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Utumi Abelian groups

Grigore Călugăreanu (∗) – Soumitra Das (∗∗)

Abstract – In a recent paper written by Y. Ibrahim and M. Yousif (Comm. in Algebra, 2018), the following class of modules is considered: a right R-module M is called a *Utumi module* if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exist direct summands K and L of M such that A is essential in K, B is essential in L and $K \oplus L$ is a direct summand of M. In this paper, all the Utumi $\mathbb{Z}$-modules (i.e. Abelian groups) and some special classes of these are determined. As an application, it is proved that all the pseudo-continuous Abelian groups are quasi-continuous.

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(∗) *Indirizzo dell'A.*: Babeş-Bolyai University, Cluj-Napoca, Romania.
E-mail: calu@math.ubbcluj.ro

(∗∗) *Indirizzo dell'A.*: KPR Institute of Engineering and Technology, Coimbatore, India.
E-mail: soumitrad330@gmail.com

1. Introduction

There is a two-way connection between Abelian group theory and Module theory. In one direction, notions and results for Abelian groups are sometimes generalized to modules (see for instance [6]), or, in the opposite direction, when notions and results arise for modules, and examples are given (more or less only) as Abelian groups, then the characterization of Abelian groups having these properties may be of interest (for both theories). Our paper falls in this last direction.

In [7] the following class of modules is considered. A right R-module M is called a *Utumi module* (U-module for short) if, whenever A and B are submodules of M with $A \cong B$ and $A \cap B = 0$, there exist direct summands K and L of M such that A is essential in K, B is essential in L and $K \oplus L$ is a direct summand of M. A useful fact which we use throughout is that *direct summands of U-modules are U-modules* (see [7, Proposition 3.2]).

The class of U-modules is a simultaneous and strict generalization of three fundamental classes of modules; namely, the *quasi-continuous*, the *square-free*, and the *automorphism-invariant* modules. The paper [7] includes a large number of examples. All these examples are $\mathbb{Z}$-modules, that is, Abelian groups.

Therefore, a natural project is to determine all the Utumi $\mathbb{Z}$-modules, that is, all the Utumi Abelian groups (U-groups for short). This is what we do in this note.

In the Abelian group case, we record the characterizations of all the special cases of U-groups listed above, and give another one: we prove that every *pseudo-continuous* group is quasi-continuous.

Our main result is the following:

Theorem. *Let G be an Abelian group. Then G is a U-group if and only if G has one of the following forms:*

(i) G is divisible (i.e. injective);

(ii) G is a torsion group, all whose primary components are isomorphic to a direct sum of copies of a cocyclic group (i.e. G is quasi-injective);

(iii) G is a torsion-free group of rank 1 (i.e. any subgroup of $\mathbb{Q}$);

(iv) G is a mixed group of torsion-free rank 1; in that case $G = Q \oplus H$, where Q is a quasi-injective torsion group and H is a mixed group of torsion-free rank 1 such that for all primes p with $T_p(H) \neq 0$ we have $T_p(H)$ is cyclic and $Q_p = 0$.

All the groups we consider are Abelian. For unexplained notions and results, we refer the reader to Laszlo Fuchs's treatise on Infinite Abelian Groups (2015, [5]). To simplify the writing, we shall use the term *homo(co)cyclic*, for direct sums of isomorphic (co)cyclic groups. By *cocyclic* p-groups we mean groups isomorphic to $\mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$ or to the (quasi-cyclic) Prüfer group $\mathbb{Z}(p^\infty)$.

For a group G, $r_0(G)$ and $r_p(G)$ denote the torsion-free rank and the p-rank of G, respectively. The term "component" will be used only for a "primary component" of some group. For a group G, G_p denotes the p-component of G and $D(G)$ denotes the maximum divisible subgroup of G. For a submodule K of a module M, $K \subseteq^{ess} M$ means K is essential in M and $K \subseteq^{\oplus} M$ means K is a direct summand of M.

2. The Abelian U-groups

First recall the following

Definitions. An R-module M is said to be *quasi-injective* if every R-homomorphism from any submodule can be extended to an R-endomorphism of M, is *square-free* (see [13]) if it contains no non-zero submodules isomorphic to a square $A \oplus A$, and is *automorphism-invariant* (auto-invariant for short) (see [10]) if it is invariant under any automorphism of its injective hull. It is *pseudo-injective* if every R-monomorphism from any submodule can be extended to an R-endomorphism of M. Clearly, quasi-injective modules are also pseudo-injective.

Next, recall that in [8] it is proved that over a principal ideal domain, all pseudo-injective modules are quasi-injective, and, that in [4] it was proved that a module is auto-invariant if and only if it is pseudo-injective.

Finally, recall that a module M is called *quasi-continuous* if every submodule of M is essential in a direct summand of M and direct sum of two direct summands of M intersecting trivially is again a direct summand of M, and *continuous* if every submodule of M is essential in a direct summand of M and every submodule of M isomorphic to a direct summand is itself a direct summand.

It is worth mentioning that the quasi-continuous Abelian groups can be traced in [5, Proposition 2.12, p. 138] and can be fully characterized using results in [15, Corollary 3.3]. Namely, an Abelian group G is *quasi-continuous* if and only if either G is quasi-injective or, if $G = T \oplus K$ where T is torsion divisible and K is a rank one torsion-free group (i.e. a proper subgroup of $\mathbb{Q}$). As for *continuous* Abelian groups, results in [15, Corollary

3.3] show that these are precisely the quasi-injective Abelian groups (see also [1]).

The quasi-injective groups were determined in [9], so we can use the

THEOREM 2.1. *The following conditions are equivalent.*

(i) the group G is pseudo-injective;

(ii) the group G is quasi-injective;

(iii) the group G is auto-invariant;

(iv) the group G is continuous;

(v) G is either injective (i.e. divisible) or is a torsion group with homococyclic components.

As for the square-free (Abelian) groups, since p-groups or torsion-free groups of rank at least two cannot be square-free we easily obtain the following characterization

THEOREM 2.2. *The square-free groups are:*

(i) torsion groups with cocyclic components, or

(ii) rank 1 torsion-free groups, or

(iii) direct sums $T \oplus F$ with torsion square-free T and torsion-free square-free F, if splitting mixed, or

(iv) groups of torsion-free rank 1 and each p-rank, at most 1, if not splitting mixed.

The determination of the Abelian U-groups is facilitated by results obtained in [7] (see Theorem 3.13, Corollary 3.18 and a special case of Corollary 3.7) and by the fact (proved in [7], Proposition 3.2) that *direct summands of U-modules are U-modules.*

The precise statement of Theorem 3.13 from [7] is the following

THEOREM 2.3. *If M is a U-module, then $M = Q \oplus T$ where*

(1) Q is a quasi-injective module,

(2) $Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a summand of $A \oplus B$,

(3) T is a square-free module,

(4) T is Q-injective, and

(5) Q and T are orthogonal.

Here a module N is called *M-injective* if every diagram in the category Mod-R with exact row

$$\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow & & \\ & & N & & \end{array}$$

can be extended commutatively by a morphism $M \to N$.

Further, the statement of Corollary 3.18 from [7] is the following

THEOREM 2.4. *If M is a non-singular right R-module, then M is a U-module if and only if $M = X \oplus Y$, where X is quasi-injective, Y is square-free, and X and Y are orthogonal.*

Finally, the statement of Corollary 3.7 from [7] is the following

THEOREM 2.5. *If $A \oplus B$ is a U-module such that A and B are subisomorphic, then $A \cong B$ and $A \oplus B$ is quasi-injective. In particular, $A \oplus A$ is a U-module if and only if A is quasi-injective.*

The previous theorems show that in order to find the U-modules we just have to look at the quasi-injective modules and at the square-free modules, and direct sums of these, respectively. Fortunately, for Abelian groups this can be done.

We are now ready to start the determination of the (Abelian) U-groups.

Since already for modules the following implications hold (see [12, Chapter 2, p. 18] and [7])

Injective ⇒ Quasi-injective ⇒ Continuous ⇒ Quasi-continuous ⇒ U-module

we obtain at once

PROPOSITION 2.6. *All the divisible groups are U-groups.*

Therefore, *arbitrary direct sums of quasicyclic groups* (i.e. $\mathbb{Z}(p^\infty)$ for some prime p) *and copies of $\mathbb{Q}$ are U-groups.* As customarily in Abelian group theory, one should expect *to reduce the study of U-groups* (via the divisible part) *to the study of reduced U-groups.*

While if $G = D(G) \oplus R$ is a U-group it follows that the (reduced) direct summand R is a U-group (direct summands of U-groups are U-groups), the converse fails.

As an example, the (genuine) mixed $\mathbb{Z}$-module $G := \mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Z}(p)\oplus\mathbb{Z}(p)$, is *not* a Utumi-module. This follows as a consequence of the special case of Theorem 2.5: *$A \oplus A$ is a U-module if and only if A is quasi-injective.* From a forthcoming result (see Proposition 2.8), it follows that $R = \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ is a U-group, by the above proposition it follows that $D(G) = \mathbb{Q} \oplus \mathbb{Q}$ is a U-group, but, since $\mathbb{Q} \oplus \mathbb{Z}(p)$ is not quasi-injective, $D(G) \oplus R$ is not a U-group. Notice that $r_0(D(G)) = 2$.

Recall that two modules are *orthogonal* if these have no nonzero isomorphic submodules.

For Abelian groups it is easy to describe which pairs of groups are orthogonal. We just gather these in the following

Lemma 2.7. *Two groups G, H are orthogonal if and only if*

(i) G is torsion-free and H is torsion;

(ii) G is mixed and H is torsion, whose components correspond to disjoint sets of primes;

(iii) G and H are torsion groups, whose components correspond to disjoint sets of primes.

We infer that (a) any two torsion-free groups are not orthogonal, (b) any torsion-free group and any (genuine) mixed group are not orthogonal, (c) any two (genuine) mixed groups are not orthogonal.

As customarily, in order to determine the reduced **torsion** U-groups we start with p-groups, for an arbitrary prime p.

Proposition 2.8. *A reduced p-group G is a U-group if and only if G is homocyclic.*

Proof. According to Theorem 2.3, $G = Q \oplus T$ with quasi-injective Q and square-free T.

By the previous characterizations, T is a cyclic p-group T_p (i.e. $\cong \mathbb{Z}(p^k)$ for $k \in \{1, 2, ..., \}$), and Q is a homocyclic p-group, that is, a direct sum of isomorphic cyclic p-groups. In what follows we refer to the conditions in Theorem 2.3.

Q satisfies condition (2). Since by (5) Q and T are orthogonal and both have (in their socle) a subgroup isomorphic to $\mathbb{Z}(p)$, one must be zero.

Therefore, in order to satisfy (1-3) and (5), G is cyclic or homocyclic. As for (4), it is readily seen that whenever $N = 0$ or $M = 0$, N is trivially M-injective. Hence, for the (two possible) cases above, (4) is also fulfilled. Since every cyclic group is also homocyclic, the statement follows. Conversely, just recall that the cyclic p-groups are square-free, and the homocyclic p-groups are (by Theorem 2.1) quasi-injective. $\square$

In what follows we show that *a p-group is a U-group only if it is divisible or reduced.* We start with an example.

LEMMA 2.9. *$\mathbb{Z}(p) \oplus \mathbb{Z}(p^\infty)$ is not a U-group.*

PROOF. The subgroup lattice of $\mathbb{Z}(p^\infty)$ is an infinite bounded chain $0 < \langle c_1 \rangle < \langle c_2 \rangle < ... < \langle c_n \rangle < ... < \mathbb{Z}(p^\infty)$ and the subgroup lattice of $\mathbb{Z}(p)$ is a two elements chain. Denote $G := \mathbb{Z}(p) \oplus \mathbb{Z}(p^\infty) = H \oplus K$ and $H = \langle a \rangle = \{0, a\}$, $K = \langle c_1, c_2, ..., c_n, ... \rangle$ with $pc_1 = 0$, $pc_2 = c_1$, ..., $pc_{n+1} = c_n$, ...

The subgroup lattice of G consists of the direct product of the chains, and the countable many "diagonals" D_n corresponding to the lattice isomorphisms of the "sections" $[0, H] \to [\langle c_{n-1} \rangle, \langle c_n \rangle]$ (for details see [2] or [16], p. 35-36).

Notice that the diagonals are cyclic subgroups, namely $D_n = \langle a + c_n \rangle \cong \langle c_n \rangle \cong \mathbb{Z}(p^n)$. Also notice that the subgroups $H \oplus \langle c_n \rangle \cong \mathbb{Z}(p) \oplus \mathbb{Z}(p^n)$ are not cyclic. It is readily seen that the choice of two isomorphic subgroups $A \cong B$ with $A \cap B = 0$ is possible only for $(A, B) \in \{(H, D_1), (H, \langle c_1 \rangle), (D_1, \langle c_1 \rangle)\}$.

Moreover, *the only direct summands of G are H* and D_1, as complements of K (two subgroups different from K have not the sum equal to G, and K is disjoint only from H and D_1).

Take the pair (H, D_1). Then $H \subseteq^{ess} K$, $D_1 \subseteq^{ess} L$ for two direct summands K, L of G, is possible only if $H = K$ and $D_1 = L$. However $K \oplus L = H \oplus D_1$ is not a direct summand of G. $\square$

In a similar way, one can show that $\mathbb{Z}(p^k) \oplus \mathbb{Z}(p^\infty)$ is not a U-group, for any positive integer k.

PROPOSITION 2.10. *A p-group is a U-group only if it is divisible or reduced.*

PROOF. Suppose G is a p-group which is not divisible nor reduced but a U-group. Since the divisible part is a direct summand, $G = D(G) \oplus R$ for some reduced subgroup R. Since, as direct summand, R is a U-group, according to Proposition 2.8 (and the structure of divisible p-groups), G has a direct summand isomorphic to $\mathbb{Z}(p^k) \oplus \mathbb{Z}(p^\infty)$, which is a U-group.

This contradicts the generalization (mentioned above) of the previous Lemma. □

Next, we need the following easy to foresee

PROPOSITION 2.11. *A torsion group is a U-group if and only if all its primary components are U-groups.*

PROOF. One way is clear since direct summands of U-groups are U-groups: the p-components of a torsion U-group are also U-groups.

Conversely, notice that the p-components of any torsion group are fully invariant direct summands. To simplify the writing, suppose $G = G_p \oplus G_q$ is the primary decomposition for a group G, with different primes p, q and suppose both G_p, G_q are U-groups. Let A, B be subgroups of G with $A \cong B$ and $A \cap B = 0$. Decompose both A and B into components, say, $A = A_p \oplus A_q$ and $B = B_p \oplus B_q$. Clearly, $A_p \cong B_p$ and $A_p \cap B_p = 0$ and the same for the q-components. Since both G_p, G_q are U-groups, there exist direct summands K_p, L_p of G_p such that $A_p \subseteq^{ess} K_p$, $B_p \subseteq^{ess} L_p$ and $K_p \oplus L_p \subseteq^{\oplus} G_p$ and the same for the q-components. Finally, if $K := K_p \oplus K_q$ and $L := L_p \oplus L_q$, it is easy to check $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L \subseteq^{\oplus} G$. □

COROLLARY 2.12. *A torsion group is a U-group if and only if it is divisible or it has homococyclic components.*

PROOF. Another similar proof (notice that the proof of the previous lemma is lattice theoretic, dealing with chains of subgroups of cocyclic groups) shows that $\mathbb{Z}(q^k) \oplus \mathbb{Z}(p^\infty)$ is not a U-group, for any different primes p, q. Therefore, *a torsion U-group is divisible or reduced* and we use Proposition 2.10. □

In closing the discussion on torsion U-groups, it is worth mentioning the following simple result observed by S. Breaz.

PROPOSITION 2.13. *Every fully invariant subgroup of a U-group is a U-group.*

PROOF. Let H be a fully invariant subgroup of a U-group G, and $A \cong B$, $A \cap B = 0$ subgroups of H. There are direct summands K, L of G such that $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L$ is a direct summand of G. Consider the subgroups $H \cap K$, $H \cap L$ of H and suppose $G = K \oplus K'$. Since H is fully invariant, $H = (H \cap K) \oplus (H \cap K')$ (see [5, Lemma 2.3, p. 51]) shows that $H \cap K$ (and similarly $H \cap L$) is a direct summand of H. Clearly, $A \subseteq^{ess} H \cap K$, $B \subseteq^{ess} H \cap L$ and $(H \cap K) \oplus (H \cap L)$ is a direct summand in H (using again the fact that H is fully invariant). □

Notice that the proof relies only on [5, Lemma 2.3, p. 51], whose proof extends *verbatim* to modules. Hence, *fully invariant submodules of U-modules are U-modules,* a result which we could not find in [7] (it was not necessary). Therefore

COROLLARY 2.14. *The torsion subgroup of any U-group is a U-group.*

The previous proposition also gives an alternative proof for: *the divisible part of any U-group is a U-group.*

Since the non-singular $\mathbb{Z}$-modules are precisely the **torsion-free** groups, for the determination of the torsion-free U-groups, we use Theorem 2.4.

Recall that X is torsion-free quasi-injective if and only if X is divisible (i.e. a direct sum of copies of $\mathbb{Q}$) and Y is torsion-free square-free if and only if it is of rank 1 (any subgroup of $\mathbb{Q}$). Since the orthogonality condition is exclusive ($\mathbb{Q}$ and any rank 1 torsion-free group have subgroups isomorphic to $\mathbb{Z}$), we obtain

PROPOSITION 2.15. *A torsion-free group G is a U-group if and only if G is a (finite or infinite) direct sum $\mathbb{Q} \oplus \mathbb{Q} \oplus ...$, or G is isomorphic to any proper subgroup of $\mathbb{Q}$.*

COROLLARY 2.16. *A reduced torsion-free group is a U-group if and only if it is isomorphic to a proper subgroup of $\mathbb{Q}$.*

As an example, $\mathbb{Z}$ is a U-group, but free groups (i.e. direct sums of $\mathbb{Z}$) of rank at least 2 are not U-groups. That $\mathbb{Z}$ is a U-group follows also from the fact that, being locally cyclic, it has a distributive subgroup lattice and, more general (see [14], Lemma 4.4), *distributive modules are square-free* (and so U-modules).

Finally we characterize the **mixed** U-groups. First we separate the mixed groups whose torsion-free rank is at least 2.

PROPOSITION 2.17. *If G is a U-group of torsion-free rank at least 2 then G is divisible.*

Proof. Let G be a U-group with $r_0(G) \geq 2$. By Theorem 2.3, $G = Q \oplus T$ with quasi-cyclic Q and square-free T. Since these two summands are orthogonal, both cannot contain infinite order elements, We go into two cases.

Case 1. Q is torsion (with homococyclic components) and $r_0(T) = r_0(G) \geq 2$. This cannot happen since T is square-free T (see Theorem 2.2).

Case 2. Q is divisible with $r_0(Q) \geq 2$ and T is square-free torsion, that is, with cocyclic components. If T has finite cocyclic components, this does not fulfill the condition (4) in Theorem 2.3: T is not Q-injective. Indeed, this reduces to the easy to check fact that $\mathbb{Z}(p^k)$ is not $\mathbb{Q}$-injective. So T is (torsion) divisible, and so is G (together with Q). □

Finally we describe the mixed U-groups of torsion-free rank 1.

Theorem 2.18. *A group G of torsion-free rank 1 is a U-group if and only if $G = Q \oplus H$, where Q is a quasi-injective torsion group and H is a mixed group of torsion-free rank 1 such that for all primes p with $T_p(H) \neq 0$ we have $T_p(H)$ is cyclic and $Q_p = 0$.*

Proof. By Theorem 2.3, $G = U \oplus V$ with U quasi-injective, V square-free, and U and V being orthogonal.

Suppose that $r_0(U) = 1$. Then (by Theorem 2.1) $U = L \oplus C$ where L is isomorphic to $\mathbb{Q}$ and C is a quasi-injective torsion group. Taking $H = L$ and $Q = C \oplus V$, it follows (by Corollary 2.14) that $Q = T(G)$ is an U-group and so (by Theorem 2.1) it is quasi-injective. In this case, the condition on H is trivially satisfied.

Suppose that $r_0(V) = 1$ and let p be a prime such that $T_p(V) \neq 0$. Notice that, by the orthogonality condition, in this case U is quasi-injective torsion. Then, since V is square-free (and so U-group), as p-group of p-rank 1, $T_p(V)$ is cocyclic (according to Proposition 2.10). Since its divisible part can be included in U, we can choose $T_p(V)$ being cyclic. Using again the orthogonality condition, it is easy to see that U cannot have elements of order p. Hence $T_p(U) = 0$. It remains to take $Q = U \oplus D(V)$ and H, any reduced part of V.

Conversely, suppose that A and B are disjoint subgroups of G such that $A \cong B$. Then A and B must be torsion subgroups. Moreover, since the p-components $T_p(H)$ are cyclic, it follows that A and B are contained in Q. Since Q is quasi-injective (and so U-group), it now follows that A and B

can be embedded as essential subgroups of some direct summands K and L of G such that $K \oplus L$ is a direct summand of G. $\square$

Open question. Are pure subgroups of U-groups also U-groups ? As mentioned above, the torsion part of a U-group is a pure U-subgroup.

3. An application

First we recall some definitions and known results.

A module M is said to satisfy the *C1* condition (or *CS* or *extending*) if every submodule of M is essential in a direct summand (equivalently, each complement submodule is a direct summand). A module M is said to satisfy the *C2* condition, if every submodule isomorphic to a summand of M is itself a summand of M. A module M satisfies the *C3* condition, if the sum of any two summands of M with zero intersection is a summand of M.

A module M is called (see [3]) a *C4-module* if, whenever A_1 and A_2 are submodules of M with $M = A_1 \oplus A_2$ and $f : A_1 \to A_2$ is an R-homomorphism with $\ker f \subseteq^{\oplus} A_1$, we have $\mathrm{Im} f \subseteq^{\oplus} A_2$.

As already mentioned in Section 2, a module is called *continuous* if it satisfies both the C1 and C2 conditions, and is called *quasi-continuous* if it satisfies both the C1 and C3 conditions.

A module M is called *pseudo-continuous* if it is both a C1-module and a C4-module. It is proved in [7] (Corollary 2.15) that *pseudo-continuous modules are U-modules*. Since C3 modules are C4, *quasi-continuous modules are pseudo-continuous.*

The characterization of quasi-continuous groups was mentioned in Section 2.

For reader's convenience, we recall the following result

THEOREM 3.1. *(a) A torsion Abelian group G is C1 if and only if it is divisible, or it is a sum of cyclic groups, such that for each prime number p there is a positive integer $n = n(p)$ such that the p-component $G_p \simeq (\bigoplus_s \mathbb{Z}(p^n)) \oplus (\bigoplus_t \mathbb{Z}(p^{n+1}))$ with (possible zero) cardinals s, t.*

(b) A reduced torsion-free Abelian group is C1 if and only if it is homogeneous completely decomposable of finite rank.

(c) An Abelian group is C1 if and only if it is torsion C1 (see (a)), or the direct sum of a torsion-free reduced C1 group (see (b)) and an arbitrary divisible group.

In [3], to find an example of a pseudo-continuous module that is not quasi-continuous was left an open question. The next proposition, which follows using our results in the previous section, shows that such an Abelian group example does not exist.

PROPOSITION 3.2. *All pseudo-continuous (Abelian) groups are quasi-continuous.*

PROOF. From Theorem 3.1 it follows that, being C1, the pseudo-continuous groups are splitting. Since these are also U-groups, by Theorem 2.18, these are direct sums of quasi-injective groups and rank 1 torsion-free groups. But (by the characterization before Theorem 2.1) such groups are indeed quasi-continuous. □

It is worth mentioning that an elaborate *example of square-free module which is not C3* was found by P. P. Nielsen (see [3], Example 2.10 and [14], Example 6.1). This also works as an example of a C4-module that is not a C3-module. As for an example of pseudo-continuous module which is not quasi-continuous (i.e. a C1 + C4 module which is not C3), this seems (so far) to be an open question (see also Question 4.4.23, [11]).

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