An exercise on isomorphic idempotents

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Isomorphic idempotents are defined via R-module isomorphisms as follows (see **Proposition (21.20)** [2]) .

Definition. Let e, f be idempotents in a ring R. We say that e and f are *isomorphic* if $eR \cong fR$ as right R-modules. Equivalently, if $Re \cong Rf$ as left R-modules.

In the proposition mentioned above, *another equivalent definition* which avoids R-modules and their isomorphisms is given: there exist $a, b \in R$ such that $e = ab$ and $f = ba$.

Chapter 21 of [2], includes an exercise, whose solution (see [1]) uses R-modules and their isomorphisms.

In the sequel we provide a solution which avoids R-modules and their isomorphisms. Obviously we use the non-module equivalent definition mentioned above. We use the notation $\overline{e} = 1 - e$ for the complementary idempotent of e.

Ex. 21.16 Let e, f be idempotents in R.

(1) Show that e and f are conjugate iff $e \cong f$ and $\overline{e} \cong \overline{f}$.

Solution. $\Longrightarrow$ If $f = (u^{-1}e)u$ then $e = u(u^{-1}e)$ so $e \cong f$ (by the equivalent definition). Moreover, $\overline{f} = 1 - f = 1 - u^{-1}eu = u^{-1}(1-e)u = u^{-1}\overline{e}u$.

$\Longleftarrow$ Suppose $e \cong f$ and $\overline{e} \cong \overline{f}$. By the definition again, let $e = ab$, $f = ba$, $\overline{e} = cd$, $\overline{f} = dc$. Then for $u = af + c\overline{f}$ we have $u^{-1} = be + d\overline{e}$ and $ufu^{-1} = e$, $u\overline{f}u^{-1} = \overline{e}$.

The details.

$uu^{-1} = (af + c\overline{f})(be + d\overline{e}) = afbe + c\overline{f}d\overline{e} + afd\overline{e} + c\overline{f}be = ababab + cdcdcd + a(1-dc)dcd + c(1-ba)bab = e + \overline{e} + 0 + 0 = 1$.

Similarly $u^{-1}u = (be + d\overline{e})(af + c\overline{f}) = 1$.

Next $ufu^{-1} = (af + c\overline{f})f(be + d\overline{e}) = afbe + afd\overline{e} = e + 0 = e$ and similarly $u\overline{f}u^{-1} = (af + c\overline{f})\overline{f}(be + d\overline{e}) = c\overline{f}(be + d\overline{e}) = c\overline{f}be + c\overline{f}d\overline{e} = 0 + \overline{e} = \overline{e}$.

Of course, there is no "miracle" in finding the above unit u.

We disclose the process. The R-module proofs are in [2] and [1], respectively.

Proposition 21.20 [Proof] (3) $\Rightarrow$ (1) Given $e = ab$, $f = ba$, the maps $\theta : eR \to fR$, $\theta(x) = bx$ and $\theta' : fR \to eR$, $\theta'(y) = ay$ are inverse to each other and so right R-modules isomorphisms.

Ex. 21.15 [special case] Let $1 = e + \overline{e} = f + \overline{f}$. If $e \cong f$ and $\overline{e} \cong \overline{f}$, show that there exist a unit u such that $f = u^{-1}eu$ and $\overline{f} = u^{-1}\overline{e}u$.

Solution. We have

$$R_R = eR \oplus \overline{e}R = fR \oplus \overline{f}R.$$

Fix an isomorphism $\phi : fR \to eR$ and $\varphi : \overline{f}R \to \overline{e}R$. Then $\phi \oplus \varphi$ is an automorphism of R_R, given by a left multiplication by some unit u.

Specifically, given a, b, c, d as above we take $\phi(y) = ay$ and $\varphi(x) = cx$. Then for any $r \in R$ we write $r = 1 \cdot r = (f + \overline{f})r = fr + \overline{f}r$ and so $(\phi \oplus \varphi)(r) = \phi(fr) + \varphi(\overline{f}r) = (af + c\overline{f})r$. Hence $u = af + c\overline{f}$ is a suitable unit.

References

[1] T. Y. Lam *Exercises in Classical Ring Theory*. Second Edition, Problem Books in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[2] T. Y. Lam *A First Course in Noncommutative Rings*. Second Edition, Graduate Texts in Math., Vol. **131**, Springer-Verlag, Berlin-Heidelberg-New York, 2001.