Zero determinants revisited

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A ring is called *GCD* if every two nonzero elements have a gretest common divisor. In such rings we can prove that if a divides a product bc and $\gcd(a;b) = 1$ then a divides c.

Proposition 1 *Let R be a GCD (commutative) domain and $A \in \mathbb{M}_2(R)$. Then* $\det(A) = 0$ *iff there exist $a, b, c, d \in R$ with* $A = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix}$.

Proof. The "only if" part is easy. For the "if" part, let $A = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ with $xt = yz$. Denote $d = \gcd(x;z)$ and $x = dx'$, $z = dz'$ with coprime x', z'. Then $x't = yz'$ and so x' divides y and z' divides t. If $y = kx'$ and $t = lz'$ it is readily seen that $k = l$ and so $\begin{bmatrix} y \\ t \end{bmatrix} = k \begin{bmatrix} x' \\ z' \end{bmatrix}$. Finally, $\begin{bmatrix} x' \\ z' \end{bmatrix} \begin{bmatrix} d & k \end{bmatrix} = A$, as desired. ■

The proof shows how we decompose any given 2×2 matrix.

Examples. 1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$.

2) $A = \begin{bmatrix} 3 & 12 \\ 9 & 36 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 12 \end{bmatrix}$.

3) $A = \begin{bmatrix} 4 & 12 \\ 6 & 24 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 6 \end{bmatrix}$.

Can we generalize this to the $n \times n$ case ? Or at least for 3×3 ? **No.**

Proposition 2 *Let R be a GDS (commutative) domain and $A \in \mathbb{M}_n(R)$. Then* $\det(A) = 0$ *iff there exist $a_1, ..., a_n, b_1, ..., b_n \in R$ with* $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$.

Proof. The "only if" part holds over any commutative ring since it follows from the properties of determinants.

Indeed, $\det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}\right) = \det\begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix} =$

$a_1a_2 \det\begin{bmatrix} b_1 & b_2 & \cdots & b_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix} = 0.$

The "if" part fails. ■

Example. Take A (over any commutative *domain*; e.g. $\mathbb{Z}$) with equal nonzero rows 1 and 2 and in row 3 we have at least one zero entry but also a nonzero entry.

That is something like $A = \begin{bmatrix} a & b & c \\ a & b & c \\ 0 & 0 & d \end{bmatrix}$ withe nonzero a, b, c, c, d. Then $\det(A) = 0$ since two rows coincide but A is not a product CR with $C = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $R = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$. Indeed, the 3-rd row of CR is $\begin{bmatrix} a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & d \end{bmatrix}$ (since $d \neq 0$, we have $a_3 \neq 0$ over any domain) implies $b_1 = b_2 = 0$. But this contradicts, for example, $a_1b_1 = a \neq 0$.

Therefore, if one row has zero and nonzero entries, and in the column of one zero we have one nonzero entry, the matrix is not a product CR. The zero determinant condition is easy to realize: two dependent (or even equal) rows. This clearly holds also in the general $n \times n$ case.