respectively. If $x_i = x(s_i)$, $y_i = y(s_i)$, $z_i = z(s_i)$ are the equations of the curve C_i and $\alpha_i = \frac{dx_i}{ds_i}$,

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$\beta_i = \frac{dy_i}{ds_i}$, $\gamma_i = \frac{dz_i}{ds_i}$ then the supporting tangents will be given by the system

$$F_1 \equiv x_2 - x_1 - u\alpha_1 = 0, F_2 \equiv y_2 - y_1 - u\beta_1 = 0, F_3 \equiv z_2 - z_1 - u\gamma_1 = 0, \quad (1)$$

where s_1, s_2 and u are the unknowns. In order to compute the number of solutions of this system we shall use the classical formulas deduced with the Kronecker characteristics [1], that is

$$4\pi N = \int\int \frac{1}{(F_1^2 + F_2^2 + F_3^2)^{3/2}} \begin{vmatrix} F_1 & \frac{\partial F_1}{\partial s_1} & \frac{\partial F_1}{\partial s_2} & \frac{\partial F_1}{\partial u} \\ F_2 & \frac{\partial F_2}{\partial s_1} & \frac{\partial F_2}{\partial s_2} & \frac{\partial F_2}{\partial u} \\ F_3 & \frac{\partial F_3}{\partial s_1} & \frac{\partial F_3}{\partial s_2} & \frac{\partial F_3}{\partial u} \\ 0 & ds_2 du & du ds_1 & ds_1 ds_2 \end{vmatrix} \quad (2)$$

where the surface integral is taken on the frontier of the infinite prism $0 \leq s_1 \leq L_1$, $0 \leq s_2 \leq L_2$, $-\infty \leq u \leq +\infty$, denoting by L_i the length of the curve C_i. The number N given in this formula is the algebraic sum of the indexes of the supporting tangents, if by *index* of such a tangent associated with a solution (s_1, s_2, u) of the system (1) we mean the corresponding number sign$\frac{D(F_1,F_2,F_3)}{D(s_1,s_2,u)}$.

This way one obtains $\frac{D(F_1,F_2,F_3)}{D(s_1,s_2,u)} = u\rho_1 S\alpha_2\alpha_1''$ with curvature ρ_1, the direction of the binormal α_1'', β_1'' γ_1'' in the point s_1 and S, a sum of three similar terms with the written one.

Further, consider the surface generated by the tangents of the curve C_1

$$X = x_1 + u\alpha_1, Y = y_1 + u\beta_1, Z = z_1 + u\gamma_1 \quad (3)$$

with parameters s_1 and u. The direction of the normal is given by $\frac{D(Y,Z)}{D(s_1,u)} = -u\rho_1\alpha_1''$ and $-u\rho_1\beta_1''$, $-u\rho_1\gamma_1''$. In a point where C_2 intersects this surface, the sign of $S\alpha_2\frac{D(Y,Z)}{D(s_1,u)} = -u\rho_1 S\alpha_2\alpha_1''$ shows that C_2 crosses the surface by entering the positive side, or conversely, which gives an interpretation of the above index. Moreover, note that N is an isotopy invariant of this linking, because, since the curves C_1 and C_2 do not cross in an isotopic deformation, the intersection points of C_2 with the surface (3), appear or disappear only in pairs, having opposite sign indexes. Formula (2) can be written

$$\begin{aligned} 4\pi N = & \int\int \frac{\begin{vmatrix} x_2 - x_1 \\ \alpha_1 \\ \alpha_2 \end{vmatrix} ds_2 du + u\rho_1 \begin{vmatrix} x_2 - x_1 \\ \alpha_1 \\ \alpha_1' \end{vmatrix} du ds_1 +}{[r_{12}^2 - 2u(\overrightarrow{r_{12}}\overrightarrow{t_1}) + u^2]^{3/2}} \\ & \frac{\left(\begin{vmatrix} x_2 - x_1 \\ \alpha_1 \\ \alpha_2 \end{vmatrix} + u\rho_1 \begin{vmatrix} x_2 - x_1 \\ \alpha_1 \\ \alpha_1' \end{vmatrix} + u^2\rho_1 \begin{vmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_1' \end{vmatrix} \right) ds_1 ds_2}{[r_{12}^2 - 2u(\overrightarrow{r_{12}}\overrightarrow{t_1}) + u^2]^{3/2}} \end{aligned} \quad (4)$$

where (to simplify the writing) we wrote only the first column of each determinant.

Since functions $x_i(s_i)$, $\alpha_i(s_i)$, ... are periodic, integrating on the sides of the above prism, the lateral sides contribution is zero, and for the bases of the prism, obtained for $u = \pm\infty$, the contribution is also zero. Hence $N = 0$, and so the number of supporting tangents to C_1, of positive index, is equal with the number corresponding to the negative index.

The tangents to C_1 being coherently oriented with the curve C_1 itself, let us call *prior supporting tangent*, a tangent to C_1 such that the supporting point on C_2, is situated after the tangency point, that is $u > 0$; similarly, we call *posterior supporting tangent*, one with $u < 0$. Since the index changes its sign together with u, we deduce that the number N_a of prior supporting tangents is equal to N_p the number or posterior supporting tangents.

Finally, let us calculate N_a; it suffices to compute the integral (4) on the prism $0 \leq s_1 \leq L_1$, $0 \leq s_2 \leq L_2$, $0 \leq u \leq +\infty$. The lateral sides have once again zero contribution, and the basis $u = 0$ leads to

$$4\pi N_a = \int_0^{L_1}\int_0^{L_2} \frac{1}{r_{12}^3} \begin{vmatrix} x_2 - x_1 \\ \alpha_1 \\ \alpha_2 \end{vmatrix} ds_1 ds_2 = 4\pi G(C_1, C_2),$$

i.e., precisely the Gauss integral representing the linking coefficient $G(C_1, C_2)$.

Therefore we have proved the following

Theorem 1 *The number of prior supporting tangents to a curve C_1 with respect to another curve C_2 is equal to the linking coefficient and equal to the number of posterior supporting tangents. If the curves are linked, the total number of supporting tangents (regardless of sign) is at least four times their linking coefficient.*

If C_1 and C_2 confound in a unique curve C, the system (1) has a singular solution $s_1 = s_2$, $u = 0$, which can be avoided, taking the integral with respect to u from $\varepsilon > 0$ to $+\infty$ and then taking $\varepsilon \to 0$. This way, one reaches the invariant K we have defined in [2].

References

[1] Picard E. *Traité d 'Analyse*, 3[e] éd., 150-156.

[2] Călugăreanu G. *Líntégrale de Gauss et l 'analyse des noeuds tridimensionels*, Revue de math. pures et appl., 1959, **IV**, 5-20.