Abelian groups whose subgroup lattice is the union of two intervals

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Abstract

In this note we characterize the abelian groups G which have two different proper subgroups N and M such that the subgroup lattice $L(G) = [0, M] \cup [N, G]$ is the union of these intervals.

For every subgroup H of an arbitrary group G, the interval $[H, G]$ is a compactly generated (algebraic) sublattice in the subgroup lattice $L(G)$.

After 1989, when Tuma ([4]) showed that *every algebraic lattice is isomorphic to an interval in the subgroup lattice of some group* (improving Whitman's theorem ([5], 1946) - *every lattice is isomorphic to a sublattice of the subgroup lattice of a group* - as far as possible), an increasing role of intervals, in subgroup lattices of groups, was noticed.

In [1], an arbitrary group G was called a *BP-group* if it has a proper subgroup H such that the subgroup lattice $L(G)$ is the union of the intervals $[1, H]$ and $[H, G]$ (i.e., any subgroup of G is either contained in H or contains H). The subgroup H was called a *breaking point* for the lattice $L(G)$. It was pointed out that the abelian BP-groups are the nonsimple cocyclic groups (i.e., up to isomorphism, $\mathbf{Z}(p^k)$ with $k > 1$ or ∞).

Roland Schmidt suggested the study of finite groups which satisfy a weaker condition: groups G having two proper subgroups N and M such that every subgroup H of G either contains N or is contained in M. In this situation the subgroup lattice $L(G)$ is again union of two intervals, namely $[1, M]$ and $[N, G]$ (such groups appeared in the study of affinities of groups - see for example **9.4.14** in [3] - but there are much more examples of this kind).

In this paper, instead of finite groups, we characterize the abelian groups which share this property. Our result is the following:

Theorem 1 *An abelian group G has two proper subgroups $N \neq M$ such that the subgroup lattice $L(G) = [0, M] \cup [N, G]$ if and only if G is a torsion group with a primary component $G_p \cong \mathbf{Z}(p^n) \oplus B$, $n \in \mathbf{N}^* \cup \{\infty\}$ such that $p^l B = 0$ holds for a nonnegative integer $l < n$.*

Additive notation is used and from now on, "group" means "abelian group". $\mathbf{N}$ denotes the set of all nonnegative integers, $\mathbf{P}$ denotes the set of all prime numbers and standard interval notation is used. $h_p(b)$ denotes the p-height of b.

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We first mention the following simple

Necessary condition: *N must be cyclic.* Indeed, take $x \in G \setminus M$. Then $\langle x \rangle \in [0, M]$ being not possible, $\langle x \rangle \in [N, G]$ or $N \leq \langle x \rangle$.

Next, notice there are three distinct possibilities with respect to subgroups N and M:
(**A**) N and M are not comparable;
(**B**) $M < N$;
(**C**) $N < M$ (e.g., the above mentioned example [3]).

1 Abelian groups with (A)

In this section we suppose M and N not comparable and $L(G) = [0, M] \cup [N, G]$. In this case $[0, M] \cap [N, G] = \emptyset$ (otherwise $N \leq M$). The following remarks are straightforward

(a) $M \cap N$ is the largest element in $[0, N)$ and $M + N$ is the smallest element in $(M, G]$.

(b) $L(M + N) = [0, M] \cup [N, N + M]$, i.e., $N + M$ has property **(A)**.

(c) $L(G/(M \cap N)) = [0, M/(M \cap N)] \cup [N/(M \cap N), G/(M \cap N)]$, i.e., $G/(M \cap N)$ has property **(A).**

(d) $(M + N)/(M \cap N)$ has property **(A)**.

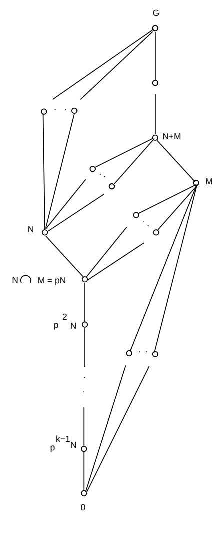
Actually, more can be proved

Lemma 1.1 *If $L(G) = [0, M] \cup [N, G]$, there is a prime number p such that*
(a) *N is a (co)cyclic p-group and $M \cap N = pN$ is maximal in N;*
(b) *G/M and $G/(M + N)$ are p-groups.*

Proof. (a) We have already noticed that N has to be cyclic. By the above remark (a), N is a (co)cyclic p-group (for a suitable prime number p). Moreover, since $M \cap N$ is its largest (proper) subgroup, $pN = M \cap N$.

To prove (b), we observe that G/M is a cocyclic group since it has a smallest subgroup, namely $(M + N)/M$. Moreover, since $(M + N)/M \cong N/(N \cap M) \cong \mathbf{Z}(p)$, G/M is a cocyclic p-group, and so $G/(M + N)$ has the same property. $\square$

Therefore the subgroup lattice is represented by the following diagram [u1.eps]



If $N \simeq \mathbf{Z}(p^k)$ it is readily seen that for $k = 1$, N is minimal and hence the sum $N + M$ is direct (otherwise $N \cap M = N$ and N, M are comparable). Actually this is the only case $N \cap M = 0$.

The following lemma will be used in the proofs of the main results of both this and next sections.

Lemma 1.2 *For a group G and $g \in G$ let p be a prime such that $K = G/\langle g \rangle$ is cocyclic p-group. If $h_p(g) \neq 0$ and G is not infinite cyclic, then $G = H_1 \oplus H_2$ for cocyclic p-group H_1 and finite cyclic group H_2 of coprime order with p such that $H_2 \leq \langle g \rangle$ ($H_2 = 0$ is not excluded).*

Proof. Since for cocyclic group G the decomposition is trivial, suppose G is not cocyclic (and so $g \neq 0$). As $r(G) \leq r(K) + r(\langle g \rangle) = 2$, we have $r(G) = 2$ and by $r_0(G) = r_0(\langle g \rangle) + r_0(K) \leq 1$, we obtain $G = H_1 \oplus H_2$ with $r(H_1) = r(H_2) = 1$ - i.e., each H_i is cocyclic or infinite cyclic (if $r_0(G) = 1$, the torsion subgroup of G is cocyclic, hence G splits). If $g = h_1 + h_2$ with $h_i \in H_i$, since $h_p(g) \geq 1$, there exist $x_1 \in H_1$ and $x_2 \in H_2$ such that $px_1 = h_1$ and $px_2 = h_2$. Moreover, $L(G/\langle g \rangle)$ is a chain and we can suppose $x_2 + \langle g \rangle \in (\langle x_1 \rangle + \langle g \rangle)/\langle g \rangle$.

Thus $x_2 \in \langle x_1 \rangle + \langle g \rangle$ and $x_2 = sx_1 + tg$ or $px_2 = spx_1 + tpg$ for suitable integers s, t. Hence $h_2 = sh_1 + tp(h_1 + h_2)$ and, the sum $H_1 \oplus H_2$ being direct, $(tp - 1)h_2 = 0$.

If $h_2 = 0$ then $g \in H_1$ and K is cocyclic if and only if $\langle g \rangle = H_1$ or $H_2 = 0$. In the first case $h_p(g) = 0$, hence $H_2 = 0$ and $G = H_1$ is a cocyclic p-group (since, by hypothesis, G is not infinite cyclic).

If $h_2 \neq 0$, the order of h_2 (say l) is finite and coprime with p. Therefore H_2 is a cocyclic q-group (if l is a power of the prime q) and this implies $H_2 \leq \langle g \rangle$ (otherwise $G/\langle g \rangle$ is not p-group). Hence there exists a nonzero integer k such that $h_2 = kh_1 + kh_2$, and so $kh_1 = 0$. Then H_1 is also cocyclic and necessarily a p-group. □

Here is the structure theorem for case **(A)**:

Theorem 1.1 *A group G satisfies* **(A)** *if and only if G is torsion with a cocyclic primary component and $r(G) > 1$.*

Proof. According to Lemma 1.1, let p be a prime such that $N = \langle a \rangle$ is cyclic of order p^k. If $m \in M \setminus N$ then $m + a \notin M$ (since $a \notin M$) and $N \leq \langle m + a \rangle$. Since $N \neq 0$ is torsion, $m + a$ and therefore m are of finite order. Hence M and, together with G/M, G are torsion.

Further, we show that $M_p \subseteq N$. Indeed, if $m \in M_p$, again, $N \subseteq \langle a + m \rangle$ so that $a = s(a + m)$ and $(1 - s)a = sm \in N \cap M = pN$ for a suitable nonzero integer s. Thus $s \equiv 1 \pmod{p}$ and let t be an inverse of s modulo the order of $m \in M_p$. Thus $m = tsm = t(1 - s)a \in N$.

Now, N and M being not comparable, $M_p \subset N$ and hence $pN = M \cap N = M_p \cap N = M_p$.

Since $M_p = pN \leq G_p$, Lemma 1.2 shows that G_p is a cocyclic group.

Conversely, suppose $G = G_p \oplus K$ with $G_p \simeq \mathbf{Z}(p^l)$, $K \neq 0$, $K_p = 0$ and take $N = G_p[p] = \langle a \rangle$ and $M = K$. If H is a subgroup of G such that $H \not\leq K$ we show $N \leq H$.

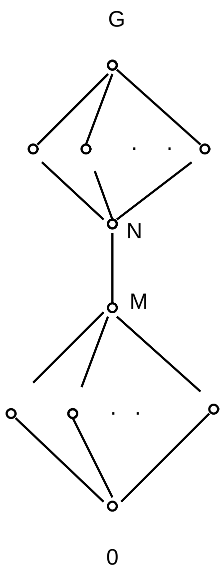
Indeed, since $H \not\leq K$, there is an element $h \in H \setminus K$. If this element decomposes as $h = g_p + k$ ($g_p \in G_p$, $k \in K$), then $g_p \neq 0$ and for a suitable multiple $p^s h = p^s(g_p + k)$ we have $0 \neq p^s g_p \in N$ respectively $p^s k \in K$. Since K is torsion and $K_p = 0$, denoting by u the order of $p^s k$, u and p are coprime and $up^s g_p \in H$. Finally, $p^s g_p \in H$ and thus $N = \langle p^s g_p \rangle \leq H$. □

Remarks. 1) The referee pointed out that a proof in Case (**A**) can be reduced to the proof of Case (**B**) using Lemma 1.1. Our proof uses Lemma 1.2 in both cases.

2) With above notations, $G/M = \bigoplus_{q\in\mathbf{P}}(G_q/M_q)$ is a p-group. Hence $G_q = M_q$ for all primes $q \neq p$ and $M = pN \oplus \bigoplus_{q\neq p, q\in\mathbf{P}} G_q$.

2 Abelian groups with (B)

Now we deal with subgroup lattices of the following type [a1.eps]



Here again $[0, M] \cap [N, G] = \emptyset$.

Although the following result was already stated in [1], we supply a specific "abelian" proof:

Lemma 2.1 *G is an abelian BP-group if and only if there is a prime p and* $k \in \mathbf{N}^* \cup \{\infty\}$, $k \geq 2$ *such that* $G \simeq \mathbf{Z}(p^k)$.

Proof. If $L(G) = [0, H] \cup [H, G]$, then (as noticed in the introduction) H is a cyclic subgroup. If p is a prime such that $pH \neq H$, then H/pH is simple, and using again $L(G) = [0, H] \cup [H, G]$, it is the smallest nonzero subgroup of G/pH. Hence G/pH is cocyclic and, having elements of order p (in H/pH), must be a p-group. Since an infinite cyclic group is not a BP-group, using Lemma 1.2, we obtain $G = H_1 \oplus H_2$ with cocyclic p-group H_1, cyclic q-group H_2, q and p are coprime and $H_2 \leq pH \leq H$. Obviously, $H_1 \not\leq H$ (otherwise $G = H$) so that $H_2 \leq H \leq H_1$. This implies $H_2 = 0$, and so G is cocyclic. Since $\mathbf{Z}(p)$ is not satisfying **(B)**, G has the requested form.

The converse is immediate (the subgroup lattice of $\mathbf{Z}(p^n)$ with $n \in \mathbf{N} \cup \{\infty\}$, $n \geq 2$ is a chain with at least 3 elements). □

Using this we obtain at once

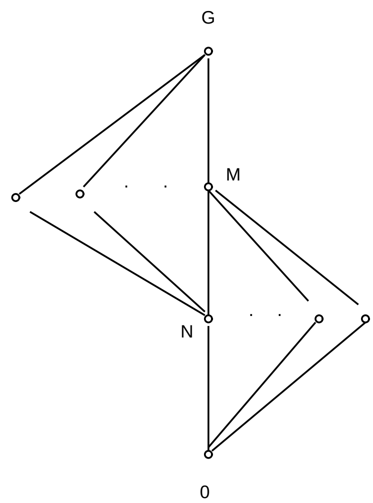
Theorem 2.1 *A group satisfies (***B***) if and only if* $G \simeq \mathbf{Z}(p^n)$ *with* $n \geq 3$.

Proof. If $L(G) = [0, M] \cup [N, G]$ and $M \leq N$ then $L(G) = [0, N] \cup [N, G]$ and so G is a BP-group. Hence G is cocyclic. Since the conditions $0 \neq M \neq N \neq G$ require at least 4 elements in $L(G)$, $G \simeq \mathbf{Z}(p^n)$ with $n \geq 3$.

The converse is obvious. □

3 Abelian groups with (C)

In this section we consider two proper subgroups $N < M$ such that $L(G) = [0, M] \cup [N, G]$. Thus the subgroup lattice looks like this [a3.eps]



Now $L(G) = [0, M] \cup [N, G]$ and $[0, M] \cap [N, G] = [M, N]$. Moreover, $[0, N] \subseteq [0, M]$ and $[M, G] \subseteq [N, G]$.

Theorem 3.1 *If a group G satisfies* **(C)** *then G is a torsion group and there exists a prime p such that G_p is a BP-group or satisfies* **(C)**. *Conversely, if G is a torsion group, $G_p \neq G$ for a prime p and G_p is a BP-group or satisfies (***C***), then G satisfies (***C***).*

Proof. Let $0 < N < M < G$ be such that $L(G) = [0, M] \cup [N, G]$.

If G is not a torsion group, there exists an infinite order element $x \in G$ such that $x \notin M$ (otherwise, since the infinite order elements generate any group, $M = G$). Then $0 < N \leq M \cap \langle x \rangle < \langle x \rangle$. If $L \leq \langle x \rangle$ then $L \leq M$ or $N \leq L$, hence $L \leq M \cap \langle x \rangle$ or $N \leq L$. Therefore $\langle x \rangle$ is a BP-group or satisfies **(C)**, but it is easy to see that no infinite cyclic group satisfies these properties (as for (**C**), if $0 < n\mathbf{Z} < m\mathbf{Z} < \mathbf{Z}$ and p is a prime not dividing n, then $p\mathbf{Z} \notin [0, m\mathbf{Z}] \cup [n\mathbf{Z}, \mathbf{Z}]$). This contradiction shows that G is a torsion group.

Suppose no component G_p is a BP-group or satisfies **(C)**. Since $M \neq G$, there exists a prime p such that $M_p \neq G_p$. If $N_p = 0$, then $G_p \subseteq M$ ($N \subseteq G_p$ is not possible,

N being a proper subgroup), hence $M_p = G_p$. Therefore $0 < N_p \leq M_p < G_p$ and $L(G_p) \neq [0, M_p] \cup [N_p, G_p]$. Then we can find $H_p \leq G_p$ such that $H_p \setminus M_p \neq \emptyset$ and $N_p \setminus H_p \neq \emptyset$. It follows $H_p \setminus M \neq \emptyset$ and $N \setminus H_p \neq \emptyset$, a contradiction.

Conversely, suppose G is torsion and G_p is a BP-group or satisfies **(C)**. Then we can find subgroups $0 < N_p \leq M_p < G_p$ such that $L(G_p) = [0, M_p] \cup [N_p, G_p]$. Set $M = M_p \oplus (\bigoplus_{q \neq p} G_q)$ and $N = N_p$. Thus $0 < N < M < G$. If $H \leq G$, then $H = H_p \oplus (\bigoplus_{q \neq p} H_q)$ with $H_p \leq G_p$ and $\bigoplus_{q \neq p} H_q \leq \bigoplus_{q \neq p} G_q$. If $N_p \leq H_p$, then $H \in [N, G]$ and if $H_p \leq M_p$, then $H \leq M_p \oplus (\bigoplus_{q \neq p} H_q) \leq M_p \oplus (\bigoplus_{q \neq p} G_q) = M$.

Actually, $G_p \neq G$ is needed only for a BP-group G_p not satisfying (**C**). □

Theorem 3.2 *A p-group G satisfies* **(C)** *if and only if* $G \cong \mathbf{Z}(p^n) \oplus B$ *such that* (i) $B \neq 0$, $n \in \mathbf{N}^* \cup \{\infty\}$ *and* $p^l B = 0$ *holds for a positive integer* $l < n$, *or,* ii) $B = 0$ *and* $n > 2$.

Proof. If G satisfies (**C**), we can suppose $N = \langle a \rangle \cong \mathbf{Z}(p)$. Let $l > 0$ be the smallest positive integer such that there exists $x \in G \setminus M$ with $p^l x = a$. Let $b \in G[p] \setminus \langle a \rangle$ and suppose $h_p(b) \geq l$. Then $b = p^l y$ for some $y \in M$ (if $y \notin M$ we have $a \in \langle y \rangle$, hence the rank of $\langle y \rangle [p]$ is at least 2, a contradiction). Thus $x + y \notin M$, and there exists a positive integer k such that $kx + ky = a$. If $k = p^r m$ with $\gcd(m; p) = 1$ then $p^r(mx + my) = a$, hence $l \leq r$. Moreover, $l \leq r$ implies $ky \in \langle a \rangle$ and $a \in \langle y \rangle$ follows, a contradiction. Then $h_p(b) < l$ for all $b \in G[p] \setminus \langle a \rangle$ and so $p^l G[p] = \langle a \rangle$. Hence $p^l G$ is a cocyclic group.

If $p^l G$ is a cyclic group then G is bounded and (using [2], **27.2**) $G = H \oplus B$ where $H \cong \mathbf{Z}(p^n)$ with $n \geq l+1$, $a \in H$ and $p^l B = 0$ (otherwise there is $b \in B[p]$ with $h_p(b) \geq l$). If $p^l G$ is a quasicyclic group, then $G = p^l G \oplus B$ and $p^l B = 0$.

Moreover, if $B = 0$ then $G \cong \mathbf{Z}(p^n)$ and condition $M \neq N$ implies $n > 2$.

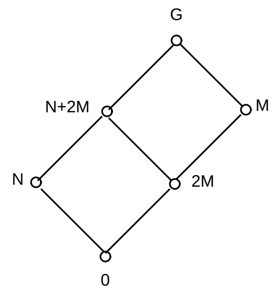
Conversely, if $B = 0$ then G satisfies condition (**C**) for $N = p^{n-1} G$ and $M = pG$. If $B \neq 0$ we choose $G = H \oplus B$ with $H \simeq \mathbf{Z}(p^n)$, $0 < l < n$ such that $p^l B = 0$, $N = H[p] = \langle a \rangle \cong \mathbf{Z}(p)$ and $M = A + B$ where A is the subgroup of H of order p^l (obviously containing N - the subgroup lattice of H being a chain with a smallest element). If X is a subgroup of G such that $X \notin [0, M]$, then there exists $x = h + b \in X \setminus M$ with $h \in H$ and $b \in B$ such that $p^r = \mathrm{ord}(h) > p^l$ (otherwise $h \in A$ and $x \in M$). By $p^l B = 0$ hypothesis, $0 \neq p^{r-1} x = p^{r-1} h \in H[p] = N$, hence $\langle p^{r-1} h \rangle = N$ is included in X. □

The only BP-groups which do not satisfy (**B**), nor (**C**) are $\mathbf{Z}(p^2)$ for any prime number p. Hence

Corollary 3.1 *A group G satisfies (***C***) if and only if it is a torsion group with a primary component* $\mathbf{Z}(p^n)$ *for* $n \geq 3$, *or* $G_p \cong \mathbf{Z}(p^n) \oplus B$ *with* $n > 1$ *or* ∞ *and* $p^l B = 0$ *holds for a nonnegative integer* $l < n$. □

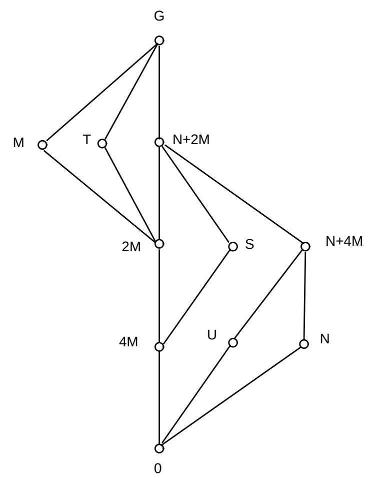
4 Comments

1. There are groups satisfying both conditions (**A**) and (**C**). As an example take $G = \mathbf{Z}(12) = \langle a, b | 3a = 4b = 0 \rangle$. Denoting by $N = \langle a \rangle$ and $M = \langle b \rangle$ the subgroup lattice looks like this [f3.eps]



Thus $L(G) = [0, M] \cup [N, G]$ for (**A**), and $L(G) = [0, N + 2M] \cup [2M, G]$ for (**C**).

2. If a group G satisfies, say, the condition (**C**) the pair M, N of subgroups is not necessarily unique. As an example, take the group $G = \mathbf{Z}(2) \oplus \mathbf{Z}(8) = \langle a, b | 2a = 8b = 0\rangle$. If we denote by $N = \langle a\rangle$, $M = \langle b\rangle$, $S = \langle a + 2b\rangle$, $T = \langle a + b\rangle$, $U = \langle a + 4b\rangle$, the subgroup lattice is now [c5.eps]



and $L(G) = [0, N + 2M] \cup [2M, G] = [0, N + 2M] \cup [4M, G]$.

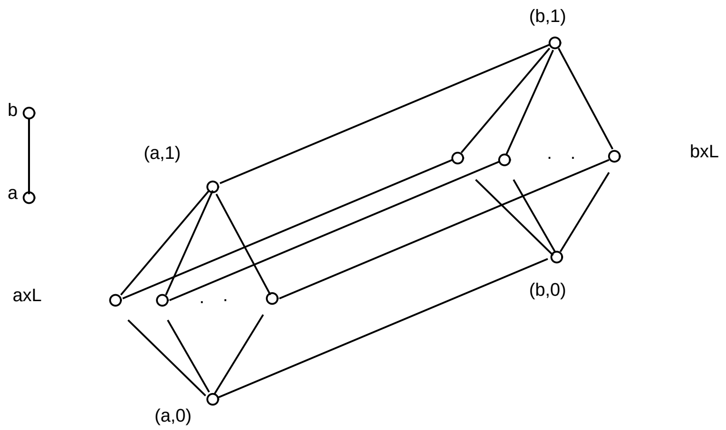
3. Our results generalize to lattices with 0 and 1, more or less arbitrary. In what follows we state some of these lattice versions.

- *If a lattice L satisfies condition* (**A**), *i.e.,* $L = [0, m] \cup [n, 1]$ *with incomparable elements* m, n *then*

(a) $[0, m \vee n] = [0, m] \cup [n, m \vee n]$ *i.e.,* $[0, m \vee n]$ *satisfies condition* (**A**);
(b) $[m \wedge n, 1] = [m \wedge n, m] \cup [n, 1]$ *i.e.,* $[m \wedge n, 1]$ *satisfies condition* (**A**);
(c) $[m \wedge n, m \vee n]$ *satisfies condition* (**A**).

- *Every direct product of two lattices, the first being a finite chain and the second having* 0 *and* 1, *satisfies condition* (**A**).

Proof. One uses the following Figure (for the sake of simplicity we have considered a chain with only two elements) [a4.eps]



Denoting the chain by $\{a, b\}$ and using elements in the Cartesian product $\{a, b\} \times L$, decomposition in the required intervals is $[(a, 0), (a, 1)] \cup [(b, 0), (b, 1)])$. $\square$

A family of torsion groups is said to be *coprime* if the orders of elements in any two members are coprime. Using an early Theorem of Suzuki (see [3]): *the groups with decomposable subgroup lattices are exactly the direct sums of coprime groups*, we have an alternative proof for sufficiency of Theorem 1.1 in the special case $k = 1$:

let G be a torsion group of rank $r(G) > 1$ with a simple p-component, i.e. $G = N \oplus M$ with $|N| = p$ and $M_p = 0$. Thus N and M are coprime, $L(G) \simeq L(N) \times L(M)$ and $L(N)$ is a chain with two elements. Applying the previous result, $L(G)$ satisfies condition (**A**). $\square$

- *Complemented lattices are not satisfying condition* (**C**).

- *Let* $\{L_i, i \in I\}$ *be an arbitrary set of bounded (i.e., with* 0_i *and* 1_i*) lattices, at least one of these satisfying condition* (**C**)*. Then the direct product* $L = \prod_{i \in I} L_i$ *satisfies condition* (**C**)*. Conversely, if* L *satisfies condition* (**C**)*, i.e.* $L = [\mathbf{0}, \alpha] \cup [\beta, \mathbf{1}]$ *and for an index* $j \in I$, $0_j < \beta_j < \alpha_j < 1_j$*, then* L_j *satisfies condition* (**C**).

- *If a lattice satisfies condition* (**C**)*, i.e.,* $L = [0, m] \cup [n, 1]$*, then* m *is essential and* n *is superfluous in* L*. Moreover, every element disjoint with* n *belongs to* $[0, m]$.

Finally we mention the lattice version of our initial proof of case (**A**):

- *Let* L *be a modular lattice,* n *an atom and* m *a dual atom in* L *such that* $1 = n \vee m$ *and* $n \wedge m = 0$*. Then* $L = [0, m] \cup [n, 1]$ *if and only if for every element* v *in* $[0, m]$*,* n *has a unique (relative) complement (namely* v*) in the sublattice* $[0, n \vee v]$.

Using this, one can show that, excepting the case $1 = n \vee m$ and $n \wedge m = 0$, (**C**) follows from (**A**).

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