# A prime connected ring

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According to Paul Cohn, the following **example** was suggested by W. S. Martindale (conversation) and appears in [6] (1973).

**Cohn** Let $k$ be a field and $R$ the $k$-algebra generated by $x$ and $y$ with the single defining relation $x^2 = 0$. Using free ring techniques we prove that the set of left zero-divisors of $R$ is $Rx$. There is a unique involution fixing $x, y$ and this makes $R$ into a prime ring with involution whose symmetric zero-divisors are nilpotent.

*$R$ is prime (and so also semiprime)*, since for any two nonzero elements $r, s \in R$ we have $rys \neq 0$.

The important subsets of $R$ are summarized in [8].

**Lemma 3.18**. Let $F$ be a field and $R = F\langle x, y : x^2 = 0\rangle$. The following hold:

(1) If $ab = 0$ for some nonzero $a, b \in R$ then $a \in Rx$ and $b \in xR$. In particular, the set of nilpotent elements of $R$ is precisely $Rx \cap xR = Fx + xRx$.

(2) The idempotents of $R$ are trivial, so $idem(R) = \{0, 1\}$.

(3) The ring $R$ is directly finite.

(4) Units in $R$ are exactly the elements of the form $\mu + a$ where $\mu \in F\backslash\{0\}$ and $a \in Rx \cap xR$. In particular, $U(R) + Fx \subseteq U(R)$, and $1 - yx \notin U(R)$.

*Proof.* **(1)** is found in [[5], **Example 9.3 - too long to reproduce here**].

**(2)**: If $e$ is a nontrivial idempotent in $R$ then $e$ is both a left and a right zero divisor, so that $e \in Rx \cap xR$ by (1). Hence $e$ is a nilpotent and thus $e = 0$, a contradiction. (Alternatively, this follows by an easy minimal degree argument - see below.)

**(3)** follows from (2) since $ab = 1$ always implies that $ba$ is a nonzero idempotent (when $1 \neq 0$).

**(4)**: Clearly $(F\backslash\{0\}) + (Rx \cap xR) \subseteq U(R)$, so it suffices to prove the other inclusion. Let $u \in U(R)$. We may write $u = \mu + u_1 + u_2 + u_3 + u_4$ with $\mu \in F$, $u_1 \in Rx \cap xR$, $u_2 \in Rx \cap yR$, $u_3 \in Ry \cap xR$ and $u_4 \in Ry \cap yR$. Clearly, $\mu \neq 0$. We need to prove that $u_2 = u_3 = u_4 = 0$.

Let $v := u - 1 = \mu + v_1 + v_2 + v_3 + v_4$, with $\mu \in F$, $v_1 \in Rx \cap xR$, $v_2 \in Rx \cap yR$, $v_3 \in Ry \cap xR$, and $v_4 \in Ry \cap yR$. We have $xu \cdot vx = 0$ and $xu, vx \neq 0$, so that (1) yields $xu \in Rx$. This gives $xu_4 \in Rx$, so that $u_4 = 0$. Similarly, $v_4 = 0$.

Suppose that $u_2 \neq 0$ and $v_2 \neq 0$. Taking any monomial $p$ in $u_2$ of the largest degree, and any monomial $q$ in $v_2$ of the largest degree, we see that the monomial $pq$ cannot cancel with any other monomial in the product $uv$, so that $uv \neq 1$, which is a contradiction. Thus $u_2 \neq 0$ forces $v_2 = 0$.

Similarly, $u_3 \neq 0$ forces $v_3 = 0$. Therefore, if both $u_2, u_3 \neq 0$ 0 then $v = \nu + v_1$, which gives $u = v^{-1} = \nu^{-1} - -\nu^{-2}v_1$, a contradiction. Hence $u_2 = 0$ or $u_3 = 0$; we may assume by symmetry that $u_3 = 0$.

Finally, suppose that $u_2 \neq 0$, so that $v_2 = 0$. Then $x = xvu = x(\nu + v_1 + v_3)(\mu + u_1 + u_2) = x(\mu + u_1 + u_2) = \mu\nu x + \nu x u_2$. Hence $xu_2 \in Fx$, which is again a contradiction. Thus $u_2 = 0$, which completes the proof (alternatively, see also (**iii**) below).

(**2**) We show (Bergman) that $R$ *is connected* (i.e., has only trivial idempotents).

Note that $R$ is graded over the natural numbers by total degree in $x$ and $y$, and that its residue field (quotient by the ideal of elements of positive degree) is $k$. This implies that the only idempotents are 0 and 1.

To see this, let $I \subset R$ be the ideal of elements of positive degree. Then $R/I \cong k$, so for every idempotent $e$ of $R$, the image of $e$ in $R/I$ is an idempotent of $k$, i.e., 0 or 1.

Replacing $e$ by $1 - e$ if its image is 1, we see that it suffices to show that the only idempotent element in $I$ is 0. And indeed, since $I$ is graded by the positive integers, for any idempotent $e$ we have $\deg(e) = \deg(e^2) \geq 2\deg(e)$, so $\deg(e) = \infty$, i.e., $e = 0$.

This example was used (at least):

**(i)** Initially in [2], after observing that *local rings with nilpotent maximal ideals and residue field* $\mathbb{F}_2$ are UU, as an

**example** of a ring which has neither of these properties but is UU. The $\mathbb{F}_2$-algebra presented by two generators $x, y$ and one relation $x^2 = 0$. It has only units of the form $1 + xax$ (with nilpotent $xax$).

(**ii**) This example was ceded as

**Example 2.5** [7] If $R$ is a commutative UU-ring, we have observed in (3) above that $rad(R) = nil(R)$. However, if $R$ is a general UU-ring, $rad(R) \subseteq nil(R)$ may be a strict inclusion.

Such a UU-ring $R$, constructed by G. Bergman, is presented here with his kind permission. Let $R$ be the $\mathbb{F}_2$-algebra generated by $x, y$ with the single relation $x^2 = 0$. Using his result on coproducts from [[1] **Corollary 2.16**], Bergman showed that $U(R) = 1 + \mathbb{F}_2 x + xRx$. Since $(\mathbb{F}_2 x + xRx)^2 = 0$, we have $\mathbb{F}_2 x + xRx \subseteq nil(R)$, so $R$ is a UU-ring. For any nonzero $r \in R$, it is easy to see that $1 + yry \notin 1 + \mathbb{F}_2 x + xRx$.

Therefore, $rad(R) = \{0\}$, which is properly contained in the set $nil(R)$.

as

(**iii**) an **example** (see [4]) of a prime ring (and so semiprime too) which is *not unit-semiprime*.

To show (Bergman) that it is not unit-semiprime, we need to know its group of units. To do this, regard $R$ as the coproduct over $k$ of $R_1 = k[x|x^2 = 0]$ and $R_2 = k[y]$. Then from Corollary 2.16 of [1], one can deduce that the units of $R$ are just the elements $c + dx + xrx$, where $c, d \in k$ and $r \in R$. We see that for any such unit, we have $x(c + dx + xrx)x = 0$, so unit-semiprimeness fails.

as

(**iv**) an **example** (see [8]) of a ring $S$ and a regular element $a \in S$ such that $a^3 = 0$, but $a$ is not unit-regular in $S$.

as

(**v**) an **example** (see [5]) of Armendariz ring which is not semi-commutative.

and as

(**vi**) **Example 2.1** [3] A prime ring which is not idempotent semiprime.

A very likely, many others.

# References


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