DUALITIES FOR SELF–SMALL GROUPS

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Abstract. We construct a family of dualities on some subcategories of the quasi-category $\mathcal{S}$ of self-small groups of finite torsion-free rank which cover the class $\mathcal{S}$. These dualities extend several of those in the literature. As an application, we show that a group $A \in \mathcal{S}$ is determined up to quasi–isomorphism by the $\mathbb{Q}$–algebras $\{\mathbb{Q}\operatorname{Hom}(C, A) : C \in \mathcal{S}\}$ and $\{\mathbb{Q}\operatorname{Hom}(A, C) : C \in \mathcal{S}\}$. We also generalize Butler's Theorem to self-small mixed groups of finite torsion-free rank.

1. Introduction

There has been considerable recent interest in the class $\mathcal{S}$ of mixed finite rank self–small abelian groups, that is, mixed abelian groups G of finite torsion–free rank such that for every index set I, the natural homomorphism from $\operatorname{Hom}(G, G^{(I)})$ to $(\operatorname{Hom}(G, G))^{(I)}$ is an isomorphism.

The reason for this interest is that $\mathcal{S}$ is one of the few extensive classes of mixed abelian groups for which it is possible to formulate plausible and interesting structure theorems. Furthermore, this class is a common generalization of several well known special classes of groups, such as finite rank torsion–free groups, quotient divisible groups, and pure subgroups of direct products over infinitely many p of finite p–groups; see for example [AM75], [AGW95], [FoW95], [FoW98(1)] and [FoW98(2)]. Several characterisations of $\mathcal{S}$ and its subclasses are presented in [ABW09, Section 2].

The class $\mathcal{S}$ is not closed with respect to finite direct sums, see for example [ABW09, Proposition 2.5]. Therefore it is convenient to represent $\mathcal{S}$, as in [AB09], as a union of a family of subclasses $\mathcal{S}(X)$, indexed by sets X of primes, such that each $\mathcal{S}(X)$ is closed with respect to quasi–isomorphisms and finite direct sums.

These are the main novelties in our approach:

(1) We represent the groups in $\mathcal{S}$ and $\mathcal{S}(X)$ as pushouts of torsion-free groups and quotient divisible mixed groups.
(2) We describe dualities between the quasi–homomorphism categories $\mathcal{S}(X)$ and $\mathcal{S}(X^{\complement})$, where $X^{\complement}$ is the complement of X in the set of primes, which enables us to transfer properties between them.

The idea of using dualities to study modules dates back to the folklore of dual finite dimensional vector spaces. This concept was first applied in the context of abelian groups by Warfield [Warf68] who described an exact duality $\mathcal{F} = \operatorname{Hom}(-, A)$:

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$\mathcal{C} \to \mathcal{C}$, where A is a rank 1 torsion–free group and $\mathcal{C}$ is the category of A–locally free groups and group homomorphisms. These kinds of dualities, called *Warfield dualities*, were recently extended in [AB10] and [Br10] to the case in which A is a mixed group of torsion–free rank 1.

A category $\mathcal{C}$ whose objects are abelian groups is called a *quasi–category* if the morphisms are quasi–homomorphisms, i.e., $\mathrm{Hom}_{\mathcal{C}}(G, H) = \mathbb{Q}\,\mathrm{Hom}(G, H)$. This concept was exploited in two papers [Ar72(1)] and [Ar72(2)] of Arnold in 1972. He constructed a duality on the quasi–category of finite rank torsion–free quotient divisible groups G, that is, those having a full free subgroup F for which G/F is divisible. Arnold's duality used in an essential way the invariants introduced in [BP61] of the localisations $\widehat{\mathbb{Z}}_pG$ of G at primes p, as well as the vector space dual $\mathrm{Hom}(\mathbb{Q}G, \mathbb{Q})$ of the divisible hull.

Fomin [Fo87] and Vinsonhaler and Wickless [VW90] showed that every finite rank torsion–free abelian group G can be decomposed as a (non–direct) sum $G_1 + G_2$ of a locally free group G_1 (in the sense of Warfield) and a quotient divisible group G_2. They constructed a duality of the quasi–category of finite rank torsion–free abelian group by combining the Warfield duality on G_1 and the Arnold duality on G_2.

The first example of functors involving a subcategory of $\mathcal{S}$ occurs in [W94], in which Wickless constructs an equivalence between the quasi–subcategory $\mathcal{G}$ of $\mathcal{S}$ of groups which are divisible modulo torsion and the quasi–category of locally free finite rank torsion–free groups. Fomin and Wickless [FoW95] used these functors to establish a duality between these quasi–categories.

Then in 1998, [FoW98(1)], the same authors discovered a duality between the quasi–category of self–small groups which are divisible modulo some full free subgroup and the quasi–category of finite rank torsion–free groups which extends both Arnold's and Wickless' duality. This duality is described in Section 3 below. Also in 1998 [FoW98(2)] Fomin and Wickless specialized their results to several other subcategories of $\mathcal{S}$.

These numerous dualities were applied both to determine properties of the categories, such as the Krull–Schmidt property, and to establish existence of arbitrarily large indecomposable objects. Recently, Fomin proved in [Fo09] that the duality from [FoW98(1)] comes from a duality between some subcategories of Ab, the category of all abelian groups, which have as objects finite rank torsion-free groups and quotient divisible groups respectively. Moreover this duality is exact.

Some problems remained unsolved, in particular, the generalization of the duality to the full quasi–category $\mathcal{S}$, and that is what we shall do in this paper. In Section 2, we characterize $\mathcal{S}$ as a union of subcategories $\mathcal{S}(X)$ indexed by sets of primes which are closed under direct sums and homomorphic images.

In Section 3 we characterize self–small groups as pushouts of torsion–free and quotient divisible groups which enables us to extend the Fomin–Wickless duality to a family of dualities $d_X : \mathcal{S}(X) \leftrightarrows \mathcal{S}(X^{\complement}) : d'_X$, where $\mathcal{S}(X)$ are considered as quasi-categories (Theorem 3.7). Moreover, it is proved that $d'_X = d_{X^{\complement}}$ and the restrictions $d_X|_{\mathcal{S}(X)\cap\mathcal{S}(Y)}$ and $d_Y|_{\mathcal{S}(X)\cap\mathcal{S}(Y)}$ coincide.

Section 4 contains our applications, firstly a version of Fuchs' Problem 34 appropriate for $\mathcal{S}(X)$ and secondly an interpretation of Butler's Theorem in the context of $\mathcal{S}(X)$.

We use standard notations and terminology presented in [Ar82], [F70], [F73]. If G is a group and p is a prime then G_p denotes the p-component of G. For every set X of primes we denote by $X^{\complement}$ the complementary set $\mathbb{P} \setminus X$. Moreover, by rank we understand the torsion-free rank. For quasi-categories we refer to [Wa64].

2. Categories of self–small mixed groups

For any sets X and Y, we say $X \subseteq^{\bullet} Y$ if $X \setminus Y$ is finite and X is *quasi–equal* to Y, denoted $X \doteq Y$, if $X \subseteq^{\bullet} Y \subseteq^{\bullet} X$.

Let $\mathbb{P}$ be the set of primes. There are two subsets of $\mathbb{P}$ associated with a mixed group G of finite torsion–free rank:

$$S(G) = \{p \in \mathbb{P} \mid G_p \neq 0\} \text{ and } D(G) = \{p \in \mathbb{P} \mid (G/F)_p \text{ is divisible}\},$$

where F is a full free subgroup of G. The set $D(G)$ is well defined up to a quasi-equality of sets.

Self-small groups of finite torsion-free rank can be characterized in many ways (see [ABW09]). We recall here that a finite torsion-free rank group G is in $\mathcal{S}$ if and only if all its p-components are finite and $S(G) \subseteq^{\bullet} D(G)$. The class of finite rank torsion-free groups coincides with the subclass $\{G \in \mathcal{S} : S(G) = \varnothing\}$, and the class of quotient divisible groups with the subclass $\{G \in \mathcal{S} : D(G) \doteq \mathbb{P}\}$.

For any groups G and H, G is a *quasi–subgroup* of H, denoted $G \leq^{\bullet} H$, if for some integer n, $nG \subseteq H$ and G is *quasi–isomorphic* to H, denoted $G \overset{\bullet}{\cong} H$, if $G \cong K \leq^{\bullet} H \leq^{\bullet} L \cong G$. More generally, a *quasi–homomorphism* from G to H is an element of $\mathbb{Q}\operatorname{Hom}(G, H) = \mathbb{Q} \otimes \operatorname{Hom}(G, H)$ and a *quasi–isomorphism* is an invertible quasi–homomorphism. If $G \in \mathcal{S}$ and $H \overset{\bullet}{\cong} G$ then $H = K \oplus B$ with $K \in \mathcal{S}$ and B a bounded group (see [Br04, Lemma 2.7]). Therefore, *if $G \in \mathcal{S}$ and $H \overset{\bullet}{\cong} G$, we will assume that $H \in \mathcal{S}$.*

For every set X of primes we denote by $\mathcal{S}(X)$ the quasi–category whose objects are the set

$$\{G \in \mathcal{S} : S(G) \subseteq^{\bullet} X \subseteq^{\bullet} D(G)\}.$$

Lemma 2.1. *For all $X \subseteq \mathbb{P}$, the category $\mathcal{S}(X)$ is closed with respect to quasi-isomorphisms and finite direct sums. Moreover, if $G, H \in \mathcal{S}(X)$ and $\alpha : G \to H$ is a homomorphism then $\operatorname{Im}(\alpha) \in \mathcal{S}(X)$.*

Proof. Let $G \in \mathcal{S}(X)$ and let $H \overset{\bullet}{\cong} G$. Then for some integers m and n, $mH \cong nG$. It follows that $S(G) \doteq S(H)$. Moreover, there are full free subgroups $F_1 \leq mH$ and $F_2 \leq nG$ such that $mH/F_1 \cong nG/F_2$. Then $(H/F_1)_p \cong (G/F_2)_p$ for all primes p which are coprime with mn, so $D(H/F_1) \doteq D(G/F_2)$.

If $G, H \in \mathcal{S}(X)$, then clearly $G \oplus H \in \mathcal{S}(X)$.

Let $\alpha : G \to H$ be a homomorphism, and $G, H \in \mathcal{S}(X)$. Then $G \oplus H \in \mathcal{S}(X)$, so $\alpha(G) \in \mathcal{S}$ by [AM75]. It is obvious that $S(\operatorname{Im}(\alpha)) \subseteq S(H) \subseteq^{\bullet} X$, and $X \subseteq^{\bullet} D(G) \subseteq^{\bullet} D(\operatorname{Im}(\alpha))$. □

Corollary 2.2. *Let H be a group and let $G \leq H$ be a subgroup such that H/G is finitely generated. If X is a set of primes, then $H \in \mathcal{S}(X)$ if and only if $G \in \mathcal{S}(X)$.*

Proof. Since H/G is finitely generated, there is a subgroup $K \leq H$ which contains G such that K/G is finite and $H = K \oplus U$ with U a free subgroup of H. Since $U \in \mathcal{S}(X)$, we observe that $H \in \mathcal{S}(X)$ if and only if $K \in \mathcal{S}(X)$. Therefore the corollary follows from [Br04, Lemma 2.5] and Lemma 2.1. □

Corollary 2.3. *For all $X \subseteq \mathbb{P}$, $\mathcal{S}(X)$ is an additive category and for all G, $H \in \mathcal{S}(X)$, $\mathbb{Q}\operatorname{Hom}(G,H)$ is a finite dimensional $\mathbb{Q}$-algebra.*

Proof. The group $G \oplus H$ is self-small, so $\operatorname{Hom}(G \oplus H, T(G \oplus H))$ is a torsion group. Therefore the group $\operatorname{Hom}(G, T(H))$ is a torsion group and it follows that $\operatorname{Hom}(G,H)$ is of finite torsion-free rank. Hence $\mathbb{Q}\operatorname{Hom}(G,H)$ is a finite dimensional $\mathbb{Q}$-algebra. $\square$

3. Self–small groups as pushouts

The homological properties of pushouts and pullbacks of abelian groups are proved in [F70, Section 10]. We start this section with some results about pushouts which will be helpful in the construction of our duality. A part of the following lemma is implicit in [VW90, p. 474].

Lemma 3.1. *Suppose that A and B are two groups with a common subgroup F, and that the groups A, B are subgroups of a group G. Let P be the pushout induced by the diagram $A \hookleftarrow F \hookrightarrow B$, where we identify A and B as subgroups of P.*

Then $G \cong P$ if and only if $G/F \cong A/F \oplus B/F$.

Proof. Since P is the pushout of the diagram $A \hookleftarrow F \hookrightarrow B$, we can suppose that $P = (A \oplus B)/U$, where $U = \{(y, -y) \mid y \in F\}$ is the diagonal subgroup induced by F. Then F can be identified, as subgroup of P, with $\widehat{F} = \{(x,0) + U : x \in F\} = \{(0,x) + U : x \in F\}$. The reader can verify directly that the map $\alpha : P/\widehat{F} \to A/F \oplus B/F$, $\alpha((a,b)+\widehat{F}) = (a+F, b+F)$, is a well defined isomorphism.

Conversely, if $F \leq A, B \leq G$ and $G/F = A/F \oplus B/F$ then $G = A + B$, and $A \cap B = F$. It is not hard to verify that $\beta : G \to P$, $\beta(a+b) = (a,b) + U$ for all $a \in A$ and $b \in B$, is well defined and is an isomorphism. $\square$

Proposition 3.2. *Let X be a set of primes. A group G of rank n is in $\mathcal{S}(X)$ if and only if it is quasi-isomorphic to a group P which can be embedded in a pushout diagram*

$$(PO_X[F,K,L]) \qquad \begin{array}{ccccc} F & \longrightarrow & L & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \| \\ K & \xrightarrow[\beta]{} & P & \longrightarrow & U \\ \downarrow & & \downarrow & & \\ V & = & V & & \end{array}$$

where

1. *The three term rows and columns are short exact sequences;*
2. *F is a free group of rank n;*
3. *L is a torsion-free group of rank n;*
4. *U is a torsion group such that $S(U) \subseteq X^{\complement}$;*
5. *K is a quotient divisible group of rank n;*
6. *V is a torsion group such that $S(V) \subseteq X$.*
   Under these conditions
7. *For all $p \in \mathbb{P}$, $P_p = \beta(K_p)$.*

Proof. Let $G \in \mathcal{S}(X)$. Then we can suppose, modulo a finite direct summand, that $S(G) \subseteq X$. If F is a full free subgroup of G then G/F is p-divisible for almost all primes $p \in X$, so the set $Y = X \setminus D(G/F)$ is finite. Let R be the reduced part of $\oplus_{p\in Y}(G/F)_p$, a direct summand of G/F. Let P be a subgroup of G containing F such that P/F is a direct complement of R in G/F. Then P is a finite index subgroup of G, and P/F is p-divisible for all $p \in X$.

For this group P we consider the subgroups $F \leq K, L \leq P$ such that $K/F = \oplus_{p\in X}(P/F)_p$ and $L/F = \oplus_{p\in X^{\complement}}(P/F)_p$. Applying Lemma 3.1, we obtain P as a pushout with the properties (1) – (7).

Conversely, if P is the pushout in the diagram $PO_X[F,K,L]$ then $P \in \mathcal{S}(X)$, so any quasi–isomorphic $G \in \mathcal{S}(X)$. □

Quasi-homomorphisms in $\mathcal{S}(X)$. Now we consider two groups G, $G' \in \mathcal{S}(X)$, presented as push-outs using the diagrams $PO_X[F,K,L]$ and $PO_X[F',K',L']$ respectively.

If the homomorphism $\alpha : G \to G'$ represents a quasi-homomorphism in $\mathcal{S}(X)$ then we can suppose that $\alpha(F) \subseteq F'$. Then α induces a homomorphism $\overline{\alpha} : G/F \to G'/F'$. By the choice of K, L, K' and L', we have $\overline{\alpha}(K/F) \subseteq K'/F'$ and $\overline{\alpha}(L/F) \subseteq L'/F'$. Therefore $\alpha(K) \subseteq K'$ and $\alpha(L) \subseteq L'$, so α induces two homomorphisms $\kappa : K \to K'$, $\kappa(x) = \alpha(x)$, and $\lambda : L \to L'$, $\lambda(x) = \alpha(x)$.

Conversely, if $\kappa : K \to K'$ and $\lambda : L \to L'$ are two homomorphisms such that $\kappa|_F = \lambda|_F$, then they induce a homomorphism $\alpha : G \to G'$, defined by $\alpha(y+z) = \kappa(y) + \lambda(z)$ for all $y \in K$ and $z \in L$.

Lemma 3.3. *With the notations above,*

(1) $\mathrm{Im}(\alpha) \doteq \mathrm{Im}(\kappa) + \mathrm{Im}(\lambda)$,
(2) $\mathrm{Ker}(\alpha) \doteq \mathrm{Ker}(\kappa) + \mathrm{Ker}(\lambda)$.

Proof. (1) is obvious since $\alpha(x) = \kappa(k) + \lambda(l)$, whenever $x = k + l$ with $k \in K$ and $l \in L$.

(2) The inclusion $\mathrm{Ker}(\kappa) + \mathrm{Ker}(\lambda) \subseteq \mathrm{Ker}(\alpha)$ is obvious.

To prove the converse quasi-inclusion, we consider the subgroup

$$U = \{k \in K : \exists l \in L \text{ such that } k + l \in \mathrm{Ker}(\alpha)\} \leq G.$$

If $x = k + l \in \mathrm{Ker}(\alpha)$ with $k \in K$ and $l \in L$ then $\alpha(k) = \alpha(-l) \in K' \cap L' = F'$. Then $\alpha(U) \subseteq F'$, so $\alpha(U)$ is a free group. It follows that $U = \mathrm{Ker}(\alpha|_U) \oplus H = (\mathrm{Ker}(\alpha) \cap K) \oplus H$ with $H \cong \alpha(U)$ a free subgroup of $U \leq G$. It follows that there is a non-zero integer n such that $nU \subseteq \mathrm{Ker}(\alpha) \cap K + F = \mathrm{Ker}(\kappa) + F$.

If $x \in n\mathrm{Ker}(\alpha)$, then $x = nk + nl$ with $k \in U$ and $l \in L$. Hence $nk = y + z$ with $y \in \mathrm{Ker}(\kappa)$ and $z \in F$. Since $F \subseteq L$, $z + nl \in L$, it follows that $x = y + (z + nl) \in \mathrm{Ker}(\kappa) + \mathrm{Ker}(\lambda)$. □

Recall that a sequence $G \overset{\alpha}{\to} G' \overset{\alpha'}{\to} G''$ is *quasi-exact* (i.e. exact in the quasi–category of abelian groups) if and only if $\mathrm{Im}(\alpha) \doteq \mathrm{Ker}(\alpha')$.

Theorem 3.4. *Let*

$$(E) \qquad G \overset{\alpha}{\to} G' \overset{\alpha'}{\to} G''$$

be a sequence of groups in $\mathcal{S}(X)$ and homomorphisms. Suppose that

i) *these groups are represented by the diagrams*

$PO_X[F,K,L]$, $PO_X[F',K',L']$ *and* $PO_X[F'',K'',L'']$ *respectively.*

ii) *the homomorphisms* α *and* α' *induce the homomorphisms*

$$K \xrightarrow{\kappa} K' \xrightarrow{\kappa'} K'', \text{ and} \tag{E_κ}$$

$$L \xrightarrow{\lambda} L' \xrightarrow{\lambda'} L', \text{ respectively.} \tag{E_λ}$$

Then the sequence (E) *is quasi-exact if and only if the sequences* (E_κ) *and* (E_λ) *are quasi-exact.*

Proof. Suppose that (E) is an exact sequence and that n is a non-zero integer such that $n\mathrm{Im}(\alpha) \subseteq \mathrm{Ker}(\alpha')$.

Let $x \in \mathrm{Im}(\kappa) \subseteq \mathrm{Im}(\alpha)$. Then $nx \in \mathrm{Ker}(\alpha') \doteq \mathrm{Ker}(\kappa') + \mathrm{Ker}(\lambda')$, so there are an integer $m > 0$ and elements $y \in \mathrm{Ker}(\kappa') \subseteq K'$, $z \in \mathrm{Ker}(\lambda') \subseteq L'$ such that $mnx = y + z$. It follows that $z \in K' \cap L' = F'$, so $\kappa'(z) = \lambda'(z) = 0$. Then $mnx \in \mathrm{Ker}(\kappa')$, and $mn\mathrm{Im}(\kappa) \subseteq \mathrm{Ker}(\kappa')$.

Let $x \in \mathrm{Ker}(\kappa') \setminus mn\mathrm{Im}(\kappa)$. Since $\mathrm{Ker}(\kappa')$ is quasi-contained in $\mathrm{Im}(\alpha) \doteq \mathrm{Im}(\kappa) + \mathrm{Im}(\lambda)$, there are an integer $r \neq 0$ for which $r\mathrm{Ker}(\kappa') \subseteq \mathrm{Im}(\kappa) + \mathrm{Im}(\lambda)$ and elements $u \in K$ and $v \in L$ such that $rx = \kappa(u) + \lambda(v)$. Since $mn\kappa(u) \in \mathrm{Ker}(\kappa')$, it follows that $mn\lambda(v) \in \mathrm{Ker}(\kappa')$. But $mn\lambda(v) \in L'$, and it follows that $mn\lambda(v) \in \mathrm{Im}(\lambda) \cap \mathrm{Ker}(\kappa') \leq K' \cap L' = F'$.

Then there exists a non-zero integer r such that for every $x \in \mathrm{Ker}(\kappa') \setminus mn\mathrm{Im}(\kappa)$ there is $y \in \mathrm{Im}(\lambda) \cap \mathrm{Ker}(\kappa')$ such that $mnrx - mny \in mn\mathrm{Im}(\kappa)$.

Since $\mathrm{Im}(\lambda) \cap \mathrm{Ker}(\kappa') \leq K' \cap L' = F'$ is a free group, $\lambda^{-1}(\mathrm{Im}(\lambda) \cap \mathrm{Ker}(\kappa')) = \mathrm{Ker}(\lambda) \oplus F_1$, where F_1 is a free subgroup of L. Then there is an integer $s \neq 0$ such that $sF_1 \subseteq F$, and it follows that $\lambda(sz) = \kappa(sz)$ for all $z \in F_1$.

For every $x \in \mathrm{Ker}(\kappa') \setminus mn\mathrm{Im}(\kappa)$ and $y \in \mathrm{Im}(\lambda) \cap \mathrm{Ker}(\kappa')$ such that $mnrx - mny \in mn\mathrm{Im}(\kappa)$, we pick an element $z \in F_1$ such that $\lambda(z) = y$. Since $sy = \lambda(zs) = \kappa(sz) \in \mathrm{Im}(\kappa)$ and $mnrsx - mnsy = s(mnrx - mny) \in mn\mathrm{Im}(\kappa)$, it follows that $mnrsx \in \mathrm{Im}(\kappa)$, so $mnrs\mathrm{Ker}(\kappa') \subseteq \mathrm{Im}(\kappa)$.

Then $\mathrm{Im}(\kappa) \doteq \mathrm{Ker}(\kappa')$, so the sequence (E_κ) is exact. The exactness for (E_λ) is proved in the same way.

If (E_κ) and (E_λ) are quasi-exact sequences, then the sequence (E) is quasi-exact as a consequence of Lemma 3.3. □

Fomin-Wickless duality. In the following we summarize some useful properties of the duality

$$^* : \mathcal{A} \rightleftarrows \mathcal{D} : ^*$$

where $\mathcal{A}$ is the quasi–category of finite rank torsion-free groups and $\mathcal{D}$ is the quasi–category of all quotient divisible groups, constructed by Fomin and Wickless in [FoW98(1), Theorem 10].

In this theorem, they associate with every group G in $\mathcal{A}$ or $\mathcal{D}$ a full free subgroup F of G, and construct a dual G^* of G and a full free subgroup F^* of G^*. Moreover, if $f : G_1 \to G_2$ is a homomorphism, and F_1, F_2 are the corresponding full free subgroups, they construct the dual homomorphism f^* by means of a rational matrix L which represents the transition from the coefficients of elements of G with respect to a basis of F and the coefficients of elements of G^* with respect to a basis of F^*.

These dual groups have the following properties:

Lemma 3.5. *Let $G \in \mathcal{A}$ or $\mathcal{D}$ and let F be a full free subgroup of G. Let $p \in \mathbb{P}$.*

1. *$(G/F)_p = 0$ if and only if G^* is p-divisible.*
2. *If $G \in \mathcal{A}$, then G^*_p is isomorphic to the reduced summand of $(G/F)_p$*
3. *Suppose that $f : G \to G'$ is a homomorphism between two groups G, G' in $\mathcal{A}$ or in $\mathcal{D}$ with full free subgroups F and F' having bases $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_m)$ respectively, used in the construction of G^* and G'^*. If $(x^*_1, \dots, x^*_n)$ and $(x'^*_1, \dots, x'^*_m)$ are the corresponding bases of the dual groups G^* and G'^* and L is a $n \times m$ matrix with integral entries such that*

$$(f(x_1), \dots, f(x_n))^t = L(x'_1, \dots, x'_m)^t, \quad \textit{then}$$
$$(f^*(x'^*_1), \dots, f^*(x'^*_m))^t = L^t(x^*_1, \dots, x^*_n)^t.$$

We will also use the following

Remark 3.6. If we restrict the functors * to the class $\mathcal{A} \cap \mathcal{D}$ we obtain Arnold's duality constructed in [Ar72(2)]. So, when we compute G^* for a group $G \in \mathcal{A} \cap \mathcal{D}$, does not matter if we consider G a torsion–free group or a quotient divisible group.

The main result. Now we are ready to prove the main result of this paper. Recall that for any set X of primes, $\mathbb{Z}_X$ is the rank 1 ring divisible by a prime p if and only if $p \in X^{\complement}$.

Theorem 3.7. *Let X be a set of primes. The quasi–categories $\mathcal{S}(X)$ and $\mathcal{S}(X^{\complement})$ are dual.*

Proof. If X is finite then the quasi–categories $\mathcal{S}(X)$ and $\mathcal{A}$, respectively $\mathcal{S}(X^{\complement})$ and $\mathcal{D}$, are equivalent. Therefore for the cases X is finite or $X^{\complement}$ is finite, the dualities are supplied by [FoW98(1), Theorem 10].

Thus we can suppose that X and $X^{\complement}$ are infinite sets of primes. Let $G \in \mathcal{S}(X)$. Then we can suppose that $S(G) \subseteq X$ and that there is a full free subgroup F of G such that $(G/F)_p$ is p-divisible for all $p \in X$. We can view G as a pushout as in the diagram $PO_X[F, K, L]$ from Proposition 3.2.

We apply the duality * to the groups $K \in \mathcal{D}$ and $L \in \mathcal{A}$ together with the (common) full free subgroup F. Then we obtain two groups $K^* \in \mathcal{A}$ and $L^* \in \mathcal{D}$, and we can suppose that they have a common full free subgroup F^* (constructed in the proof of [FoW98(1), Theorem 10]). Note that K^* is p-divisible for all $p \in X^{\complement}$ and L^* is p-divisible for all $p \in X$ (and $L^*_p = 0$ for all $p \in X$).

For these groups we consider the subgroups $F^* \leq K^*_{(X)} \leq K^*$ and $F^* \leq L^*_{(X^{\complement})} \leq L^*$ such that $K^*_{(X)}/F^* = \oplus_{p \in X}(K^*/F^*)_p$ and $L^*_{(X^{\complement})}/F^* = \oplus_{p \in X^{\complement}}(L^*/F^*)_p$. We define the dual $d(G)$ of G to be the pushout of the diagram $K^*_{(X)} \hookleftarrow F^* \hookrightarrow L^*_{(X^{\complement})}$:

$$(PO_{X^{\complement}}[F^*, L^*_{(X^{\complement})}, K^*_{(X)}]) \qquad \begin{array}{ccc} F^* & \longrightarrow & K^*_{(X)} \\ \downarrow & & \downarrow \\ L^*_{(X^{\complement})} & \longrightarrow & d(G) \end{array}$$

Observe that $d(G) \in \mathcal{S}(X^{\complement})$ as a consequence of Proposition 3.2.

Now, we consider two groups G, $G' \in \mathcal{S}(X)$ with full free subgroups F with basis $(x_1, \dots, x_n)$ and F' with basis $(x'_1, \dots, x'_n)$ respectively. Constructing the dual groups $d(G)$ and $d(G')$ as above, we will use the subgroups K, L for G and K', L' for G'.

If $\alpha : G \to G'$ is a homomorphism then we can suppose that $\alpha(F) \subseteq F'$. Then α induces a homomorphism $\overline{\alpha} : G/F \to G'/F'$. By the choice of K, L, K' and L', we have $\overline{\alpha}(K/F) \subseteq K'/F'$ and $\overline{\alpha}(L/F) \subseteq L'/F'$. Therefore $\alpha(K) \subseteq K'$ and $\alpha(L) \subseteq L'$, so α induces two homomorphisms $\kappa : K \to K'$, $\kappa(x) = \alpha(x)$, and $\lambda : L \to L'$, $\lambda(x) = \alpha(x)$. If $\kappa^* : K'^* \to K^*$ and $\lambda^* : L'^* \to L^*$ are the dual maps constructed in [FoW98(1), Theorem 10], then $\kappa^*(x) = \lambda^*(x) \in F^*$ for all $x \in F'^*$ as a consequence of Lemma 3.5. Therefore, $\kappa^*(K^*_{(X)}) \subseteq K^*_{(X)}$ and $\lambda^*(L^*_{(X^{\complement})}) \subseteq L^*_{(X^{\complement})}$. These data induce a homomorphism $d(\alpha) : d(G') \to d(G)$ by $d(\alpha)(k'^* + l'^*) = \kappa^*(k'^*) + \lambda^*(l'^*)$ (this is well defined since $K'^* \cap L'^* = F'^*$ and $\kappa^*(x) = \lambda^*(x) \in F^*$ for all $x \in F'^*$).

It is not hard to see that we have defined a contravariant functor $d : \mathcal{S}(X) \to \mathcal{S}(X^{\complement})$. We shall prove that this is a duality.

We consider a group $H \in \mathcal{S}(X^{\complement})$ and a pushout diagram

$$(PO_{X^{\complement}}[F, V, U]) \qquad \begin{tikzcd} F \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & H \end{tikzcd}$$

with $V \in \mathcal{D}$ and $U \in \mathcal{A}$ such that $(V/F)_p = 0$ for all $p \in X$ and $(U/F)_p = 0$ for all $p \in X^{\complement}$.

Then we consider the groups $V \otimes \mathbb{Z}_{X^{\complement}} \in \mathcal{D}$ and $U \otimes \mathbb{Z}_X \in \mathcal{A}$ with the same common full free subgroup F.

We apply the duality * to obtain two groups $(U \otimes \mathbb{Z}_X)^* \in \mathcal{D}$ and $(V \otimes \mathbb{Z}_X)^* \in \mathcal{A}$ with a common full free subgroup F^* such that $(U^*/F^*)_p = 0$ for all $p \in X^{\complement}$ and $(V^*/F^*)_p = 0$ for all $p \in X$. Then the pushout $d'(H)$ of the diagram $U^* \hookleftarrow F^* \hookrightarrow V^*$ can be embedded in a pushout diagram of the form $PO_X[F^*, U^*, V^*]$.

To define d' on quasi-homomorphisms, we consider a homomorphism $\beta : H \to H'$, with H, $H' \in \mathcal{S}(X^{\complement})$. Let F, U, V groups associated to H as in the construction of $d'(H)$, and let F', U', V' be corresponding subgroups associated to H'. We can suppose that $\beta(F) \subseteq F'$, so β induces two homomorphisms $\mu : U \to U'$ and $\nu : V \to V'$. Using again Lemma 3.5 we deduce that the dual maps $\kappa : (\mu \otimes 1_{\mathbb{Z}_X})^* : U'^* \to U^*$ and $\lambda : (\nu \otimes 1_{\mathbb{Z}_{X^{\complement}}})^* : V'^* \to V^*$ have the property $\kappa(x) = \lambda(x)$ for all $x \in F^*$. Therefore they induce a homomorphism $d'(\beta) : d'(H') \to d'(H)$, $d'(\beta)(y + z) = \kappa(y) + \lambda(z)$ for all $y \in U'^*$ and $z \in V'^*$.

A direct computation shows that d is a duality with inverse d'. $\square$

In [AW04] the authors observed that the dualities $^* : \mathcal{A} \rightleftarrows \mathcal{D} :^*$ preserve the torsion-free rank and the quasi-exactness in $\mathcal{A}$ and $\mathcal{D}$ (see also [Fo09]). Using Theorem 3.4 and the proof of Theorem 3.7, we have the following

Corollary 3.8. *The dualities d and d' constructed in Theorem 3.7 preserve torsion-free rank and quasi-exactness.*

If X is a set of prime, then we will denote by $d_X : \mathcal{S}(X) \to \mathcal{S}(X^{\complement})$ and $d'_X : \mathcal{S}(X^{\complement}) \to \mathcal{S}(X)$ the functors constructed in Theorem 3.7.

Corollary 3.9. *If X is a set of primes, then $d'_X = d_{X^{\complement}}$.*

Proof. Using the notations from the proof of Theorem 3.7, and using Lemma 3.5 we observe that $(U \otimes \mathbb{Z}_X)^* = U^*_{(X)}$ and $(V \otimes \mathbb{Z}_X)^* = V^*_{(X^{\complement})}$. Therefore $d'_X = d_{X^{\complement}}$. $\square$

Example 3.10. If X is any set of primes then $\mathbb{Q} \in \mathcal{S}(X)$. We can view $\mathbb{Q}$ as a pushout $PO_X(Z, \mathbb{Z}_X, \mathbb{Z}_{X^{\complement}})$. Since $\mathbb{Z}_X^* = \mathbb{Z}_{X^{\complement}}$ and $\mathbb{Z}_{X^{\complement}}^* = \mathbb{Z}_X$, $(\mathbb{Z}_X^*)_{(X)} = \mathbb{Z} = \mathbb{Z}_{X^{\complement}}^*$ and $d_X(\mathbb{Q})$ will be the pushout of the diagram $PO_{X^{\complement}}[\mathbb{Z}, \mathbb{Z}, \mathbb{Z}]$. Therefore $d_X(\mathbb{Q}) = \mathbb{Z}$.

Example 3.11. Let X be a set of primes and $G \in \mathcal{S}(X)$. If G is a quotient divisible group or a torsion–free group then $d_X(G) \stackrel{\cdot}{\cong} G^*$.

Proof. Suppose that G is a quotient divisible group. From the previous example we know that $d_X(\mathbb{Q}) = \mathbb{Z} = \mathbb{Q}^*$. Therefore, we can suppose that G is reduced. We view G as a pushout $PO_X(G, K, L)$ as in Proposition 3.2, where we fixed a full free subgroup $F \leq G$. Using the notations from [FoW98(1)] for the quotient divisible groups G, K and L, we observe that $M_p^G = M_p^K$ for all $p \in X$ and $M_p^G = M_p^L$ for all $p \in X^{\complement}$. From the construction of G^* presented in [FoW98(1), p.49], it follows that $G^* = K_{(X)}^* + L_{(X^{\complement})}^*$. Since $K_{(X)}^* \cap L_{(X^{\complement})}^* = F^*$, it follows from Lemma 3.1 that G^* is the pushout of the diagram $K_{(X)}^* \hookleftarrow F^* \hookrightarrow L_{(X^{\complement})}^*$. Hence $G^* = d_X(G)$.

If G is a torsion–free group then $d_X(G)$ is a quotient divisible group, so

$$G^{**} \stackrel{\cdot}{\cong} G \stackrel{\cdot}{\cong} d_{X^{\complement}}(d_X(G)) = (d_X(G))^*,$$

and it follows that $G^* \stackrel{\cdot}{\cong} d_X(G)$. □

This example will be generalized in the following results.

Lemma 3.12. *Let $X \subseteq Y$ be two sets of primes. If $G \in \mathcal{S}(X) \cap \mathcal{S}(Y)$ then $d_X(G) = d_Y(G)$.*

Proof. Let F be a full free subgroup of $G \in \mathcal{S}(X) \cap \mathcal{S}(Y)$. We consider the pushout diagrams $PO_X(F, K_X, L_X)$ and $PO_Y(F, K_Y, L_Y)$ constructed as in Proposition 3.2. Let $Z = Y \setminus X$ and consider the subgroup $F \leq G_{(Z)} \leq G$ such that $G_{(Z)}/F = \oplus_{p \in Z}(G/F)_p$. Then $K_Y = K_X + G_{(Z)}$ and $L_X = G_{(Z)} + L_Y$. We note that $G_{(Z)}$ is a torsion-free quotient divisible group. By Remark 3.6, $(G_{(Z)})^*$ is the Arnold dual of $G_{(Z)}$, so it does not matter whether we consider it as an element of $\mathcal{A}$ or of $\mathcal{D}$. To simplify the notation, we let M be the subgroup of $(G_{(Z)})^*$ such that $F^* \leq M$ and $M/F^* = \oplus_{p \in Z}(G_{(Z)})_p^*$. From the construction of d_X and d_Y it follows that $(K_Y^*)_{(X)} = (K_X^*)_{(X)} + M$ and $(L_X)_{(X^{\complement})} = (L_Y^*)_{(X^{\complement})} + M$, so both $d_X(G)$ and $d_Y(G)$ are the colimit of the diagram

$$(F^* \hookrightarrow (K_X^*)_{(X)};\ F^* \hookrightarrow (L_Y^*)_{(X^{\complement})};\ F^* \hookrightarrow M),$$

so that $d_X(G) = d_Y(G)$. □

Theorem 3.13. *Let X and Y be two sets of primes. If $G \in \mathcal{S}(X) \cap \mathcal{S}(Y)$ then $d_X(G) = d_Y(G)$.*

Proof. If $G \in \mathcal{S}(X) \cap \mathcal{S}(Y)$ then $G \in \mathcal{S}(X \cap Y)$. By Lemma 3.12 we deduce that $d_X(G) = d_{X \cap Y}(G) = d_Y(G)$. □

4. Applications

Fuchs Problem 34. In [AB09] the authors prove a quasi–isomorphism criterion for groups in $\mathcal{S}(X)$ which is related to Problem 34 posed in [F70].

This result states that two groups $A, B \in \mathcal{S}(X)$ are quasi-isomorphic provided that $\mathbb{Q}\operatorname{Hom}(A, C) \cong \mathbb{Q}\operatorname{Hom}(B, C)$ for all $C \in \mathcal{S}(X)$. In the theory of finitely generated modules over artinian algebras, a similar theorem was observed by Auslander in [Aus82] and Bongartz [Bo89] found a nice simple proof for full subcategories of abelian categories which are closed with respect direct sums and kernels. Since the classes $\mathcal{S}(X)$ are not closed with respect to kernels, Bongartz's idea is not directly applicable to the full subcategory $\mathcal{S}(X)$ of the quasi–category $\mathbb{Q}Ab$ of all abelian groups and quasi-isomorphisms. However, his idea can be applied for the dual problem (even though $\mathcal{S}(X)$ is not closed with respect to cokernels):

Theorem 4.1. *Let X be a set of primes and $A, B \in \mathcal{S}(X)$. The following statements are equivalent:*

(1) $A \overset{\cdot}{\cong} B$.
(2) $\mathbb{Q}\operatorname{Hom}(C, A) \cong \mathbb{Q}\operatorname{Hom}(C, B))$ *for all groups* $C \in \mathcal{S}(X)$.

Proof. It is trivial that (1) implies (2).

Conversely, assume that (2) holds. We claim that there exists a strongly essentially indecomposable direct quasi-summand of A which is isomorphic to a quasi-summand of B. Let $\alpha_1, \dots, \alpha_n \in \operatorname{Hom}(A, B)$ be a family of homomorphisms such that $1 \otimes \alpha_1, \dots, 1 \otimes \alpha_n$ is a basis of the $\mathbb{Q}$-linear space $\mathbb{Q}\operatorname{Hom}(A, B)$. If $\alpha : A \to B^n$ is the homomorphism induced by $\alpha_1, \dots, \alpha_n$, we consider the exact sequence $A \xrightarrow{\alpha} B^n \to G \to 0$, where $G = B^n/\operatorname{Im}(\alpha)$. Applying the contravariant functors $\mathbb{Q}\operatorname{Hom}(-, A)$ and $\mathbb{Q}\operatorname{Hom}(-, B)$ we obtain the exact sequences

$$0 \to \mathbb{Q}\operatorname{Hom}(G, A) \to \mathbb{Q}\operatorname{Hom}(B^n, A) \to \mathbb{Q}\operatorname{Hom}(A, A)$$

and

$$0 \to \mathbb{Q}\operatorname{Hom}(G, B) \to \mathbb{Q}\operatorname{Hom}(B^n, B) \to \mathbb{Q}\operatorname{Hom}(A, B) \to 0,$$

where the latter sequence is exact on the right since $\mathbb{Q}\operatorname{Hom}(A, B)$ is generated by the $1 \otimes \alpha_i$ described above.

We denote by D the torsion divisible subgroup of G and let $K = \oplus_{p \in X^{\complement}} G_p$. If $H = G/(D+K)$, we observe that $\mathbb{Q}\operatorname{Hom}(G, A) = \mathbb{Q}\operatorname{Hom}(H, A)$ and $\mathbb{Q}\operatorname{Hom}(G, B) = \mathbb{Q}\operatorname{Hom}(H, B)$ since the torsion parts of A and B are reduced and $S(A), S(B) \subseteq^{\bullet} X$. Moreover, if F is a full free subgroup of G then $F' = (F + (D + K))/(D + K)$ is a full free subgroup of H and H/F' is p-divisible for almost all $p \in X$ since H/F' is a torsion epimorphic image of B^n.

Therefore $H \in \mathcal{S}(X)$ so

$$\mathbb{Q}\operatorname{Hom}(G, A) = \mathbb{Q}\operatorname{Hom}(H, A) = \mathbb{Q}\operatorname{Hom}(H, B) = \mathbb{Q}\operatorname{Hom}(G, B).$$

It follows that the sequence

$$0 \to \mathbb{Q}\operatorname{Hom}(G, A) \to \mathbb{Q}\operatorname{Hom}(B^n, A) \to \mathbb{Q}\operatorname{Hom}(A, A) \to 0$$

is exact, so the sequence $0 \to A \xrightarrow{\alpha} B^n \to G \to 0$ splits in the category $\mathbb{Q}Ab$. Therefore A is quasi-isomorphic to a quasi-direct summand of B^n. Since $\mathcal{S}(X)$ is an additive Krull-Schmidt category (see [Br04]), there is an essentially strongly indecomposable quasi-summand of A which is not torsion and is quasi-isomorphic to a quasi-summand of B.

To complete the proof we can proceed as in the proofs of [AB09, Theorem 2.8] and [Bo89], using induction on the torsion–free rank m of A. If $m = 0$ then A is finite, and it follows that $\operatorname{Hom}(B, B)$ is a torsion group. This is possible only if

B is also a finite group. Suppose that the property (2) implies (1) is valid for all groups in $\mathcal{S}(X)$ of torsion–free rank $< m$. By what we just proved, it follows that $A \doteq A_1 \oplus A'$ and $B \doteq B_1 \oplus B'$ with A_1 and B_1 quasi-isomorphic essentially strongly indecomposable groups of torsion free rank ≥ 1.

For all $C \in \mathcal{S}(X)$, we have

$$\mathbb{Q}\operatorname{Hom}(C, A) \cong \mathbb{Q}\operatorname{Hom}(C, A_1) \oplus \mathbb{Q}\operatorname{Hom}(C, A'),$$

$$\mathbb{Q}\operatorname{Hom}(C, B) \cong \mathbb{Q}\operatorname{Hom}(C, B_1) \oplus \mathbb{Q}\operatorname{Hom}(C, B')$$

and

$$\mathbb{Q}\operatorname{Hom}(C, A_1) \cong \mathbb{Q}\operatorname{Hom}(C, B_1)$$

so that A' and B' satisfy (2).

Consequently $\mathbb{Q}\operatorname{Hom}(C, A') \cong \mathbb{Q}\operatorname{Hom}(C, B')$ for all $C \in \mathcal{S}(X)$ so by induction $A' \stackrel{\cdot}{\cong} B'$. Hence $A \stackrel{\cdot}{\cong} B$. □

Now, using our duality Theorem 3.7 we can prove Theorem 2.8 from [AB09].

Theorem 4.2. *Let X be a set of primes and $A, B \in \mathcal{S}(X)$. The following statements are equivalent:*

(1) $A \stackrel{\cdot}{\cong} B$.
(2) $\mathbb{Q}\operatorname{Hom}(A, C) \cong \mathbb{Q}\operatorname{Hom}(B, C))$ *for all groups* $C \in \mathcal{S}(X)$.

Proof. Once again, (1) implies (2) is trivial, so assume $\mathbb{Q}\operatorname{Hom}(A, C) \cong \mathbb{Q}\operatorname{Hom}(B, C))$ for all groups $C \in \mathcal{S}(X)$. Then we have $\mathbb{Q}\operatorname{Hom}(D, d(A)) \cong \mathbb{Q}\operatorname{Hom}(D, d(B))$ for all groups $D \in \mathcal{S}(X^{\complement})$. Therefore by Theorem 4.1, $d(A) \stackrel{\cdot}{\cong} d(B)$, so $A \stackrel{\cdot}{\cong} B$. □

Mixed Butler groups. We shall prove, using Theorem 3.7, a version of Butler's Theorem, [But65], for groups in $\mathcal{S}(X)$. In order to do this we use an idea from [AW04].

Recall that the main result proved by Butler states that a torsion-free group G is a pure subgroup of a finite rank completely decomposable group (i.e. there is an exact sequence $0 \to G \to C \to H \to 0$ in $\mathcal{A}$ with C a completely decomposable group) if and only if G is an epimorphic image of a finite rank completely decomposable group (i.e. there is an exact sequence $0 \to H \to C \to G \to 0$ in $\mathcal{A}$ with C a completely decomposable group). Moreover, it is not hard to see that the class of Butler groups is closed with respect quasi-isomorphisms, so the exactness of the above sequences can be replaced with quasi-exactness.

Theorem 4.3. *The following are equivalent for a group $G \in \mathcal{S}(X)$:*

(1) *There is a quasi-exact sequence in $\mathcal{S}(X)$ of the form*

$$(\sharp) \qquad 0 \to G \to \oplus_{i=1}^{m} A_i \to H \to 0,$$

such that all groups A_i are of torsion-free rank 1;

(2) *There is a quasi-exact sequence in $\mathcal{S}(X)$ of the form*

$$(\flat) \qquad 0 \to H \to \oplus_{i=1}^{m} A_i \to G \to 0,$$

such that all groups A_i are of torsion-free rank 1.

Proof. (1) $\Rightarrow$ (2). Suppose that each group U involved in the short exact sequence ($\sharp$) are given by the diagrams $PO_X[F_U, K_U, L_U]$. Then we obtain two quasi-exact sequences

$$(\sharp_L) \qquad 0 \to L_G \to \oplus_{i=1}^m L_{A_i} \xrightarrow{\varphi} L_H \to 0$$

and

$$(\sharp_K) \qquad 0 \to K_G \to \oplus_{i=1}^m K_{A_i} \to K_H \to 0.$$

For the exact sequence $(\sharp_L)$ we will use the proof of Butler's Theorem presented in [Ri90] to observe that $\mathrm{Ker}(\varphi)$ is an epimorphic image of a finite direct sum $\oplus B_i$ where all B_i are rank 1 subgroups of $\oplus_{i=1}^m L_{A_i}$. Since $\mathrm{type}(L_{A_i})$ is 0 on every component which corresponds to a prime $p \in X$, the types of groups B_i have the same property. Therefore all groups B_i and $\mathrm{Ker}(\alpha)$ are in $\mathcal{S}(X)$. Since $L_G \doteq \mathrm{Ker}(\varphi)$ we can replace α by an integral multiple of it such that $\mathrm{Im}(\alpha) \subseteq L_G$, so we have a quasi-epimorphism $\alpha : \oplus B_i \to L_G$.

For the exact sequence $(\sharp_K)$, a careful analysis of the proof of [AW04, Theorem 8] shows that there is a quasi-epimorphism $\beta : \oplus C_j \to K_K$ such that each C_j is a finite direct sum of torsion–free rank 1 quotient divisible groups $C_i \in \mathcal{S}(X)$ and $\mathrm{Ker}(\beta) \in \mathcal{S}(X)$. For the reader's convenience we give here the details for this proof: We apply the functor $d_X = (-)^*$ to the exact sequence $\sharp_K$ to obtain a quasi-exact sequence

$$(\sharp_K^*) \qquad 0 \to K_H^* \to \oplus_{i=1}^m K_{A_i}^* \to K_G^* \to 0.$$

In this sequence all groups are $X^{\complement}$-divisible torsion-free groups and all groups $K_{A_i}^*$ are of rank 1. Therefore K_G^* is a Butler group, and it follows that there is an exact sequence

$$0 \to K_G^* \to \oplus_{i=1}^m U_i \to V \to 0.$$

of $X^{\complement}$-divisible torsion-free groups with all groups U_i of rank 1. Now, applying the functor $(-)^* = d_{X^{\complement}}$ we obtain a quasi-exact sequence

$$0 \to V^* \to \oplus_{i=1}^m U_i^* \to K_G^{**} \to 0.$$

with all groups quotient divisible groups in $\mathcal{S}(X)$, and the existence of β is proved.

Let $\delta : (\oplus B_i) \oplus (\oplus C_j) \to G$ be the homomorphism induced by α and β. Since $G = K_G + L_G$, it follows that δ is a quasi-epimorphism. To complete the proof, we need to prove that $\mathrm{Ker}(\delta) \in \mathcal{S}(X)$. By Corollary 2.2, it is enough to prove that $\mathrm{Ker}(\delta)/(\mathrm{Ker}(\alpha) \oplus \mathrm{Ker}(\beta))$ is finitely generated.

Let $F_1 \leq \oplus B_i$ and $F_2 \leq \oplus C_j$ be finitely generated free subgroups such that such that $\alpha^{-1}(F_G) = \mathrm{Ker}(\alpha) \oplus F_1$ and $\beta^{-1}(F_G) = \mathrm{Ker}(\beta) \oplus F_2$. Let $x = b + c \in \mathrm{Ker}(\delta)$ with $b \in \oplus B_i$ and $c \in \oplus C_j$. Since $\alpha(b) + \beta(c) = 0$, $\alpha(b) = \beta(-c) \in F_G$. Then $b \in \alpha^{-1}(F_G) = \mathrm{Ker}(\alpha) \oplus F_1$, where F_1 is a finitely generated free subgroup of $\oplus B_i$. In the same way we deduce that $c \in \mathrm{Ker}(\beta) \oplus F_2$. Note that F_1 and F_2 are independent of x, b and c so $\mathrm{Ker}(\delta) \subseteq (\mathrm{Ker}(\alpha) \oplus \mathrm{Ker}(\beta)) + (F_1 \oplus F_2)$. Then $\mathrm{Ker}(\delta)/\mathrm{Ker}(\alpha) \oplus \mathrm{Ker}(\beta)$ is finitely generated, and it follows that $\mathrm{Ker}(\delta) \in \mathcal{S}(X)$.

(2) $\Rightarrow$ (1) We apply the functor d to the quasi-exact sequence ($\flat$) to obtain a quasi-exact sequence

$$(d(\flat)) \qquad 0 \to d(G) \to \oplus_{i=1}^m d(A_i) \to d(H) \to 0$$

in $\mathcal{S}(X^{\mathtt{C}})$. Applying the implication (1) $\Rightarrow$ (2) we observe that there exists a quasi-exact sequence

$$0 \to H' \to \oplus_{i=1}^{m} A_i' \to d(G) \to 0$$

in $\mathcal{S}(X^{\mathtt{C}})$. Applying again the duality d we obtain the required short quasi-exact sequence. □

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